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A NIL-IMPLIES-NILPOTENT RESULT IN ARTINIAN RINGS

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It is shown that if the ring A is left Artinian and L_1 and L_2 are left ideals of A then L_1 is nilpotent modulo L_2 if L_1 is nil modulo L_2.

An easy consequence of Levitzki's theorem is that if the ring A is left Noetherian and L is a left ideal which is nil modulo I, where I is a two-sided ideal, then L will be nilpotent modulo I. This can be proved by considering the ring A/I. The problem is: what if I is a left ideal?

In this note I solve the problem for left Artinian rings, but a proof (or counter example) is still lacking in the left Noetherian case:

THEOREM *Suppose the ring A is left Artinian (not necessarily with 1). Let L_1 and L_2 be left ideals of A. Then L_1 is nil mod L_2 if and only if L_1 is nilpotent mod L_2.*

Proof $\Leftarrow$ is trivial.
$\Rightarrow$: Suppose L_1 is nil mod L_2. Consider the descending chain:

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$$L_1 \supseteq L_1^2 \supseteq L_1^3 \supseteq \dots \supseteq L_1^n = L_1^{n+1} = B = B^2 = \dots .$$

Suppose $B^2 = B \not\subseteq L_2$. Let H be a minimal left ideal contained in L_1 such that $BH \not\subseteq L_2$. Then there is an element $h \in H$ such that $Bh \not\subseteq L_2$. But now $B(Bh) = B^2h = Bh \not\subseteq L_2$ and $Bh \subseteq H$ which forces $Bh = H$ by the minimality of H. Let $b \in B$ be such that $h = bh$. Hence, $h = bh = b^2h = \dots = b^qh \in L_2h$ for q large enough since $b \in B \subseteq L_1$ and L_1 is nil mod L_2. Consequently, there is an element $\ell \in L_1 \cap L_2$ such that $h = \ell h$ and the following relation holds for all integers $i,j \geq 1$:

$$h^j = \ell^i h^j$$

Let p and r be the smallest integers such that $(\ell + h)^p \in L_2$ and $h^r \in L_2$ $(p,r \geq 2)$. Let $t = \max\{p,r\}$. Then

$$(\ell+h)^t = h + k_2h^2 + k_3h^3 + \dots + k_{r-1}h^{r-1} + k_rh^r + \dots$$
$$+ h^t + \text{terms that end with an } \ell$$

is an element of L_2, where the k_i are integers. Hence

$$h + k_2h^2 + k_3h^3 + \dots + k_{r-1}h^{r-1} \in L_2 \dots \qquad (*)$$

If $r = 2$, then (*) would imply that $h \in L_2$, a contradiction.
If $r > 2$, then

$$h^{r-2}(h + k_2h^2 + \dots + k_{r-1}h^{r-1})$$
$$= h^{r-1} + k_2h^r + \dots + k_{r-1}h^{2r-3} \in L_2$$

which imply that $h^{r-1} \in L_2$, contradicting the minimality of r.

Hence we must have $L_1^n = B \subseteq L_2$.

Examples suggest the validity of the result for left Noetherian rings (with 1, of course). However, to the best of my knowledge this remains an unsolved open problem.

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References

[1] N. Divinsky, *Rings and radicals*, (University of Toronto Press, 1965).

[2] A.P.J. Van der Walt, "A variation on nil and nilpotent", *Quaestiones Math.* 7 (1984), 101 - 104.

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