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KORÁNYI'S LEMMA FOR HOMOGENEOUS SIEGEL DOMAINS OF TYPE II. APPLICATIONS AND EXTENDED RESULTS

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Abstract

We show that the modulus of the Bergman kernel $B(z, \zeta)$ of a general homogeneous Siegel domain of type II is 'almost constant' uniformly with respect to z when ζ varies inside a Bergman ball. The control is expressed in terms of the Bergman distance. This result was proved by A. Korányi for symmetric Siegel domains of type II. Subsequently, R. R. Coifman and R. Rochberg used it to establish an atomic decomposition theorem and an interpolation theorem by functions in Bergman spaces A^p on these domains. The atomic decomposition theorem and the interpolation theorem are extended here to the general homogeneous case using the same tools. We further extend the range of exponents p via functional analysis using recent estimates.

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1. Introduction

Let $\mathcal{D} \subset \mathbb{C}^n$ be a homogeneous Siegel domain of type II and let $B : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ denote the Bergman kernel of $\mathcal{D}$. The Bergman distance function $d : \mathcal{D} \times \mathcal{D} \to \mathbb{R}^+$ is defined from the Bergman metric on $\mathcal{D}$. In this article, we prove the following statement.

THEOREM 1.1 (Korányi's lemma). *For every $\rho > 0$, there exists a constant $M_\rho > 0$ such that*

$$\left| \frac{B(z, \zeta_2)}{B(z, \zeta_1)} - 1 \right| < M_\rho d(\zeta_1, \zeta_2)$$

for all $z, \zeta_1, \zeta_2 \in \mathcal{D}$ with $d(\zeta_1, \zeta_2) < \rho$.

This theorem was proved by Korányi (see [8]) for symmetric Siegel domains of type II and was used respectively in [8] and [19] to establish successively an atomic decomposition theorem and an interpolation theorem by functions in Bergman spaces A^p on these domains (see also [7] and [1]). It was then clear that if Korányi's lemma

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could be extended to general homogeneous Siegel domains of type II, these theorems may be extended with the same proofs to the general case.

The validity of these results was restricted to exponents $p > 1$ for which there exists L^p-boundedness of the integral operator with kernel given by the modulus of the Bergman kernel; a range of such p was given in [6]. It was proved in [5] and [2] that there is an extended range of p for which the weighted Bergman projector is bounded in L^p on tube domains over symmetric cones (symmetric Siegel domains of type I). Moreover, in [4], the atomic decomposition theorem of functions in L^p was extended via functional analysis to this extended range of p on these domains. Recently, Nana [16] proved that results of [2] and [5] above remain valid in the case of general homogeneous Siegel domains of type II. This gives the way to the extension with the same functional analysis proof of the atomic decomposition theorem of functions in A^p for this extended range of $p > 1$ on these general domains.

The paper is organised as follows: in Sections 2 and 3, we shall give two proofs of Theorem 1.1. The uniform boundedness of Cayley transform images of the homogeneous Siegel domain $\mathcal{D}$ of type II plays a key role in both proofs. In Section 4, we state the Coifman–Rochberg atomic decomposition theorem and Rochberg's interpolation theorem on general homogeneous Siegel domains of type II. Finally, in Section 5, we state the extension of the atomic decomposition theorem for an extended range of $p > 1$ on these general domains and sketch the functional analysis proof.

2. First proof of Theorem 1.1

It was shown in [22] that $B(w,z) \neq 0$ for all $w, z \in \mathcal{D}$. Since $\mathcal{D}$ is convex and hence simply connected, it is well known (see [15, Proposition 12.7.1, page 254]) that the holomorphic function $w \mapsto B(w,z)$ possesses a holomorphic logarithm $w \mapsto \log B(w,z)$, that is, a holomorphic function on $\mathcal{D}$ whose exponential is $w \mapsto B(w,z)$.

Since $B(z,w) = \overline{B(w,z)}$ for all $z, w \in \mathcal{D}$, our purpose is to show that

$$\left|\frac{B(\zeta_2,z)}{B(\zeta_1,z)} - 1\right| = |e^{\log B(\zeta_2,z)-\log B(\zeta_1,z)} - 1| \leq M_\rho d(\zeta_1,\zeta_2)$$

for all $z, \zeta_1, \zeta_2 \in \mathcal{D}$ with $d(\zeta_1,\zeta_2) < \rho$. We shall rely on the following elementary inequality:

$$|e^s - 1| \leq |s|e^{|s|} \quad (s \in \mathbb{C}).$$

It therefore suffices to prove that there exists a constant $C_{\mathcal{D}} > 0$ such that

$$|\log B(\zeta_2,z) - \log B(\zeta_1,z)| \leq C_{\mathcal{D}} d(\zeta_1,\zeta_2)$$

for all $z, \zeta_1, \zeta_2 \in \mathcal{D}$ with $d(\zeta_1,\zeta_2) < \rho$.

Let $\gamma : [0,1] \to \mathcal{D}$ be a geodesic curve in the Bergman metric connecting $\zeta_1 = \gamma(0)$ to $\zeta_2 = \gamma(1)$. Then, for every holomorphic function f on $\mathcal{D}$,

$$f(\zeta_2) - f(\zeta_1) = \int_0^1 \frac{d}{dt}(f(\gamma(t)))\,dt.$$

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We denote by $\Phi_{\gamma(t)} : \mathcal{D} \to \mathcal{D}$ a biholomorphic automorphism on $\mathcal{D}$ which maps $\gamma(t)$ to the base point $(ie, 0)$ of $\mathcal{D}$. (For the notation $(ie, 0)$, see the beginning of Section 3.) For every $t \in [0, 1]$,

$$\begin{aligned}\frac{d}{dt}(f(\gamma(t))) &= \left(\frac{d}{d\tau}\right)_{\tau=t}((f \circ \Phi_{\gamma(t)}^{-1})(\Phi_{\gamma(t)} \circ \gamma)(\tau))\\ &= (\partial_{w=(ie,0)}(f \circ \Phi_{\gamma(t)}^{-1})(w)) \cdot \left(\frac{d}{d\tau}\right)_{\tau=t}((\Phi_{\gamma(t)} \circ \gamma)(\tau)).\end{aligned}$$

So,

$$\left|\frac{d}{dt}(f(\gamma(t)))\right| \le \|\partial_{w=(ie,0)}(f \circ \Phi_{\gamma(t)}^{-1})(w)\| \cdot \left\|\left(\frac{d}{d\tau}\right)_{\tau=t}((\Phi_{\gamma(t)} \circ \gamma)(\tau))\right\|,$$

where $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{C}^n$, which is canonically identified with the tangent space $T_{(ie,0)}\mathcal{D}$ at $(ie, 0)$. The Bergman Hermitian metric $H_\zeta(u, v) := \frac{1}{2}\sum_{j,k=1,\dots,n}(\partial^2/\partial z_j\partial\bar{w}_k)\log B(\zeta, \zeta)u_j\bar{v}_k$ at $\zeta = \gamma(t) \in \mathcal{D}$ enjoys the following invariance property:

$$\begin{aligned}H_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) &= H_{(ie,0)}\left(\left(\frac{d}{d\tau}\right)_{\tau=t}((\Phi_{\gamma(t)} \circ \gamma)(\tau)), \left(\frac{d}{d\tau}\right)_{\tau=t}((\Phi_{\gamma(t)} \circ \gamma)(\tau))\right)\\ &\approx \left\|\left(\frac{d}{d\tau}\right)_{\tau=t}((\Phi_{\gamma(t)} \circ \gamma)(\tau))\right\|^2.\end{aligned}$$

The latter relation relies on the equivalence between the Bergman metric and the Euclidean metric on $\mathbb{C}^n$. The conclusion is

$$\left|\frac{d}{dt}(f(\gamma(t)))\right| \le \|\partial_{w=(ie,0)}(f \circ \Phi_{\gamma(t)}^{-1})(w)\| H_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))^{1/2}.$$

In particular, for $f(w) = \log B(w, z)$ with $z \in \mathcal{D}$,

$$\begin{aligned}&|\log B(\zeta_2, z) - \log B(\zeta_1, z)|\\ &\quad\le \left(\int_0^1 H_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))^{1/2}\,dt\right)\sup_{t\in[0,1],\ z\in\mathcal{D}}\|\partial_{w=(ie,0)}(\log B(\Phi_{\gamma(t)}^{-1}(w), z))\|\\ &\quad= d(\zeta_1, \zeta_2)\sup_{t\in[0,1],\ z\in\mathcal{D}}\|\partial_{w=(ie,0)}(\log B(\Phi_{\gamma(t)}^{-1}(w), z))\|.\end{aligned}$$

The next step is to prove that the following estimate holds:

$$\sup_{t\in[0,1],\ z\in\mathcal{D}}\|\partial_{w=(ie,0)}(\log B(\Phi_{\gamma(t)}^{-1}(w), z))\| < \infty.$$

In view of the standard change of variables formula in the Bergman kernel [9], we obtain

$$\log B(\Phi_{\gamma(t)}^{-1}(w), z) = \log B(w, \Phi_{\gamma(t)}(z)) - \log J\Phi_{\gamma(t)}^{-1}(w) - \log \overline{J\Phi_{\gamma(t)}^{-1}(\Phi_{\gamma(t)}(z))},$$

where $J\Phi_{\gamma(t)}^{-1}(\zeta)$ denotes the Jacobian of $\Phi_{\gamma(t)}^{-1}$ at ζ. We note that here $\Phi_{\gamma(t)}^{-1}$ may be an affine transform because $\mathcal{D}$ is an (affine)-homogeneous Siegel domain of type II; then

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$\partial_w \log J\Phi^{-1}_{\gamma(t)}(w) \equiv 0$ since $J\Phi^{-1}_{\gamma(t)}(w)$ is independent of w. It therefore suffices to show the following estimate:

$$\sup_{z\in\mathcal{D}} \|\partial_{w=(ie,0)}(\log B(w,z))\| < \infty. \tag{2.1}$$

Let $\sigma : \mathcal{D} \to \mathbb{C}^n$ be a holomorphic mapping defined by

$$\sigma(\zeta) := \bar\partial_{w=(ie,0)}(\log B(\zeta,w)) - \bar\partial_{w=(ie,0)}(\log B((ie,0),w)). \tag{2.2}$$

The mapping σ is called the Bergman mapping (see [12] and [22]), which was shown in [13] to be equal to $\frac{1}{2}$ times the Cayley transform of the Siegel domain $\mathcal{D}$, introduced by Penney [18]. The following theorem is fundamental (see [13], [17], [18] and [22]).

Theorem 2.1. *The Bergman mapping σ gives a biholomorphic mapping from $\mathcal{D}$ onto the image $\sigma(\mathcal{D})$ which is a bounded domain in $\mathbb{C}^n$.*

We first point out that

$$\bar\partial_{w=(ie,0)}\,(\log B(\zeta,w)) = \sigma(\zeta) + \bar\partial_{w=(ie,0)}(\log B((ie,0),w))$$

and $\log B(w,\zeta) = \overline{\log B(\zeta,w)}$ for all $\zeta, w \in \mathcal{D}$. We deduce at once from Theorem 2.1 and (2.2) that (2.1) is valid.

3. Second proof of Theorem 1

Here we rely heavily on [14, Proposition 6.1]. Let $\sigma : \mathcal{D} \to \sigma(\mathcal{D})$ be the biholomorphic mapping defined in (2.2). The domain $D := \sigma(\mathcal{D})$ is a homogeneous bounded domain. Moreover, D is minimal with centre 0 by [13, Proposition 3.8], so that the Bergman kernel B_D of the domain D has the property

$$B_D(w,0) = \frac{1}{\mathrm{Vol}(D)} \quad (w \in D). \tag{3.1}$$

Note that $\sigma(ie,0) = 0$.

For every $z \in \mathcal{D}$, let ϕ_z be the function on $\mathcal{D}$ given by

$$\phi_z(w) := \frac{B(w,z)}{B((ie,0),z)} \quad (w \in \mathcal{D}),$$

where B denotes again the Bergman kernel of $\mathcal{D}$. Let δ be the Euclidean distance from $(ie,0)$ to the boundary of $\mathcal{D}$, and put

$$\mathcal{K} := \left\{ w \in \mathcal{D} : |w - (ie,0)| \le \frac{\delta}{2} \right\}.$$

It was pointed out in [6, page 157] that Theorem 1.1 follows from the following lemma.

Lemma 3.1. *There exists a positive constant $C > 0$ such that*

$$|\phi_z(w)| \le C$$

for all $w \in \mathcal{K}$ and $z \in \mathcal{D}$.

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Proof. In view of the change of variables formula in the Bergman kernel and (3.1), we obtain (see also [14, Lemma 5.2])

$$\mathrm{Vol}(D)B_D(\sigma(w), \sigma(z)) = \frac{B(w, z)B((ie, 0), (ie, 0))}{B(w, (ie, 0))B((ie, 0), z)},$$

so that

$$\phi_z(w) = \mathrm{Vol}(D)B_D(\sigma(w), \sigma(z))\frac{B(w, (ie, 0))}{B((ie, 0), (ie, 0))}. \tag{3.2}$$

Since $\mathcal{K} \subset \mathcal{D}$ is compact,

$$C_1 := \sup_{w\in\mathcal{K}}\left|\frac{B(w, (ie, 0))}{B((ie, 0), (ie, 0))}\right| < \infty.$$

We denote again by d the Bergman distance on $\mathcal{D}$. Put

$$\rho := \sup_{w\in\mathcal{K}} d(w, (ie, 0)) < \infty.$$

By [14, Proposition 6.1], there exists $N_\rho > 0$ such that

$$d_D(\zeta_1, 0) \le \rho \Rightarrow N_\rho^{-1} \le |B_D(\zeta_1, \zeta_2)| \le N_\rho$$

for all $\zeta_2 \in D$, where d_D stands for the Bergman distance on D. By virtue of the equivalence of the Euclidean and Bergman distances on the compact set $\mathcal{K}$, there exists a positive constant C such that the following implication holds:

$$|w - (ie, 0)| \le \frac{\delta}{2} \Rightarrow d(w, (ie, 0)) \le C\delta.$$

If $w \in \mathcal{K}$, the invariance of the Bergman metrics under biholomorphisms tells us that $d_D(\sigma(w), 0) = d(w, (ie, 0)) \le C\delta$. Therefore,

$$|B_D(\sigma(w), \sigma(z))| \le N_{C\delta},$$

so that (3.2) implies

$$|\phi_z(w)| \le \mathrm{Vol}(D)N_{C\delta}C_1 \quad (w \in \mathcal{K}).$$

This finishes the proof of Lemma 3.1. □

4. Atomic decomposition and interpolation: the constructive approach

Coifman and Rochberg [8] used Korányi's lemma to establish atomic decompositions of functions in Bergman spaces on symmetric Siegel domains of type II. Subsequently, Rochberg [19] applied these atomic decompositions to prove interpolation results by functions in Bergman spaces. Their proofs were constructive: see also [1] and [7] where these results were extended to two homogeneous, nonsymmetric Siegel domains of type II to which it has been possible to generalise Korányi's lemma. Since, in the present paper, Korányi's lemma is proved in full generality, these atomic decomposition and interpolation results can now be extended

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to general homogeneous Siegel domains of type II. In this section, we state these general results.

We use the same notation as in [16]. We consider the T-algebra $\mathcal{U}$ in which an involution is given by $x \mapsto x^\star$. We have the following canonical decomposition of $\mathcal{U}$:

$$\mathcal{U} = \bigoplus_{1 \le i,j \le r} \mathcal{U}_{ij},$$

where r is a positive integer, the subspaces $\mathcal{U}_{ij}$ satisfy $\mathcal{U}_{ii} = \mathbb{R}c_i$ with $c_i^2 = c_i$ and $\dim \mathcal{U}_{ij} = n_{ij} = n_{ji}$. We denote by e the unit element of the matrix T-algebra $\mathcal{U}$, that is,

$$e = \sum_{j=1}^{r} c_j.$$

Let ρ be the canonical isomorphism from $\mathcal{U}_{ii}$ onto $\mathbb{R}$. We consider the subalgebra

$$\mathcal{T} = \bigoplus_{1 \le i \le j \le r} \mathcal{U}_{ij}$$

of $\mathcal{U}$ consisting of upper triangular matrices. Let

$$H = \{t \in \mathcal{T} : \rho(t_{ii}) > 0, i = 1, \dots, r\}$$

be the subgroup of upper triangular matrices whose diagonal elements are positive, and let

$$V = \{x \in \mathcal{U} : x^\star = x\}$$

and

$$\Omega = \{ss^\star : s \in H\}.$$

Then V is an n-dimensional Euclidean vector space (the dimension n will be given below) and $\Omega \subset V$ is an irreducible, open, convex and homogeneous cone of rank r, which contains no straight line. The group H acts simply transitively on Ω through the transformations

$$\pi(w) : uu^\star \mapsto (wu)(u^\star w^\star) \quad (w, u \in H),$$

which correspond to the left translations of Ω [21, page 383].

We define

$$n_i = \sum_{j=1}^{i-1} n_{ji}, \quad m_i = \sum_{j=i+1}^{r} n_{ij};$$

then

$$\dim V = n = r + \sum_{i=1}^{r} m_i = r + \sum_{i=1}^{r} n_i.$$

For $k = 1, \dots, r$, we consider the homogeneous functions of degree 1 denoted by χ_k in [1, 6, 10], defined by

$$Q_k(x) = \frac{p_k(x)}{\prod_{j=k+1}^{r} p_j(x)},$$

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where p_k, $k = 1, \dots, r$ are homogeneous polynomials of degree 2^{r-k} called determinant-type polynomials associated with the cone Ω (see [11, Proposition 1.4]).

The convex homogeneous cone Ω is therefore defined by

$$\Omega = \{x \in V : Q_j(x) > 0,\ j = 1, \dots, r\}$$

and all irreducible, open, convex and homogeneous cones are obtained in this manner (see [21]).

We shall use the following notation.

(1) For all $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$ and $x \in \Omega$, we put

$$Q^\alpha(x) = \prod_{j=1}^{r} Q_j^{\alpha_j}(x).$$

(2) We put $\tau = (\tau_1, \dots, \tau_r) \in \mathbb{R}^r$ with

$$\tau_i = 1 + \tfrac{1}{2}(m_i + n_i)$$

(in the notation of [1, 6, 10], τ corresponds to $-d$).

Let now $\mathcal{D} = \{(z, u) \in \mathbb{C}^n \times \mathbb{C}^m : \operatorname{Im} mz - F(u, u) \in \Omega\}$ be the homogeneous Siegel domain of type II associated with the open, convex and homogeneous cone Ω and with the homogeneous Ω-Hermitian form F. We have the direct sum decomposition $\mathbb{C}^m = \prod_{i=1}^{r} \mathbb{C}_i$, where $\mathbb{C}_i$ is the subspace of $\mathbb{C}^m$ on which the projection F_{ii} of the form F on the subspace $\mathcal{U}_{ii}$ of $\mathcal{U}$ is positive definite. We set $b_i = \dim_{\mathbb{C}} \mathbb{C}_i$, each b_j is a nonnegative integer and we denote by b the vector $(b_1, \dots, b_r)$.

In the sequel, we denote by dV the Lebesgue measure on $\mathcal{D}$. For $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{R}^r$, dV_ν stands for the weighted measure

$$dV_\nu(z, u) := Q^{\nu - b/2 - \tau}(\operatorname{Im} mz - F(u, u))\, dV(z, u).$$

We denote by $L^p_\nu(\mathcal{D})$, $0 < p < \infty$, the weighted Lebesgue space $L^p(\mathcal{D}, dV_\nu(z, u))$. The weighted Bergman space $A^p_\nu(\mathcal{D})$ is the closed subspace of $L^p_\nu(\mathcal{D})$ consisting of holomorphic functions on $\mathcal{D}$. To have $A^p_\nu(\mathcal{D}) \neq \{0\}$, we must take $\nu_j > \frac{1}{2}(m_j + b_j)$ for all $j = 1, \dots, r$.

The orthogonal projector of the Hilbert space $L^2_\nu(\mathcal{D})$ onto its closed subspace $A^2_\nu(\mathcal{D})$ is the weighted Bergman projector P_ν. For $\alpha \in \mathbb{R}^r$ and $z \in V + i\Omega$, we denote by $Q^\alpha(z/i)$ the holomorphic determination of the power on the tube domain $V + i\Omega$ over Ω which is positive on $i\Omega$. Then P_ν is the integral operator

$$P_\nu f(z, u) = \int_{\mathcal{D}} B_\nu((z, u), (w, t)) f(w, t)\, dV_\nu(w, t),$$

where

$$B_\nu((z, u), (w, t)) = d_\nu Q^{-\nu - b/2 - \tau}\Big(\frac{z - \bar{w}}{2i} - F(u, t)\Big)$$

(note that [10] gives the value of d_ν) is the weighted Bergman kernel of $\mathcal{D}$. The unweighted Bergman kernel of $\mathcal{D}$, which was denoted by B in the previous three sections, is given by $B = B_{(b/2)+\tau}$.

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Definition 4.1. Suppose that η is a positive number and $\{\zeta_k\}_{k\in\mathbb{Z}_+}$ is a sequence of points of $\mathcal{D}$. Let d denote again the Bergman distance on $\mathcal{D}$ and define the Bergman balls

$$\widehat{B_k} = \Big\{\zeta \in \mathcal{D} : d(\zeta,\zeta_k) < \frac{\eta}{2}\Big\},$$
$$B_k = \{\zeta \in \mathcal{D} : d(\zeta,\zeta_k) < \eta\},$$
$$\widetilde{B_k} = \{\zeta \in \mathcal{D} : d(\zeta,\zeta_k) < 4\eta\}.$$

The sequence $\{\zeta_j\}_{j\in\mathbb{Z}_+}$ is called an η-lattice if:

(1) $\widehat{B_k} \cap \widehat{B_l} = \emptyset,\ (k \neq l)$;
(2) $\bigcup_{k\in\mathbb{Z}_+} B_k = \mathcal{D}$;
(3) there is a positive integer M which depends neither on η nor on the ζ_k such that no point of $\mathcal{D}$ belongs to more than M balls $\widetilde{B_k}$.

Coifman and Rochberg [8] proved the existence of an η-lattice for each $\eta > 0$ (see also [4]). For a point $\zeta = (z,u)$ of $\mathcal{D}$ and for $\mu \in \mathbb{R}^r$, we adopt the notation

$$\Delta_\mu(\zeta) := Q^{\mu+(b/2)+\tau}(\operatorname{Im} mz - F(u,u)).$$

Put

$$\Theta = \max_{1\le j\le r} \frac{\frac{n_j}{2}}{\nu_j - \frac{m_j}{2} - \frac{b_j}{2}}.$$

We first state the atomic decomposition theorem.

Theorem 4.2. *Let $\mathcal{D}$ be a general homogeneous Siegel domain of type II. Let p be a positive number and $\nu = (\nu_1,\dots,\nu_r)$ a vector in $\mathbb{R}^r$ such that $\nu_j > \frac{1}{2}(m_j + b_j)$ for all $j = 1,\dots,r$. Let α be a vector of $\mathbb{R}^r$. The following two assertions hold.*

(1) *Assume that $0 < p \le 1$ and, for $j = 1,\dots,r$,*

$$\alpha_j > \max\Big\{\frac{p}{2} + \frac{1}{2} + \frac{p}{2}\frac{m_j + b_j}{b_j + 2\tau_j} + \frac{\nu_j}{b_j + 2\tau_j}, \frac{\nu_j + \frac{n_j}{2}}{b_j + 2\tau_j} + \frac{1}{2}\Big\}$$

(respectively, $1 < p < 1 + 1/\Theta$ and, for $j = 1,\dots,r$,

$$\alpha_j > \max\Big\{\frac{p-1}{2} + (p-1)\frac{\nu_j + \frac{n_j}{2}}{b_j + 2\tau_j}, p + p\frac{\nu_j}{b_j + 2\tau_j} + \frac{p}{2}\frac{m_j + b_j}{b_j + 2\tau_j}\Big\}\Big).$$

Then there is a positive constant η_0 such that for each $\eta \in (0,\eta_0)$ and each η-lattice $\{\zeta_k\}$ in $\mathcal{D}$, every $F \in A^p_\nu(D)$ can be decomposed in the form

$$F(\zeta) = \sum_k \lambda_k B_\mu(\zeta,\zeta_k)\Delta_{\mu'}(\zeta_k) \quad (\zeta \in D), \tag{4.1}$$

where

$$\mu = \Big(\frac{2\alpha}{p} - 1\Big)\Big(\frac{b}{2} + \tau\Big), \quad \mu' = \mu - \frac{\nu + \frac{b}{2} + \tau}{p}$$

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and $\{\lambda_k\}$ is an l^p-sequence of complex numbers. Moreover, there is a constant $C = C(\alpha, p, \nu)$ such that for each $\eta \in (0, \eta_0)$, each η-lattice $\{\zeta_k\}$ and each $F \in A^p_\nu(\mathcal{D})$, the sequence $\{\lambda_k\}$ of (4.1) satisfies

$$\sum_k |\lambda_k|^p \leq C\|F\|^p_{A^p_\nu} \eta^{2(n+m)(p-1)} \quad \textit{if } p > 1$$

(respectively, $\sum_k |\lambda_k|^p \leq C\|F\|^p_{A^p_\nu}$ if $p \in (0, 1]$).

(2) *Assume that $p \in (0, 1]$ and*

$$\alpha_j > \frac{\nu_j + \frac{n_j}{2}}{2\tau_j + b_j} + \frac{1}{2}$$

for every $j = 1, \ldots, r$ (respectively, $1 < p < 1 + 1/\Theta$ and

$$\alpha_j > p\Big(\frac{1}{2} + \frac{\nu_j + \frac{n_j}{2}}{2\tau_j + b_j}\Big)$$

for every $j = 1, \ldots, r$). Then there is a constant $C = C(\alpha, p, \nu)$ such that for each $\nu > 0$, each sequence $\{\zeta_k\}$ satisfying $\inf_{k \neq l} d(\zeta_k, \zeta_l) \geq \eta$ and each l^p-sequence $\{\lambda_k\}$, the function F defined by (4.1) belongs to $A^p_\nu(\mathcal{D})$ and satisfies the estimate $\|F\|^p_{A^p_\nu} \leq C \sum_k |\lambda_k|^p$ (respectively, $\|F\|^p_{A^p_\nu} \leq C \sum_k |\lambda_k|^p \eta^{-2(n+m)(p-1)}$).

The proof of this theorem relies on the following three propositions, which are established in [7]. In Theorem 4.2, we take $\mu = (2\alpha/p - 1)(b/2 + \tau)$.

Proposition 4.3. *Let ν be a vector in $\mathbb{R}^r$ such that $\nu_j > \frac{1}{2}(m_j + b_j)$ for all $j = 1, \ldots, r$. Let $0 < p < 1 + 1/\Theta$. Then the weighted projector P_μ reproduces functions in the weighted Bergman space $A^p_\nu(\mathcal{D})$ if the following condition is fulfilled:*

$$\mu_j > \max\Big\{\frac{1}{2}(m_j + b_j) + \frac{1}{p}\Big(\nu_j + \frac{b_j}{2} + \tau_j\Big), \frac{1}{2}(m_j + b_j) + \Big(\nu_j + \frac{b_j}{2} + \tau_j\Big)\Big\}$$

for all $j = 1, \ldots, r$.

Proposition 4.4. *Let μ, ν be two vectors in $\mathbb{R}^r$ such that $\nu_j > \frac{1}{2}(m_j + b_j)$ and $\mu_j > \nu_j + n_j/2$ for all $j = 1, \ldots, r$. Then the following estimate holds:*

$$\int_{\mathbb{D}} |B_\mu(z, w)| \, dV_\nu(w) = C_{\mu,\nu} \Delta_{-(\mu-\nu)-b/2-\tau}(z).$$

Proposition 4.5. *Let ν be a vector in $\mathbb{R}^r$ such that $\nu_j > \frac{1}{2}(m_j + b_j)$ for all $j = 1, \ldots, r$. Then the integral operator P^+_μ with kernel given by the modulus of the weighted Bergman kernel is bounded on the weighted Lebesgue space $L^p_\nu(\mathcal{D})$ if the following conditions are fulfilled:*

$$1 < p < 1 + \frac{1}{\Theta} \quad \textit{and} \quad \mu_j \geq \nu_j + \frac{n_j}{2}, \ j = 1, \ldots, r.$$

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The sufficient conditions given in the latter proposition are not known to be necessary even for tubes over symmetric cones; in this particular case, the conditions were only proved to be necessary for $\nu = \mu$ real, that is, $\nu = (\nu, \dots, \nu)$ [4]. (The reader should also look at the remark following [16, Theorem 2.3].)

Let us next move to the announced interpolation result. Let $S = \{\zeta_k\}_{k\in\mathbb{Z}_+}$ be a sequence of points of $\mathcal{D}$ and define the operator $T : A^p_\nu(\mathcal{D}) \to l^\infty(S)$ by $(TF)(\zeta_k) = F(\zeta_k)\Delta_\nu^{1/p}(\zeta_k)$. We can now state the interpolation theorem.

Theorem 4.6. *Let $\mathcal{D}$ be a general homogeneous Siegel domain of type II. Let $S = \{\zeta_k\}_{k\in\mathbb{Z}_+}$ be a sequence of points of $\mathcal{D}$ satisfying $\inf_{k\neq l} d(\zeta_k, \zeta_l) \geq \eta > 0$. Let p be a positive number and $\nu = (\nu_1, \dots, \nu_r)$ a vector in $\mathbb{R}^r$ such that $\nu_j > \frac{1}{2}(m_j + b_j)$ for all $j = 1, \dots, r$. Then:*

(1) *T is bounded from $A^p_\nu(\mathcal{D})$ into $l^p(S)$;*

(2) *moreover, if $0 < p < 1 + 1/\Theta$, there exists a positive number $\eta_0 = \eta_0(p, \nu)$ such that if $\eta > \eta_0$, T maps $A^p_\nu(\mathcal{D})$ onto $l^p(S)$; more precisely, there is a continuous linear operator $R : l^p(S) \to A^p_\nu(\mathcal{D})$ such that $TR = Id_{l^p(S)}$.*

5. Extension of the atomic decomposition theorem: a functional analysis approach

The extension of the atomic decomposition theorem to other values of $p > 1$ was obtained in [4] on tube domains over symmetric cones (symmetric Siegel domains of type I). The proof uses functional analysis and it also works here. The results in the general case are the following.

Theorem 5.1. *Let $\mathcal{D}$ denote a homogeneous Siegel domain of type II and let $p \in (1, \infty)$. Assume that the topological dual of $A^p_\nu(\mathcal{D})$ identifies with $A^{p'}_\nu(\mathcal{D})$, $1/p + 1/p' = 1$ by means of the map*

$$L_G : G \in A^{p'}_\nu(\mathcal{D}) \mapsto L_G(F) = \int_{\mathcal{D}} F(z)\overline{G(z)}\, dV_\nu(z). \tag{5.1}$$

Let $\{\zeta_k\}$ be an η-lattice in $\mathcal{D}$. Then the following two assertions hold.

(1) *For every complex sequence $\{\lambda_k\}$ such that*

$$\sum_k |\lambda_k|^p < \infty,$$

the series $\sum_k \lambda_j B_\nu(\zeta, \zeta_k)\Delta_\nu^{1-(1/p)}(\zeta_k)$ is convergent in $A^p_\nu(\mathcal{D})$. Moreover, its sum F satisfies the inequality

$$\|F\|^p_{A^p_\nu} \leq C \sum_k |\lambda_k|^p.$$

(2) *For η small enough, every function $F \in A^p_\nu(\mathcal{D})$ may be written as*

$$F(\zeta) = \sum_k \lambda_k B_\nu(\zeta, \zeta_k)\Delta_\nu^{1-1/p}(\zeta_k)$$

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with

$$\sum_k |\lambda_k|^p \le C\|F\|^p_{A^p_\nu}.$$

The functional analysis proof of this theorem is identical to the proof for symmetric Siegel domains of type I [4]. The idea is to examine the adjoint operator of the interpolation mapping

$$F \mapsto \{F(\zeta_k)\Delta_\nu^{1/p}(\zeta_k)\}_{k\in\mathbb{Z}_+}$$

on the respective bases of assertion (1) of Theorem 4.6 and the following sampling theorem [4, Theorem 5.6].

THEOREM 5.2. *Let $\{\zeta_j\}$ be an η-lattice in $\mathcal{D}$ with $\eta \in (0,1)$. If η is small, then there is a positive constant C_η such that every $F \in A^p_\nu(\mathcal{D})$ satisfies*

$$\|F\|^p_{A^p_\nu} \le C \sum_k |F(\zeta_k)|^p \Delta_\nu(\zeta_k).$$

To adapt the proof of [4], the reader should keep in mind that the measure $dV(\zeta)/\Delta_{(b/2)+\tau}(\zeta)$ is invariant under automorphisms of $\mathcal{D}$.

We now explain why the range of $p > 1$ for which the topological dual of $A^p_\nu(\mathcal{D})$ identifies with $A^{p'}_\nu(\mathcal{D})$ by means of the map L_G given in (5.1) extends the interval of validity of Theorem 4.2, that is, $(1, 1 + \min_{1\le i\le r}((\nu_i - m_i/2 - b_i/2)/(n_i/2)))$ or, equivalently, $((\Theta - 1)/(2(\Theta + 1))) < 1/p - 1/2 < 1/2$ with $\Theta = \max_{1\le i\le r}((n_i/2)/(\nu_i - m_i/2 - b_i/2))$. It is routine to show that this dual property holds whenever the weighted Bergman projector P_ν is bounded on $L^p_\nu(\mathcal{D})$. The conclusion follows from the next theorem.

THEOREM 5.3 [16, Theorem 2.3]. *Let $\nu \in \mathbb{R}^r$ be such that $\nu_j > \frac{1}{2}(m_j + n_j + b_j)$, $j = 1, \ldots, r$. The weighted Bergman projector P_ν extends to a bounded operator on $L^p_\nu(\mathcal{D})$ for*

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1-\Lambda}{2(1+\Lambda)},$$

where $\Lambda = \max_{1\le i\le r}((n_i/2)/(\nu_i - m_i/2 - b_i/2 + n_i/2))$.

REMARK 5.4. (1) Let $\nu \in \mathbb{R}^r$ be such that $\nu_j > \frac{1}{2}(m_j + b_j)$, $j = 1, \ldots, r$. It was shown in [6, Theorem II.8] that P_ν does not extend to a bounded operator on L^p_ν for $1/p - 1/2 \ge (1 - 2\Gamma)/2$ or $1/p - 1/2 \le (2\Gamma - 1)/2$, where $\Gamma = \min_{1\le i\le r}((n_i/2)/(\nu_i + 1 + (m_i + b_i)/2 + n_i))$. For $\nu_j > \frac{1}{2}(m_j + n_j + b_j)$, $j = 1, \ldots, r$, there remain two intervals of exponents p for which it is not known in general whether P_ν extends to a bounded operator on $L^p_\nu(\mathcal{D})$. A conjecture is stated in [3] for the particular case of tubes over symmetric cones.

(2) This functional analysis approach can also be applied to smaller values of p, namely $1 \le p \le 1 + \Lambda$ with

$$\Lambda = \max_{1\le i\le r} \frac{\frac{n_i}{2}}{\nu_i - \frac{m_i}{2} - \frac{b_i}{2} + \frac{n_i}{2}}$$

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or, equivalently,

$$\frac{1-\Lambda}{2(1+\Lambda)} \leq \frac{1}{p} - \frac{1}{2} < \frac{1}{2}.$$

In this case, it was proved in [20] (see also [3] for tubes over symmetric domains) that the topological dual of $A^p_\nu(\mathcal{D})$ identifies with a (Banach) analytic Besov space.

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