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A NOTE ON THE REALIZATION OF TYPES

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Let L be a countable first-order language and T a fixed complete theory in L. If $\underline{M}$ is a model of T, $\bar{v}=\langle v_1,\ldots,v_n\rangle$ is an n-sequence of variables, and $\bar{a}=\langle a_1,\ldots,a_n\rangle$ is an n-sequence of elements of M, the universe of $\underline{M}$, we let $Tp_{\bar{v}}(\bar{a},\underline{M})=\{\phi(\bar{v}):\underline{M}\models\phi^{\bar{v}}_{\bar{a}}\}$ where $\phi(\bar{v})$ ranges over formulas of L containing freely at most the variables $v_1,\ldots,v_n$. $\bar{a}$ is said to realize $Tp_{\bar{v}}(\bar{a},\underline{M})$ in $\underline{M}$. We let $Tp(\bar{a},\underline{M})$ be $Tp_{\bar{v}}(\bar{a},\underline{M})$ where $\bar{v}$ is the sequence of the first n variables of L. Sets of the form $Tp_{\bar{v}}(\bar{a},\underline{M})$ as above are called n-types; p, q, r will range over n-types in T below, and will sometimes be written $p(\bar{v})$, etc., to indicate their free variables. We say p is a type in T if it is an n-type in T for some n.

Theorem 1 below can easily be derived from Theorem 6.1 of [1]. For completeness, and as the object of later comment, we present a direct proof.

If $\underline{M}$ is a model of T, we let $\mathcal{T}(\underline{M})$ be the set of types realized in $\underline{M}$ by sequences of elements of M.

THEOREM 1. *A countable set Y of types in T is $\mathcal{T}(\underline{M})$ for some (countable) model $\underline{M}$ of T if and only if Y satisfies all the following conditions:*

(1) *Y is closed under* 1-1 *substitution of variables; if $p\in Y$, $q\subseteq p$, then $q\in Y$.*

(2) *If $p(v_0,\ldots,v_k)\in Y$, $q(v_{k+1},\ldots,v_n)\in Y$, there is $r(v_0,\ldots,v_n)\in Y$ such that $p\subseteq r$ and $q\subseteq r$.*

(3) *If $p(v_0,\ldots,v_k)\in Y$ and $\exists v_{k+1}\theta(v_0,\ldots,v_k,v_{k+1})=\psi\in p$ then there is $q\in Y$ such that $p\subseteq q$ and $\theta\in q$.*

Proof. $\rightarrow$: It is easy to check that (1), (2), (3) are always satisfied by $\mathcal{T}(\underline{M})$.

$\leftarrow$: Let $C=\{c_0,c_1,\ldots,c_n,\ldots\}$ be a collection of new constants not in L and let L_1 be the language obtained by adjoining these to L. Let ψ_n, $n\in\omega$, be an enumeration of all sentences of L_1 of the form $\exists v_i\phi$ for some $i\in\omega$, such that each appears infinitely often.

Let $p_n(v_0,\ldots,v_{s_n})$, $n\in\omega$, be an enumeration of Y.

We shall define by induction on n a sequence of theories T_n, such that for each $n>0$, there will be a $q_n(v_0,\ldots,v_k)\in Y$ and constants $c_{i_0},\ldots,c_{i_k}$ C such that $T_n=q_n(c_{i_0},\ldots,c_{i_k})$.

Let $T_0=T$. Suppose T_n has been defined and is $q_n(c_{i_0},\ldots,c_{i_k})$.

If $n=2m+1$, let $\ell=k+s_m+1$ and use (2) to obtain a type $r(v_0,\ldots,v_\ell)$ such that $n\in Y$, $q_n(v_0,\ldots,v_k)\subseteq r$, and $p_m(v_{k+1},\ldots,v_\ell)\subseteq r$. Let $c_{i_{k+1}},\ldots,c_{i_\ell}$ be

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the first elements in the enumeration of C not among $c_{i_0}, \ldots, c_{i_k}$. Finally, let $T_{n+1} = r(c_{i_0}, \ldots, c_{i_\ell})$.

If $n = 2m+2$, let $\psi_m = \exists v_i \phi$. If ϕ contains an occurence of a constant of C not among $c_{i_0}, \ldots, c_{i_k}$ or if $\sim\exists v_i\phi \in q_n(c_{i_0}, \ldots, c_{i_k})$ Let $T_{n+1} = T_n$. Otherwise there is $\phi'(v_0, \ldots, v_k, v_{k+1})$ such that ψ_m is equivalent to $\exists v_{k+1}\phi'(c_{i_0}, \ldots, c_{i_k}, v_{k+1})$ and so $\exists v_{k+1}\phi'(v_0, \ldots, v_k, v_{k+1}) \in q_n(v_0, \ldots, v_k)$. By (3), there is a $p(v_0, \ldots, v_{k+1}) \in Y$ such that $\phi' \in p$ and $q_n \subseteq p$. Let c' be the first element of C not in $\{c_{i_0}, \ldots, c_{i_k}\}$ and let $T_{n+1} = p(c_{i_0}, \ldots, c_{i_k}, c')$.

Let $T' = \bigcup_n T_n$. (Note that by the construction $T_n \subseteq T_{n+1}$ for all n.) Form a structure $\underline{M}$ from T' in the usual way whose universe is $\{[c]: c \in C\}$ where $[c] = \{d \in C: c = d \in T'\}$, and for all relation symbols R, $(R[c_1] \cdots [c_n] \leftrightarrow Rc_1 \cdots c_n \in T')$. Then a simple induction will verify that this structure is a model of every sentence in T'. It is therefore a model of T.

If $c_{i_0}, \ldots, c_{i_n} \in E$, the type realized by $[c_{i_0}], \ldots, [c_{i_n}]$ in $\underline{M}$ is a subtype of some q_m and hence by (1) is in Y. But by the steps at the odd stages of the construction, $Y \subseteq \mathcal{T}(\underline{M})$. Hence we have obtained a countable model $\underline{M}$ of T such that $\mathcal{T}(\underline{M}) = Y$, as required.

A structure $\underline{M}$ is homogeneous if whenever $\bar{a}$, $\bar{b}$ are sequences from M, $c \in M$, *and* $Tp(\bar{a}, \underline{M}) = Tp(\bar{b}, \underline{M})$, then there is $d \in M$ such that $Tp(\bar{a}^\frown c, M) = Tp(\bar{b}^\frown d, \underline{M})$.

THEOREM 2. *A countable set Y of n-types in T is $\mathcal{T}(\underline{M})$ for some (countable) homogeneous model of T if and only if Y satisfies all the following conditions.*

(1) *as in Theorem* 1

(2′) *If* $p(v_0, \ldots, v_k) \in Y$, $q(v_0, \ldots, v_n) \in Y$. $-1 < \ell < k < n$, *and for all* $\phi(v_0, \ldots, v_\ell)$, $(\phi \in p \leftrightarrow \phi \in q)$ (*if* $\ell = -1$ *this condition is vacuous*), *then there is* $r(v_0, \ldots, v_m) \in Y$ *where* $m = (n + k - \ell)$ *such that* $p(v_0, \ldots, v_k) \subseteq r$ *and* $q(v_0, \ldots, v_\ell, v_{k+1}, \ldots, v_m) \subseteq r$.

(3) *as in Theorem* 1.

REMARK. The reader should observe that (2′) above is essentially (2) with "amalgamation of a common subtype".

Proof. Suppose $\underline{M}$ is homogeneous. We show that $\mathcal{T}(\underline{M})$ satisfies (2′). Let p, q be given as in (2′), and suppose $\bar{a}$, $\bar{b}$ satisfy p, q, respectively, in $\underline{M}$. Thus $Tp(\bar{a}\restriction\ell, \underline{M}) = Tp(\bar{b}\restriction\ell, \underline{M})$ and so by homogeneity, there is $\bar{c} \in M^{n-\ell}$ such that $Tp(\bar{a}\restriction\ell^\frown\bar{c}, \underline{M}) = Tp(\bar{b}, \underline{M})$. But then $Tp(\bar{a}^\frown\bar{c}, \underline{M})$ satisfies the conclusion of (2′).

$\leftarrow$ Note that (2′) implies condition (2) of Theorem 1, since T is complete (take $\ell = 0$). We show how to modify the construction in Theorem 1 so the resulting model is homogeneous. Let s_n, $n \in \omega$, be an enumeration of all 3-tuples $\langle \bar{c}_1, \bar{c}_2, c\rangle$ where $\bar{c}_1, \bar{c}_2 \in C^\ell$ for some ℓ (depending on n), $c \in C$ and c does not appear in the sequences $\bar{c}_1$, $\bar{c}_2$, such that each such 3-tuple appears infinitely often in the list. Now there will be three ways of constructing T_{n+1}

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from T_n corresponding to the value of n (mod 3). If $n \equiv 1$ or 2 (mod 3), we do exactly as in Theorem 1 for $n \equiv 1$ or 2 (mod 2). Suppose $n = 3m$ and $s_m = \langle c_1, \bar{c}_2, c \rangle$; if not all the constants in s_m appear among $c_{i_0}, \ldots, c_{j_k}$, set $T_{n+1} = T_n$; if $\{\phi : \phi(\bar{c}_1) \in T_b\} \neq \{\phi : \phi(\bar{c}_2) \in T_n\}$ set $T_{n+1} = T_n$; otherwise use (2′) to get $T_{n+1} = q_{n+1}(\bar{c}_2, c', \bar{c}_1, c, d)$ where c' is the first new constant in C, $q_{n+1} \in Y$, $T_n \subseteq T_{n+1}$, and $\{\phi : \phi(c_1, c) \in T_{n+1}\} = \{\phi : \phi(\bar{c}_2, c') \in T_{n+1}\}$. This will make the final model homogeneous.

We show how to do this when $\bar{d}$, $\bar{c}_1$, and $\bar{c}_2$ are all sequences of length 1; the argument in the general case is easily constructed from this. So we suppose $T_n = q(c_2, c_1, c, d)$ with $q \in Y$, and let $p = \{\psi(v_0, v_1) : \psi(v_1, v_2 \in q\}$; thus p is the type to be realized by $\langle c_1, c \rangle$ in the model being constructed. Note that p is obtained by a 1-1 substitution into a subtype of q, so is in Y. Further, the hypotheses of (2′) are satisfied when $\ell = 0$, so by (2′) there is $r(v_0, \ldots, v_4) \in Y$ such that $p \subseteq r$ and $q(v_0, v_2, v_3, v_4) \subseteq r$. Let $q_{n+1} = r$.

EXAMPLE 1. Let $\underline{M} = \mathbf{Z} \times \mathbf{2}$ (the product of order types). Let $a_0 = \langle 0, 0 \rangle$, $a_1 = \langle 0, 1 \rangle$, $p = Tp(\langle a_0, a_1 \rangle, \underline{M})$, and $q = p(v_1, v_0)$. Then p, q violate (2′) for $Y = \mathcal{T}(\underline{M})$ and $\ell = 0$. Hence $\mathcal{T}(\underline{M})$ is not $\mathcal{T}(\underline{M}')$ for any homogeneous $\underline{M}$.

EXAMPLE 2. Let $\underline{M}_1 = \mathbf{Z} \times \mathbf{Z}$ and $\underline{M}_2 = \mathbf{Z} \times \mathbf{Q}$. Then $\underline{M}_1$ is not homogeneous, $\underline{M}_2$ is homogeneous, and $\mathcal{T}(\underline{M}_1) = \mathcal{T}(\underline{M}_2)$.

We remark that a homogeneous structure $\underline{M}$ is determined up to isomorphism by $\mathcal{T}(\underline{M})$. This has been combined with the fact that an uncountable $\mathbf{\Sigma}_1^1$ sets of reals has the power of the continuum to conclude the same for the set of isomorphism types of homogeneous models of a complete theory. Using Theorem 2, it is not hard to see that one can get by with the classical theorem for $\mathbf{\Pi}_2^0$ sets.

COROLLARY 1. *Let $\underline{M}_n$, $n \in \omega$, be structures such that $\mathcal{T}(\underline{M}_n) \subseteq \mathcal{T}(\underline{M}_{n+1})$ for all n. Then there is $\underline{M}$ such that $\mathcal{T}(\underline{M}) = \bigcup_n \mathcal{T}(\underline{M}_n)$.*

Proof. It is simple to verify that the conditions (1), (2), (3) of Theorem 1 are preserved under directed unions.

Note that if the structures in Corollary 1 were homogeneous, they would form an elementary chain (with appropriate embeddings) and the conclusion would be a simple consequence of Tarksi's theorem on elementary chains. However, there does not appear to be a direct model-theoretic construction which will yield the $\underline{M}$ of the conclusion from the $\underline{M}_n$'s of the hypothesis. We remark that an application of the general Omitting Types Theorem of [3] can be used to prove Corollary 1 directly. The author does not know whether Theorems 1 and 2 have simple proofs from the Omitting Types Theorem.

We conclude with some questions. (i) Is there a reasonable characterization of the pairs $\langle Y_1, Y_2 \rangle$ of sets of types for which there exist structures $\underline{M}_1$ and $\underline{M}_2$ such that $Y_1 = \mathcal{T}(\underline{M}_1)$, $Y_2 = \mathcal{T}(\underline{M}_2)$ and $\underline{M}_1 < \underline{M}_2$? (As far as the author knows

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it might be all pairs $\langle Y_1, Y_2\rangle$ such that $Y_1 \subseteq Y_2$, and both Y_1 and Y_2 satisfy the hypotheses of Theorem 1). (ii) Is there a characterization similar to Theorem 1 of the sets of n-types (for a fixed n) realized in some structure? This seems interesting even for $n = 1$.

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