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On the maximal normal prime-nilpotent subgroup of a prime-solvable group

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A characterization of the maximal normal p-nilpotent subgroup of a finite p-solvable group is obtained for primes $p \neq 2$ or 3.

1. Introduction

A result of Baer provides a characterization of the largest normal p-subgroup, $O_p(G)$, of a finite group G where p is a prime: an element x belongs to $O_p(G)$ if and only if the subgroup $\langle x, x' \rangle$ is a p-group for all conjugates x' of x in G [2, Theorem 3.8.2].

The purpose of this note is to provide a similar description of the largest normal p-nilpotent subgroup $O_{p',p}(G)$ of a finite p-solvable group. A finite group G is p-nilpotent if there is a normal subgroup G complementing a Sylow p-subgroup.

The following example complicates our conclusion. Let Qd(3) denote the natural semi-direct product of a 2-dimensional vector space V over GF(3) with SL(2, 3), the group of all linear transformations on V of determinant 1. A counting argument, a generalization of which appears in step (7) below, shows that Qd(3) is generated by a class K of 3-elements such that any two elements in K generate a 3-nilpotent group, but Qd(3) is not 3-nilpotent.

THEOREM 1. *Let G be a finite p-solvable group, where p is an odd prime. If for some element x of G, $\langle x, x' \rangle$ is p-nilpotent for*

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all conjugates x' *of* x *in* G *, then* $x \in O_{p',p}(G)$ *or* $p = 3$ *,* 3 *divides* $o(x)$ *, and* $\langle x^g : g \in G \rangle$ *involves* Qd(3) .

Theorem 1 will be derived from the following two results. Let π denote a set of primes and π' the set of primes not in π . A finite group is π-separable if each composition factor is a π-group or a π'-group. The first result includes the Theorem of Baer for π-separable groups.

PROPOSITION 1. *Let* G *be a finite* π*-separable group. Let* x *be a* π*-element of* G *such that* $\langle x, x' \rangle$ *is a* π*-group for all* x' *conjugate to* x *. Then* $x \in O_{\pi}(G)$.

THEOREM 2. *Let* G *be a finite* p*-solvable group,* p *an odd prime. Let* x *be a* p*-element in* G *such that* $\langle x, x' \rangle$ *is* p*-nilpotent for all* x' *conjugate to* x *. Then* $x \in O_{p',p}(G)$ *unless* $p = 3$ *and* $\langle x^g : g \in G \rangle$ *involves* Qd(3) .

REMARK. The following example shows that Theorem 1 is false if $p = 2$. For any odd q and any $n \geq 1$, let G be the group generated by $x, y, x_1, \ldots, x_q$, subject to the relations:

$$x^{2^{n+1}} = y^q = x_i^{2^n} = [x_i, x_j] = 1 \text{ for all } i, j ,$$

$$y^x = y^{-1} , \quad x^2 = x_1 x_2 \ldots x_q ,$$

$$x_i^x = x_{i\alpha} , \quad x_i^y = x_{i\beta} ,$$

where

$$\alpha = (1)(2, q) \ldots \left(\frac{q-1}{2}, \frac{q+1}{2}\right)$$

and

$$\beta = (1, 2, \ldots, q) .$$

A counting argument similar to that used in (7) below shows that any two conjugates of x generate a 2-nilpotent group, but $x \notin O_{2',2}(G)$.

The reader is referred to [2] or [4] for terminology and notations, which are standard.

2. Proofs

Proof of Proposition 1. Let G be a minimal counter-example. Then $O_\pi(G) = 1$, for otherwise $xO_\pi(G) \in O_\pi\big(G/O_\pi(G)\big) = O_\pi(G)$. By π-separability, $O_{\pi'}(G) \neq 1$. Let q be a prime in π' ; there is an x-invariant Sylow q-subgroup Q of $O_{\pi'}(G)$, by [2, Theorem 6.2.2]. For any $y \in Q$, $[x, y] = x^{-1}x^y \in \langle x, x^y\rangle \cap Q$. Since $\langle x, x^y\rangle$ is a π-group and Q is a π'-group, $[x, y] = 1$ and thus x centralizes Q . It follows that x centralizes $O_{\pi'}(G)$. By [2, Theorem 6.3.2], $C_G\big(O_{\pi'}(G)\big) \leq O_{\pi'}(G)$, a contradiction.

Proof of Theorem 2. Let G be a minimal counter-example. We may assume that G is generated by the conjugates of x in G , for if the conjugates of x generate a proper subgroup M , M either involves Qd(3) or $x \in O_{p',p}(M)$. Since $M \trianglelefteq G$, in the latter case $x \in O_{p',p}(G)$.

As G is a counter-example, no section of G involves Qd(3) if $p = 3$, a fact used implicitly in each induction step below.

(1). $O_{p'}(G) = 1$.

For otherwise, let $\overline{G} = G/O_{p'}(G)$. Then $\langle \overline{x}, \overline{x}'\rangle$ is p-nilpotent for all conjugates x' of x . Thus $\overline{x} \in O_{p',p}(\overline{G})$ and we are done.

Let $P = O_p(G)$. By p-solvability of G , $P \neq 1$.

(2). *P is elementary abelian and $C_G(P) = P$.*

Let $\overline{G} = G/\Phi(P)$. By [2, Theorem 6.3.4], $C_{\overline{G}}(\overline{P}) = \overline{P}$. Then again $\langle \overline{x}, \overline{x}'\rangle$ is p-nilpotent for all conjugates x' of x . Thus if $\Phi(P) \neq 1$, $\overline{x} \in O_{p',p}(\overline{G})$. Since $O_{p'}(\overline{G})$ centralizes $\overline{P}$, $O_{p'}(\overline{G}) = 1$. Thus $\overline{x} \in O_p(\overline{G})$, and hence $x \in O_p(G)$, a contradiction. We therefore have $\Phi(P) = 1$.

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(3). *P contains a unique minimal normal subgroup of G.*

Otherwise, if P_1 and P_2 are disjoint normal subgroups of G contained in P, G/P_i is p-nilpotent. Therefore $G \cong G/P_1 \cap P_2$ is p-nilpotent.

For some conjugate x' of x, $\langle x, x' \rangle$ is not a p-group, since otherwise $x \in O_p(G)$ by Proposition 1. Thus x normalizes some non-trivial p'-subgroup. If x centralizes every p'-subgroup it normalizes, then x centralizes $O_{p'}(\langle x, x' \rangle)$ and therefore lies in $O_p(\langle x, x' \rangle)$. It follows that $\langle x, x' \rangle$ is a p-group, for all x' conjugate to x.

Among all p'-subgroups of G, normalized but not centralized by x, choose Q of minimal order. By a standard Hall-Higman reduction, given in [3, p. 5], Q is a q-group for some prime $q \neq p$ and x acts irreducibly on Q/Q'. Furthermore, Q is either elementary abelian, or $Q' = Z(Q) = \Phi(Q) = C_Q(x)$ and Q' is elementary abelian. In particular, if $g \in Q \backslash Q'$, $\langle x, x^g \rangle = \langle x, Q \rangle$.

(4). *$G = PQ\langle x \rangle$, P is a faithful irreducible $Q\langle x \rangle$-module, $\langle x \rangle \cap P = 1$, and $x^p = 1$.*

If $H = PQ\langle x \rangle \neq G$, then $x \in O_{p',p}(H)$. Since $C_G(P) = P$, $O_{p'}(H) = 1$. Thus $x \in O_p(H)$, and $[x, Q] \leq O_p(H) \cap Q = 1$, a contradiction. Therefore $G = PQ\langle x \rangle$.

Let P^* be the unique minimal normal subgroup of G contained in P. If $P^* \neq P$, then G/P^* is p-nilpotent, by induction. Thus Q centralizes P/P^*. The subgroup $\langle x \rangle \cap P$ is central in G, and since $\langle x \rangle \cap P$ has order at most p, $\langle x \rangle \cap P \leq P^*$. Therefore $L = P^*Q\langle x \rangle$ is a proper subgroup of G and $x \in O_{p',p}(L)$. Since Q is generated by Q-conjugates of x, $Q \leq O_{p',p}(L)$ and therefore $[Q, P^*] = 1$. Thus Q centralizes P/P^* and P^*; by [2, Theorem 5.3.2], Q centralizes P, a contradiction. Thus $P^* = P$, and since $C_G(P) = P$, P is faithful and irreducible when viewed as a $Q\langle x \rangle$-module.

Since $\langle x\rangle \cap P$ is central in G, $\langle x\rangle \cap P = 1$. If $x^p \neq 1$, $PQ\langle x^p\rangle$ is a proper normal subgroup of G, and $\langle x^p, x'^p\rangle$ is a p-nilpotent group for all x' conjugate to x. By induction, $x^p \in O_{p',p}(PQ\langle x^p\rangle) \trianglelefteq G$. Thus $x^p \in O_{p',p}(G) = P$ and therefore $x^p = 1$.

(5). *Let* $C = C_P(x)$. *Then* $|C|^2 \leq |P|$.

If not, then for any conjugate x' of x, $D = C_P(x) \cap C_P(x') \neq 1$. Thus for x and x' generating $Q\langle x\rangle$, D is a $Q\langle x\rangle$-invariant subgroup of P. It follows that $D = P$. Since $Q\langle x\rangle$ centralizes C, we have a contradiction.

(6). $C_G(x) = CC_Q(x)\langle x\rangle$.

By modularity, $C_G(x) = \langle x\rangle \times C_{QP}(x)$. Since x acts irreducibly on Q/Q', $C_{QP}(x) \leq Q'P$. By modularity again, $C_{PQ}(x) = Q'C_P(x) = C_Q(x)C_P(x)$.

(7). $|C|^2 = |P|$.

We count the G-conjugates of x in two ways. First, by (6), x has $|G : C_G(x)| = |P : C||Q : C_Q(x)|$ conjugates.

For any x' conjugate to x, let $L = \langle x, x'\rangle$. Suppose L is not a p-group. Then $LP/P \cong Q\langle x\rangle$. Since L is p-nilpotent, $L \neq G$, and therefore $L \cap P = 1$, by the irreducibility of $Q\langle x\rangle$ on P. Thus $L \cong L/L\cap P \cong Q\langle x\rangle$. Since $Q\langle x\rangle = N_G(Q)$, L is either a p-group or conjugate to $N_G(Q)$.

Since $\langle x, P\rangle$ is the unique Sylow P-subgroup of G containing x, $\langle x, x'\rangle$ is p-group only if $x' \in \langle x, P\rangle$. There are $|P : C||Q : C_Q(x)|$ conjugates of x in G and $|Q : C_Q(x)|$ Sylow p-subgroups in G. Hence each Sylow p-subgroup contains

$$\big(|P : C||Q : C_Q(x)|\big)\big/\big(|Q : C_Q(x)|\big) = |P : C|$$

conjugates of x.

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If x and x' normalize Q, $\langle x\rangle$ and $\langle x'\rangle$ are Q-conjugate, and since $N_G(\langle x\rangle) = C_G(x)$, x and x' are Q-conjugate. Therefore $Q\langle x\rangle$ contains $|Q : C_Q(x)| - 1$ conjugates of x apart from x. Now if x normalizes Q and Q^y, then $x^{y^{-1}}$ normalizes Q, and $x^t = x^{y^{-1}}$ for some $t \in Q$. Thus $x^{ty} = x$ and $Q^{ty} = Q^y$; that is, the conjugates of Q normalized by x are conjugate under $C_G(x)$. Thus there are

$$\left(|Q : C_Q(x)|-1\right)|C_G(x) : C_G(x) \cap N_G(Q)| = \left(|Q : C_G(x)|-1\right)|C|$$

conjugates x' of x which with x generate a group isomorphic to $Q\langle x\rangle$. This completes the second count of the conjugates of x.

Comparing, we have

$$|P : C| + |C|\left(|Q : C_Q(x)|-1\right) = |P : C|\,|Q : C_Q(x)| ,$$

from which we have

$$0 = \left(|Q : C_Q(x)|-1\right)\left(|P : C|-|C|\right) .$$

Since $C_Q(x) \neq Q$, the conclusion follows.

Let K be a finite splitting field for $Q\langle x\rangle$. Viewing P as a $\mathrm{GF}(p) = F$ module for $Q\langle x\rangle$, we consider the $KQ\langle x\rangle$-module $V = K \otimes_F P$. By [4, Satz V, 13.3], V is a direct sum of absolutely irreducible $KQ\langle x\rangle$-modules, $V_1, \ldots, V_u$. It is routine to check that $\dim_K C_V(x) = \dim_F C_P(x) = \frac{1}{2}\dim_F P$ and $C_V(x) = \sum C_{V_i}(x)$. For the argument below, let W denote any one of the submodules $\{V_i\}$. Since the representations of $Q\langle x\rangle$ on the modules $\{V_i\}$ are algebraically conjugate, the representation on W is faithful.

(8). *Q is not abelian.*

Suppose Q is abelian. By the remarks preceeding step (4) above, Q is elementary abelian and x acts irreducibly on Q. Thus $Q\langle x\rangle$ is a Frobenius group with cyclic complement. By Clifford's Theorem [2, Theorem 3.4.1], W is a direct sum of Q-submodules $\{W_i\}$, each of which is a

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direct sum of isomorphic irreducible Q-submodules. Further, $\langle x\rangle$ permutes the homogeneous components $\{W_i\}$ transitively. By Theorem 3.4.3 of [2], the number of homogeneous components $\{W_i\}$ is the order of $\langle x\rangle$, in our present case. Thus $\langle x\rangle$ permutes the components regularly and therefore $|C_W(x)| = |W_1|$. Hence $\dim_K C_W(x) = (1/p) \dim_K W$ and so $\dim_K C_V(x) = (1/p) \dim_K V$. Since $p > 2$, this contradicts (7).

(9). *G does not exist.*

By (8) and the remarks above (4), we are left with the case that Q is special. We argue that W is an irreducible Q-module. Again, by Clifford's Theorem, W is a sum of e homogeneous Q-components. If $e \neq 1$, choose E to be an irreducible Q-submodule of the first component. Then $\sum Ex^i$ is a $Q\langle x\rangle$-submodule of W. Thus $W = \sum Ex^i$ and each component is irreducible. Since $\langle x\rangle$ acts transitively on the set of components and x has order p, $\langle x\rangle$ acts regularly. As in (8), the centralizer of x in W is too small. Thus $e = 1$ and W is a sum of isomorphic irreducible Q-modules. By a theorem of Green [2, Theorem 3.5.6], the number of distinct irreducible Q-submodules of W is congruent to 1 modulo p and so $\langle x\rangle$ fixes a Q-submodule of W. Thus W is irreducible as a Q-module.

Since the representation on W is faithful and absolutely irreducible, $Z(Q)$ is cyclic, that is, Q is extra-special. The computations of Section 8 of [3] show that W, viewed as an $\langle x\rangle$-module, is a sum of t copies of the regular representation and one copy of the indecomposable representation of degree $p - 1$. Thus W has K-dimension $tp + p - 1$ and $C_W(x)$ has K-dimension $t + 1$. By (7), $2(t+1)u = (tp+p-1)u$, forcing $p = 2 + 1/(t+1)$. Thus $t = 0$ and $p = 3$. Hence W is a 2-dimensional K-space and $Q\langle x\rangle$ is isomorphic to a subgroup of $SL(2, K)$. A 3-nilpotent subgroup of $SL(2, K)$ which is not a 3-group is isomorphic to $SL(2, 3)$, as follows from [4, Hauptsatz II, 8.27]. We now see that G is a semi-direct product of P with $SL(2, 3)$, the latter being represented faithfully and irreducibly on P. By [1], $SL(2, 3)$ has only one faithful irreducible representation over $GF(3)$, the natural representation of dimension 2. Hence $G = Qd(3)$, a

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final contradiction.

Proof of Theorem 1. The element x can be written uniquely in the form $x = yz = zy$, where y and z are powers of x, and y is a p'-element and z is a p-element. For any $g \in G$, $\langle x, x^g \rangle$ is p-nilpotent, and so $y \in O_{p'}(\langle x, x^g \rangle)$. Thus $\langle y, y^g \rangle$ is a p'-group for all $g \in G$. By Proposition 1, $y \in O_{p'}(G)$. Similarly, $\langle z, z^g \rangle$ is p-nilpotent for all $g \in G$. Applying Theorem 2, $z \in O_{p',p}(G)$ unless $p = 3$ and $\langle z^g : g \in G \rangle$ involves Qd(3). Thus $x = yz \in O_{p',p}(G)$ unless $p = 3$ and $\langle x^g : g \in G \rangle \geq \langle z^g : g \in G \rangle$ involves Qd(3).

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