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PRODUCTS OF LOCALLY COMPACT GROUPS WITH ZERO AND THEIR ACTIONS

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1. Introduction. In [**4**], Hofmann defines a *locally compact group with zero* as a Hausdorff locally compact topological semigroup, S, with a non-isolated point, 0, such that $G = S - \{0\}$ is a group. He shows there that 0 is indeed a zero for S, G is a locally compact topological group, and the identity of G is the identity of S. The author has investigated actions of such semigroups on locally compact spaces in [**1**; **2**]. In this paper, we are investigating direct products of semigroups of the above type and actions of these products; for a special case of this, the reader is referred to [**3**].

We show here that if S is the direct product of n locally compact groups with zero, then S has 2^n idempotents, the idempotents of S are central, and S is the disjoint union of 2^n locally compact topological groups, one of which is open and dense in S. This is done in Section 2. In Section 3, which is the main body of this paper, we consider actions of S on Hausdorff spaces. We have dropped our usual hypothesis that the spaces also be locally compact. The results in this section give S as an H-closed semigroup, as defined by Stepp in [**8**]. We conclude this section with a theorem which partially characterizes those Hausdorff spaces which can serve as state spaces for actions of S.

Throughout this paper, all topological spaces are to be Hausdorff. We use $\emptyset$ to denote the null set, and A^* to denote the closure of the set A. For the notation and terminology of semigroups, we make reference to [**5**], and for that of actions, we make reference to [**6**].

We are indebted to the referee for his very useful suggestions, especially those which gave rise to Lemma 1 and Lemma 4.

Dedication. This paper is dedicated in appreciation to Dr. W. T. Hanson.

2. Structure of S. We let $N = \{1, \ldots, n\}$ and for each $j \in N$, we let S_j be a locally compact group with zero, 0_j. We set $G_j = S_j - \{0_j\}$, and let 1_j be the identity for G_j, and hence for S_j. Throughout the remainder of this work, $S = S_1 \times \ldots \times S_n$, $1 = (1_1, \ldots, 1_n)$, and $0 = (0_1, \ldots, 0_n)$. Clearly 1 is the identity for S and 0 is a zero for S. For each subset, A, of N, we set $e_A = (s_1, \ldots, s_n) \in S$, where $s_i = 0_i$ whenever $i \in A$, and $s_i = 1_i$ otherwise.

PROPOSITION 1. $E(S) = \{e_A : A \subset N\} \subset Z(S)$; *if* A *and* B *are subsets of* N, $e_A e_B = e_{(A \cup B)}$, *and* $e_A = e_B$ *if and only if* $A = B$.

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Proof. Since $\{0_i, 1_i\} = E(S_i) \subset Z(S_i)$, it follows that $E(S) = \{e_A : A \subset N\} \subset Z(S)$. The rest follows directly from the definition of e_A.

PROPOSITION 2. *If $e = e_A \in E(S)$, then $H(e) = T_1 \times \ldots \times T_n$ where $T_i = \{0_i\}$ if $i \in A$, and $T_i = G_i$ otherwise. Also, $H(e)$ is a locally compact topological group, and $H(1)$ is open and dense in S.*

Proof. If $e = e_A = (s_1, \ldots, s_n) \in E(S)$, then $H(e) = H(s_1) \times \ldots \times H(s_n)$, where $H(s_i) = \{0_i\}$ if $i \in A$, and $H(s_i) = H(1_i) = G_i$ if $i \notin A$. Hence, $H(e)$ is as described, and since each $H(s_i)$ is locally compact, $H(e)$ is a locally compact topological group.

$H(1) = H(e_\emptyset) = G_1 \times \ldots \times G_n$, so it is open in S. Also, $H(1)^* = G_1^* \times \ldots \times G_n^* = S_1 \times \ldots \times S_n$, so $H(1)$ is dense.

Before proceeding, let us remark here that Proposition 2 shows that S is a tame W-semigroup, as defined in [**9**]. Proposition 2 also describes $H(1)^*$; the next result describes $H(e)^*$ for every idempotent in S.

PROPOSITION 3. *If $e = e_A \in E(S)$, then $H(e)^* = \bigcup\{H(e_B) : A \subset B \subset N\}$, and is a direct product of n-*card(A) *locally compact groups with zero.*

Proof. If $e = e_A \in E(S)$, $H(e)^* = T_1^* \times \ldots \times T_n^*$, where $T_i^* = \{0_i\}$ if $i \in A$ and $T_i^* = G_i^* = S_i$ otherwise. This shows immediately that $H(e)^*$ is isomorphic to the product of $\{S_j : j \notin A\}$.

Let $y = (t_1, \ldots, t_n) \in S$. If $y \in H(e)^*$, $t_i = 0_i$ for $i \in A$. If $B = \{j : t_j = 0_j\} \subset N$, we have $A \subset B \subset N$ and $y \in H(e_B)$ since $t_j \in G_j$ for $j \notin B$. Thus, $H(e)^* \subset \bigcup\{H(e_B) : A \subset B \subset N\}$. If $y \in H(e_B)$ with $A \subset B \subset N$, then $t_j = 0_j$ for $j \in A$, and from above we have $y \in H(e)^*$. Thus,

$$\bigcup\{H(e_B) : A \subset B \subset N\} = H(e)^*.$$

The following is an easy consequence of the above Propositions.

THEOREM 1. *If $S = S_1 \times \ldots \times S_n$, then*

(a) *S contains 2^n idempotents, and $E(S) \subset Z(S)$;*

(b) *for each $e \in E(S)$, $H(e)$ is a locally compact topological group, and $H(e)^*$ is a direct product of $m \leqq n$ locally compact groups with zero;*

(c) *$H(1)$ is dense and open in S; and*

(d) *S is a locally compact Clifford semigroup; more specifically, S is the union of 2^n disjoint locally compact topological groups, one of which is dense and open in S.*

3. Actions of S.

If T is a topological semigroup and X is a topological space, we say T *acts* on X if there is a continuous function $\phi : T \times X \to X$ *onto* X such that if $x \in X$ and $t_1, t_2 \in T$, then

$$\phi(t_1, \phi(t_2, x)) = \phi(t_1 t_2, x).$$

If T acts on X, then we say that ϕ is an *action* of T on X; X is called the *state space* of the action. The assumption that the function be onto assures us that whenever T has an identity, e, then for every $x \in X$, $\phi(e, x) = x$. If T

acts on X via the function ϕ, and when there is no possibility of confusion, we will write tx in lieu of $\phi(t, x)$, i.e., $\phi(t, x) = tx$. In this context the action equation above becomes the more familiar $t_1(t_2x) = (t_1t_2)x$. If $A \subset T$ and $B \subset X$, then $AB = \{tx : t \in A \text{ and } x \in B\}$, and if $x \in X$, $Ax = A\{x\}$. Notice that if $Y \subset X$ such that $TY = Y$, then T acts on Y via $\phi|(T \times Y)$. If ϕ is an action of T on X, then for each $x \in X$, we define

$$\phi_x : T \to X \text{ via } \phi_x(t) = tx = \phi(t, x),$$

and notice that for every $x \in X$, ϕ_x is continuous. We let

$$X' = \{x \in X : \phi_x \text{ is } 1 - 1\}.$$

Suppose each of T_1 and T_2 is a topological semigroup with identity 1_1 and 1_2 respectively. Let $T = T_1 \times T_2$, and suppose T acts on X via ϕ. Then ${}_1\phi : T_1 \times X \to X$ given by

$${}_1\phi(t, x) = \phi((t, 1_2), x) = (t, 1_2)x$$

and ${}_2\phi : T \times X \to X$ given by

$${}_2\phi(s, x) = \phi((1_1, s), x) = (1_1, s)x$$

give actions of T_1 and T_2 respectively, on X. These are called the actions of T_i on X induced by the action of T on X. When this occurs, we set $X_i' = \{x \in X : {}_i\phi_x : T_i \to X \text{ is } 1 - 1\}$. Observe that X' is a subset of each X_i'.

Let T be a semigroup (all semigroups are to be topological) acting on a space X. If Y is a subset of X, we say that T satisfies *Property N* on Y provided that whenever $\{t_\rho\}$ is a net in T and $y \in Y$ such that $\lim_\rho(t_\rho y) = x$ for some $x \in X$, then there is a $t \in T$ such that $\lim_\rho t_\rho = t$.

If X is a space and $\{x_\rho\}$ is a net in X, we say $\lim_\rho x_\rho = \infty$ in X provided that $\{x_\rho\}$ has no convergent subnets in X.

As our first result in this section, we have

LEMMA 1. *Let T_1 be a locally compact group with zero and let T_2 be any semigroup with identity. If $T_1 \times T_2$ acts on the space X such that under the induced action T_2 satisfies* Property N *on X_2', then $T_1 \times T_2$ satisfies* Property N *on X'.*

Proof. If $x \in X'$, then from above $x \in X_2'$. Furthermore, $(0, 1_2)x \in X_2'$, where 0 is the zero of T_1. For, suppose $(1_1, t)[(0, 1_2)x] = (1_1, t')[(0, 1_2)x]$. Then

$$(0, t)x = [(1_1, t)(0, 1_2)]x = (1_1, t)[(0, 1_2)x] = (1_1, t')[(0, 1_2)x] = [(1_1, t')(0, 1_2)]x = (0, t')x.$$

Since $x \in X'$, we conclude that $(0, t) = (0, t')$, so $t = t'$. From this we see that $(0, 1_2)x$ is indeed in X_2'.

Let $x \in X'$, and suppose $\{(s_\rho, t_\rho)\}$ is a net in $T_1 \times T_2$ such that $\lim_\rho(s_\rho, t_\rho)x = y$ for some $y \in X$. Then

$$\lim_\rho(0, t_\rho)x = \lim_\rho(0, 1_2)[(s_\rho, t_\rho)x] = (0, 1_2)y.$$

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However, for each ρ, $(0, t_\rho)x = (1_1, t_\rho)[(0, 1_2)x]$, so $\lim_\rho(1_1, t_\rho)[(0, 1_2)x] = (0, 1_2)y$. But, from above, $(0, 1_2)x$ is in X_2', so, since T_2 satisfies Property N on X_2', there is a $t \in T_2$ such that $\lim_\rho t_\rho = t$ in T_2. Hence, $\lim_\rho(1_1, t_\rho) = (1_1, t)$ in $T_1 \times T_2$.

If $\lim_\rho s_\rho = \infty$ in T_1, then, from [**1**], $\lim_\rho s_\rho^{-1} = 0$ in T_1. Hence

$$\begin{aligned}(1_1, t)x &= \lim_\rho(1_1, t_\rho)x = \lim_\rho(s_\rho^{-1}, 1_2)[(s_\rho, t_\rho)x] \\ &= (0, 1_2)y \\ &= [(0, 1_2)(0, 1_2)]y = (0, 1_2)[(0\ 1_2)y] \\ &= (0, 1_2)[(1_1, t)x] = [(0, 1_2)(1_1, t)]x \\ &= (0, t)x.\end{aligned}$$

But, $x \in X'$, and $(1_1, t) \neq (0, t)$. Thus, there is a subnet $\{s_\alpha\}$ of $\{s_\rho\}$ and an $s \in T_1$ such that $\lim_\alpha s_\alpha = s$ in T_1. Hence, $\{(s_\alpha, t_\alpha)\}$ is a subnet of $\{(s_\rho, t_\rho)\}$ such that $\lim_\alpha(s_\alpha, t_\alpha) = (s, t)$ in $T_1 \times T_2$. Also, since $\{(s_\alpha, t_\alpha)\}$ is a subnet of $\{(s_\rho, t_\rho)\}$, we have $(s, t)x = \lim_\alpha(s_\alpha, t_\alpha)x = y$.

We shall now show that we in fact have $\lim_\rho(s_\rho, t_\rho) = (s, t)$ in $T_1 \times T_2$. To this end, suppose $\{(s_\beta, t_\beta)\}$ is a subnet of $\{(s_\rho, t_\rho)\}$. Since $\lim_\beta(s_\beta, t_\beta)x = y$, we apply the above argument to find a subnet $\{(s_\gamma, t_\gamma)\}$ of $\{(s_\beta, t_\beta)\}$ and an $(s', t') \in T_1 \times T_2$ such that $\lim_\gamma(s_\gamma, t_\gamma) = (s', t')$. Then, $(s', t')x = \lim_\gamma(s_\gamma, t_\gamma)x = y = (s, t)x$. Since $x \in X'$, $(s', t') = (s, t)$. We have shown that every subnet of $\{(s_\rho, t_\rho)\}$ has a subnet of itself which converges to (s, t). From this it follows that $\lim_\rho(s_\rho, t_\rho) = (s, t)$.

Therefore $T_1 \times T_2$ satisfies Property N on X'.

The next result is an immediate consequence of the concept of *equivalent* actions as defined by Stadtlander in [**7**]. For this reason, no proof will be given.

LEMMA 2. *Suppose the semigroup T_1 acts on the space X via ϕ. If T_2 is a semigroup isomorphic to T_1, the isomorphism induces an action, ψ, of T_2 on X equivalent to ϕ. If T_2 satisfies* Property N *on a subset Y of X relative to this induced action, then T_1 satisfies* Property N *on Y. Furthermore, ϕ_x is* 1-1 *if and only if ψ_x is* 1-1.

As a consequence of the above Lemmas, we have

THEOREM 2. *If S acts on the space X, then S satisfies* Property N *on X'.*

Proof. For $n = 1$, the result follows directly from [**1**; Theorem 3 and Corollary 4.2]. Assume the result for $n - 1$.

Now,

$$S = S_1 \times \ldots \times S_n \simeq T_1 \times T_2, \text{ where } T_1 = S_1 \text{ and } T_2 = S_2 \times \ldots \times S_n.$$

From Lemma 2 above, the set X' for the action of S on X is the same as the set X' for the induced action of $T_1 \times T_2$ on X. By the inductive assumption, T_2 satisfies Property N on X_2', so by Lemma 1, $T_1 \times T_2$ satisfies Property N on X'. Hence, by Lemma 2, S satisfies Property N on X'.

Using Theorem 2, we can present

THEOREM 3. *Suppose S acts on X and $x \in X'$. Then Sx is a closed subset of X, and ϕ_x maps S homeomorphically onto Sx.*

Proof. Suppose first that $y \in (Sx)^*$. Then, there is a net $\{s_\rho\}$ in S such that $\lim_\rho s_\rho x = y$. From Theorem 2, there is an $s \in S$ such that $\lim_\rho s_\rho = s$. Then, $y = \lim_\rho s_\rho x = sx$, and hence $y \in Sx$, and Sx is indeed a closed subset of X.

Clearly ϕ_x is a 1-1 continuous function from S onto Sx. If $\{s_\rho\}$ is a net in S such that for some $s \in S$ $\lim_\rho \phi_x(s_\rho) = \phi_x(s)$, then $\lim_\rho s_\rho x = sx \in X$, so by Theorem 2 there is a $t \in S$ such that $\lim_\rho s_\rho = t$. Then, $tx = \lim_\rho s_\rho x = sx$. But, $x \in X'$, so $t = s$. Thus, $\lim_\rho s_\rho = s$, and we see that ϕ_x is a homeomorphism.

Theorem 3 of [**1**] shows that if $n = 1$, the hypothesis that x be in X' is not needed to obtain Sx as a closed subset of X. However, an example in [**3**] shows that in general we do need this hypothesis.

Theorem 3 has two interesting corollaries. The first is straightforward, so no proof will be given.

COROLLARY 3.1 [**8**]. *S is an H-closed semigroup.*

COROLLARY 3.2. *Suppose S acts on the space X and $x \in X$. If $e \in E(S)$ such that $\phi_x|H(e)^*$ is 1-1, then $[H(e)x]^* = [H(e)^*]x$, and is homeomorphic to $H(e)^*$. If $x \in X'$, we obtain this for every $e \in E(S)$.*

Proof. Suppose $e \in E(S)$ such that $\phi_x|H(e)^*$ is 1-1. Since e is the identity of $H(e)^*$, it follows that $H(e)^*$ acts on eX via $\psi = \phi|(H(e)^* \times eX)$; $\psi(t, ez) = \phi(t, ez) = t(ez) = (te)z = tz$. It is easy to see that relative to this action of $H(e)^*$ on eX, $ex \in (eX)'$. By Theorem 1(b), $H(e)^*$ is a direct product of locally compact groups with zero, so, by Theorem 3, $[H(e)^*]ex = \psi(H(e)^* \times \{ex\})$ is a closed subset of eX homeomorphic to $H(e)^*$. However, if $t \in H(e)^*$, $tx = (te)x = t(ex) = \psi(t, ex)$, so $[H(e)^*]x = [H(e)^*]ex$ is a closed subset of eX homeomorphic to $H(e)^*$.

Since $eX = \{y \in X : ey = y\}$, eX is a closed subset of X. Thus, $[H(e)^*]x$ is a closed subset of X. Since $H(e)x \subset [H(e)^*]x$, we have $[H(e)x]^* \subset [H(e)^*]x$. But, by the continuity of the function ϕ_x, $[H(e)^*]x \subset [H(e)x]^*$, so we indeed have $[H(e)^*]x = [H(e)x]^*$, and this is, from above, homeomorphic to $H(e)^*$.

If $y \in X'$, then clearly $\phi_y|H(e)^*$ is 1-1 for every $e \in E(S)$, and the proof is concluded.

We now present

LEMMA 3. *Let T be a semigroup with identity, e, and let H be the maximal subgroup of T which contains e. If T acts on X, then H acts on X'.*

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Proof. Since $e \in H$, we need only show that $HX' \subset X'$. Now, if $g \in H$, then $\rho_g : S \to S$ given by $\rho_g(s) = sg$ is a homeomorphism. If $x \in X$, then $\phi_{gx} = (\phi_x) \circ (\rho_g)$, so if ϕ_x is 1-1 so is ϕ_{gx}. Hence, if $x \in X'$ and $g' \in H$, then $gx \in X'$. Thus, $HX' \subset X'$, and H acts on X'.

If the semigroup T acts on the space X, we say that T acts IP on X provided that whenever $\{x_\rho\}$ is a convergent net in X and $\{t_\rho\}$ is a net in T such that $\lim_\rho t_\rho = \infty$ in T, then $\lim_\rho t_\rho x_\rho = \infty$ in X.

LEMMA 4. *Let T_1 be a locally compact group with zero having H_1 as the dense maximal group. Suppose T_2 is a semigroup with identity, 1_2, such that the maximal subgroup of T_2 which contains 1_2, call it H_2, is a topological group. If $T_1 \times T_2$ acts on the space X such that H_2 acts IP on X_2', then $H_1 \times H_2$ acts IP on X'.*

Proof. By Lemma 3, $H_1 \times H_2$ acts on X'. To show that it acts IP on X', we need only show that if $\{x_\rho\}$ is a net in X' and $\{(g_\rho, h_\rho)\}$ is a net in $H_1 \times H_2$ such that for some $x, y \in X'$ we have $\lim_\rho x_\rho = x$ and $\lim_\rho (g_\rho, h_\rho)x_\rho = y$ in X', then $\{(g_\rho, h_\rho)\}$ has a subnet which converges in $H_1 \times H_2$.

If $\lim_\rho (g_\rho, h_\rho)x_\rho = y$, then

$$\lim_\rho (0, h_\rho)x_\rho = \lim_\rho (0, 1_2)[(g_\rho, h_\rho)x_\rho] = (0, 1_2)y.$$

Now, in the proof of Lemma 1, it is shown that if $z \in X'$, then $(0, 1_2)z \in X_2'$. Hence, $\{(0, 1_2)x_\rho\}$ is a net in X_2' and $(0, 1_2)x \in X_2'$ is such that $\lim_\rho (0, 1_2)x_\rho = (0, 1_2)x$. Also, $(0, 1_2)y \in X_2'$, and $\{h_\rho\}$ is a net in H_2 such that

$$\lim_\rho (1_1, h_\rho)[(0, 1_2)x_\rho] = \lim_\rho [(1_1, h_\rho)(0, 1_2)]x_\rho = \lim_\rho (0, h_\rho)x_\rho = (0, 1_2)y.$$

Since H_2 acts IP on X_2', we conclude that there is an $h \in H_2$ and a subnet $\{(g_\alpha, h_\alpha)\}$ of $\{(g_\rho, h_\rho)\}$ such that $\lim_\alpha (1_1, h_\alpha) = (1_1, h)$. Since H_2 is a topological group, $\lim_\alpha (1_1, h_\alpha^{-1}) = (1_1, h^{-1})$.

Now, $z = (1_1, h^{-1})y \in X' \subset X_1'$, and $\{x_\alpha\}$ is a net in $X' \subset X_1'$ such that $\lim_\alpha x_\alpha = x \in X' \subset X_1'$. Also, $\{g_\alpha\}$ is a net in H_1 such that

$$\lim_\alpha (g_\alpha, 1_2)x_\alpha = \lim_\alpha (1_1, h_\alpha^{-1})[(g_\alpha, h_\alpha)x_\alpha] = (1_1, h^{-1})y = z,$$

because $\lim_\alpha (g_\alpha, h_\alpha)x_\alpha = \lim_\rho (g_\rho, h_\rho)x_\rho = y$. From Theorem 4 in [**1**], it follows that H_1 acts IP on X_1', so there is a $g \in H_1$ and a subnet $\{g_\beta\}$ of $\{g_\alpha\}$ such that $\lim_\beta (g_\beta, 1_2) = (g, 1_2)$. Then, $(g, h) \in H_1 \times H_2$ and $\{(g_\beta, h_\beta)\}$ is a subnet of $\{(g_\rho, h_\rho)\}$ such that $\lim_\beta (g_\beta, h_\beta) = (g, h)$ in $H_1 \times H_2$.

Hence, $\{(g_\rho, h_\rho)\}$ has a subnet which converges in $H_1 \times H_2$, so we conclude that $H_1 \times H_2$ acts IP on X'.

We now present another Lemma, without proof, which is a direct consequence of equivalent actions.

LEMMA 5. *Suppose T_1 is a semigroup and P_1 is a subsemigroup of T_1. Assume that T_1 acts on X, and Y is a subset of X such that $P_1 Y = Y$ (i.e., P_1 acts on Y).*

Let T_2 be a semigroup isomorphic to T_1 and let P_2 be the image of P_1. Under the induced action of T_2 on X, P_2 acts on Y. If P_2 acts IP on Y, then P_1 acts IP on Y.

The above Lemmas yield

THEOREM 4. *If S acts on X, then $G = H(1)$ acts IP on X'.*

Proof. For $n = 1$, the result follows from [**1**; Theorem 4]. Assume the result for $n - 1$.

Let $T_1 = S_1$, $H_1 = G_1$, $T_2 = S_2 \times \ldots \times S_n$, and $H_2 = G_2 \times \ldots \times G_n$. Then $S \simeq T_1 \times T_2$ and $H_1 \times H_2$ is the image of G. By Lemma 2, the set X' for the induced action of $T_1 \times T_2$ on X is the same as the set X' for the action of S on X. By the inductive assumption, H_2 acts *IP* on X_2'. Hence, by applying Lemmas 4 and 5, we conclude that G acts *IP* on X.

If a topological group K acts on a topological space X, then for every $x, y \in X$, either $Kx = Ky$ or they are disjoint. Hence, $\{Kx : x \in X\} = X/K$ is given the quotient topology and is called the *orbit space* of K acting on X. If $\nu : X \to X/K$ is the natural map, $\nu(x) = Kx$, then ν is onto, continuous and open. A set C contained in X is a *cross-section* to the orbits of K in X if there is a 1-1 continuous function $f : X/K \to C$ onto C such that $f(Kx) \in Kx$. It follows that $f \circ \nu : C \to C$ and $\nu \circ f : X/K \to X/K$ are the identities, so $f : X/K \to C$ is in fact a homeomorphism. This leads us to

THEOREM 5. *Suppose S acts on X. If there is a cross-section, C, to the orbits of $G = H(1)$ acting on X', then X' is homeomorphic to $G \times X'/G$.*

Proof. Theorem 4 says that G acts *IP* on X'. Furthermore, if $g_1, g_2 \in G$ such that for some $x \in X'$ we have $g_1 x = g_2 x$, then $g_1 = g_2$. The Theorem now follows from [**1**; Theorem 2].

In [**1**] it is in fact shown that under the hypotheses of Theorem 5, the map $\psi : G \times C \to X'$ defined by $\psi(g, c) = gc$ is a homeomorphism from $G \times C$ onto X'. We shall use this fact to prove the main result of this section.

THEOREM 6. *Suppose S acts on the topological space X such that*

(a) *there is a cross-section, $C \subset X'$, to the orbits of $G = H(1)$ in X';*

(b) *$X = SX'$;*

(c) *if $x, y \in X'$ such that $0x = 0y$, then $Gx = Gy$; and*

(d) *if U is open in S and V is open in X', then UV is open in X.*

Then, X is homeomorphic to $S \times X/G$.

Proof. By virtue of (a) and the preceeding comment and Theorem, we know that if we define $\phi : S \times C \to X$ by $\phi(s, c) = sc$, then $\phi|(G \times C)$ is a homeomorphism from $G \times C$ onto X'. Since ϕ is simply the restriction of the action function to $S \times C$, ϕ is a continuous function. Let $f : X'/G \to C$ be the homeomorphism guaranteed by the cross-section.

Suppose $x \in X$. By (b), $x \in SX'$, so there is a $y \in X'$ and an $s \in S$ such that $x = sy$. Since $\phi|(G \times C)$ maps onto X', there is a $(g, c) \in G \times C$ such that $y = gc = \phi(g, c)$. Thus, $(sg, c) \in S \times C$ such that $\phi(sg, c) = (sg)c = s(gc) = sy = x$, so ϕ is onto. Now suppose $\phi(s, c) = \phi(s', c')$. Then, $sc = s'c'$, so $0c = (0s)c = 0(sc) = 0(s'c') = (0s')c' = 0c'$. From (c), $Gc = Gc'$, so $c = f(Gc) = f(Gc') = c'$, and $sc = s'c' = s'c$. Since $c \in X'$, $s = s'$, and we have $(s, c) = (s', c')$, so ϕ is 1-1.

Let $U \times V$ be open in $S \times C$, and let $sc = \phi(s, c)$ be in $\phi(U \times V)$. Since ϕ is 1-1, we know that $s \in U$ and $c \in V$. Since $s = s1 \in U$, there are open sets W_s and W_1 in S which contain s and 1 respectively, such that $W_s W_1 \subset U$. Since G is open in S, by Proposition 2, we may assume that $W_1 \subset G$. Then, $W_1 \times V$ is open in $G \times C$, so, by previous remarks, $W_1 V = \phi(W_1 \times V)$ is open in X', and $c = \phi(1, c) \in W_1 V$. Since $W_1 V$ is open in X' and W_s is open in S, (d) gives $W_s(W_1 V)$ as an open set in X. Also, $s \in W_s$ and $c \in W_1 V$, so $sc \in W_s(W_1 V)$. Since $W_s W_1 \subset U$, we have $W_s(W_1 V) = (W_s W_1)V \subset UV = \phi(U \times V)$. Hence, $W_s(W_1 V)$ is an open set in X which contains sc and lies in $\phi(U \times V)$. Therefore, $\phi(U \times V)$ is open, and ϕ is an open map. Hence, ϕ is a homeomorphism from $S \times C$ onto X.

Defining $\Psi: S \times X'/G \to X$ by $\Psi(s, Gx) = \phi(s, f(Gx))$ gives the desired homeomorphism from $S \times X'/G$ onto X.

We conclude this paper with the following easy corollary to Theorem 6.

COROLLARY 6.1. *Suppose $\eta: S \times X \to X$ is an action of S on X satisfying the hypotheses of Theorem 6. If we define $\mu: S \times (S \times X'/G) \to S \times X'/G$ by*

$$\mu(s, (t, Gx)) = (st, Gx),$$

where st is the multiplication in S, then μ is an action of S on $S \times X'/G$ which is equivalent to η.

4. Questions. We conclude this paper with some questions. First, are all the conditions in Theorem 6 really necessary? Can one obtain results similar to Theorems 5 and 6 if we consider W-semigroups [**9**]? What about tame W-semigroups? Finally, what, if anything, can be said in this vein concerning actions of an arbitrary locally compact Clifford semigroup?

REFERENCES

1. T. H. McH. Hanson, *Actions of a locally compact group with zero*, Can. J. Math. *23* (1971), 413–420.

2. ——— *Actions of a locally compact group with zero. II*, Semigroup Forum *3* (1972), 371–374.

3. ——— *Actions that fiber and vector semigroups*, Can. J. Math. *24* (1972), 29–37.

4. K. H. Hofmann, *Locally compact semigroups in which a subgroup with compact complement is dense*, Trans. Amer. Math. Soc. *106* (1963), 19–51.

5. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups* (Charles E. Merrill Books, Inc., Columbus, Ohio, 1966).

6. D. Montgomery and L. Zippin, *Topological transformation groups* (Interscience, New York, 1955).
7. D. P. Stadtlander, *Semigroup actions and dimension*, Aequationes Math. *3* (1969), 1–14.
8. J. W. Stepp, *D-semigroups*, Proc. Amer. Math. Soc. *22* (1969), 402–406.
9. T. S. Wu, *Locally compact semigroups with dense maximal subgroups*, Trans. Amer. Math. Soc. *113* (1964), 151–168.

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