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Radius of comparison and mean topological dimension: $\mathbb{Z}^d$-actions

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Abstract. Consider a minimal-free topological dynamical system $(X, \mathbb{Z}^d)$. It is shown that the radius of comparison of the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is at most half the mean topological dimension of $(X, \mathbb{Z}^d)$. As a consequence, the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is classified by the Elliott invariant if the mean dimension of $(X, \mathbb{Z}^d)$ is zero.

1 Introduction

Let (X, Γ) be a topological dynamical system, where X is a compact Hausdorff space and Γ is a discrete amenable group. The mean (topological) dimension of (X, Γ), denoted by $\mathrm{mdim}(X, \Gamma)$, was introduced by Gromov [11], and then was developed and studied systematically by Lindenstrauss and Weiss [26]. It is a numerical invariant, taking values in the range $[0, +\infty]$, to measure the complexity of (X, Γ) in terms of dimension growth with respect to partial orbits. Applications of mean dimension theory can be found in the theory of topological dynamical systems (see [12, 13, 15, 16, 22, 24, 26]), geometric analysis (see [3, 27, 39, 40]), operator algebras (see [7, 23, 28–30]), and information theory (see [25]).

On the other hand, for a general unital stably finite C*-algebra A, the radius of comparison, introduced by Toms [36] and denoted by $\mathrm{rc}(A)$, is also a numerical invariant, to measure the regularity of the C*-algebra A. It can be regarded as an abstract measure of the dimension growth of A. A heuristic example is $\mathrm{M}_n(\mathrm{C}(X))$, the C*-algebra of (complex) $n \times n$ matrix valued continuous functions on a finite CW-complex X; its radius of comparison is around $\frac{1}{2}\frac{\dim(X)}{n}$, half the dimension ratio of $\mathrm{M}_n(\mathrm{C}(X))$.

For the given topological dynamical system (X, Γ), the canonical C*-algebra to be considered is the transformation group C*-algebra, $\mathrm{C}(X) \rtimes \Gamma$. A natural question to ask then is how the radius of comparison of the C*-algebra is connected to the mean dimension of the dynamical system. Phillips and Toms have made the following conjecture:

Conjecture (Phillips–Toms). *Let (X, Γ) be a minimal and free topological dynamical system, where X is a compact Hausdorff space, and Γ is a discrete amenable group.*

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Then

$$\mathrm{rc}(\mathrm{C}(X) \rtimes \Gamma) = \frac{1}{2}\mathrm{mdim}(X, \Gamma).$$

This conjecture is closely related to the classification of C*-algebras. In general, the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ can be wild and not determined by (i.e., classified by) the Elliott invariant (even with $\Gamma = \mathbb{Z}$; see [8]). So, an important question in the classification program for C*-algebras is to determine which transformation group C*-algebras are classifiable. Now, a special case of this conjecture is that $\mathrm{mdim}(X, \Gamma) = 0$ implies $\mathrm{rc}(\mathrm{C}(X) \rtimes \Gamma) = 0$, which is the same as strict comparison of positive elements; by the Toms–Winter conjecture, this should imply that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is Jiang–Su stable; and hence belongs to the classifiable class of [2, 4, 6, 9, 10, 35].

Many researches have been done on the classifiability of transformation group C*-algebras. Under the assumption that X is finite dimensional (in which case the mean dimension is automatically zero), it was shown in [38] that the algebra $\mathrm{C}(X) \rtimes \mathbb{Z}$ has finite nuclear dimension, and therefore is Jiang–Su stable. Using Rokhlin dimension, this result was generalized to $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ by [33], and then to the actions of residually finite groups with box spaces of finite asymptotic dimension (see [34]). And recently, substantial progresses have been made on the Jiang–Su stability using almost finiteness and dynamical comparison (see [18–20]).

In the case that X is not necessary finite dimensional, which this paper mainly focuses on, so far the only result is [7] where $\mathbb{Z}$-actions were considered, and zero mean dimension was shown to imply zero radius of comparison (classifiability of the C*-algebra). Note that this result in particular covers all strictly ergodic dynamical systems. Beyond the case of mean dimension zero, Phillips considered $\mathbb{Z}$-actions in [30] and showed that (in the minimal case) the radius of comparison of $\mathrm{C}(X) \rtimes \mathbb{Z}$ is at most $1 + 36\mathrm{mdim}(X, \mathbb{Z})$.

In this paper, the results of [7] are both strengthened and generalized to minimal and free $\mathbb{Z}^d$-actions.

Theorem A (Theorem 5.6). *Let $(X, T, \mathbb{Z}^d)$ be a minimal-free dynamical system. Then*

$$\mathrm{rc}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) \leq \frac{1}{2}\mathrm{mdim}(X, T, \mathbb{Z}^d). \tag{1.1}$$

As a consequence, one obtains classifiability of $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ if $(X, T, \mathbb{Z}^d)$ has mean dimension zero.

Theorem B (Theorem 5.8). *Let $(X, T, \mathbb{Z}^d)$ be a minimal-free dynamical system with mean dimension zero. Then $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ absorbs the Jiang–Su algebra tensorially, and hence is classified by its Elliott invariant. In particular, if $\dim(X) < \infty$, or $(X, T, \mathbb{Z}^d)$ has at most countably many ergodic measures, or $(X, T, \mathbb{Z}^d)$ has finite topological entropy, then $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is classified by its Elliott invariant.*

The argument in [7, 30, 38] relies on Putnam's orbit-cutting algebra (or what Phillips termed the large subalgebra) A_y, associated with a point $y \in X$. In the case of zero mean dimension, the argument in [7] also heavily depends on the small boundary property (which is equivalent to mean dimension zero in the case of $\mathbb{Z}$-actions). However, beyond the case of $\mathbb{Z}$-actions, it is not clear in general how to construct large subalgebras; moreover, once the dynamical system does not have mean dimension zero, the small boundary property does not hold any more. So, instead of large subalgebra and small boundary property, the proofs of Theorems A and B depend on the Uniform Rokhlin Property (URP) and Cuntz-comparison of open sets (COS).

Definition 1.1. (Definitions 3.1 and 4.1 of [29]) A topological dynamical system (X, Γ), where Γ is a discrete amenable group, is said to have the URP if, for any $\varepsilon > 0$ and any finite set $K \subseteq \Gamma$, there exist closed sets $B_1, B_2, \ldots, B_S \subseteq X$ and (K, ε)-invariant sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_S \subseteq \Gamma$ such that the sets

$$B_s\gamma, \quad \gamma \in \Gamma_s,\ s = 1, \ldots, S,$$

are mutually disjoint and

$$\operatorname{ocap}(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in \Gamma_s} B_s\gamma) < \varepsilon,$$

where ocap denotes the orbit capacity (see, for instance, Definition 5.1 of [26]).

The dynamical system (X, Γ) is said to have (λ, m)-COS, where $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, if for any open sets $E, F \subseteq X$ with

$$\mu(E) < \lambda\mu(F), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

where $\mathcal{M}_1(X, \Gamma)$ is the simplex of all invariant probability measures on X, then

$$\varphi_E \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{m} \quad \text{in } (\mathrm{C}(X) \rtimes \Gamma) \otimes \mathcal{K},$$

where φ_E and φ_F are continuous functions supported on E and F, respectively (see Example 2.1).

The dynamical system (X, Γ) is said to have COS if it has (λ, m)-COS for some λ and m.

Remark 1.2. If Γ is monotilable, e.g., $\Gamma = \mathbb{Z}^d$, then one can take $S = 1$ in the definition of URP.

These properties were introduced in [29], and it was shown (Theorem 4.8 of [29]) that the URP and COS together imply

$$\operatorname{rc}(\mathrm{C}(X) \rtimes \Gamma) \leq \frac{1}{2}\operatorname{mdim}(X, \Gamma),$$

if (X, Γ) is free and minimal. It was shown in [28] (Theorem 4.8) that if, in addition, (X, Γ) has mean dimension zero, then the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is classifiable. Thus, both Theorems A and B follow from the following result.

Theorem C (Theorems 4.2 and 5.5). *Any dynamical system $(X, T, \mathbb{Z}^d)$ with the marker property (see Definition 3.1) has the URP and COS.*

The proof of the theorem above uses the adding-one-dimension and going-down arguments of [14]. With these arguments, one constructs a Rokhlin tower with base set open, such that the tower almost cover the whole space in the sense that the its

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complement is uniformly small under all the invariant probability measures. This implies the property URP, and is essentially contained in [14].

For the property COS, one actually uses the adding-one-dimension and going-down arguments to construct two Rokhlin towers. Although these two Rokhlin towers do not in general even taken together cover the space X (even approximately), they can be constructed so that the complement of the first tower can be broken into finitely many pieces so that each piece can be translated into the support of the second tower, in such a way that the (order of the) overlaps of the translations of those pieces can be bounded by a constant which only depends on the group $\mathbb{Z}^d$. Together with other things, this eventually leads to the property COS.

Remark 1.3. In [21], it is shown that the URP and COS imply that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ always has stable rank one (classifiable or not), and satisfies the Toms–Winter conjecture (i.e., it is classifiable if, and only if it has strict comparison of positive elements) if (X, Γ) is free and minimal. Thus, by Theorem C, the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ always has stable rank one (classifiable or not), and satisfies the Toms–Winter conjecture. (In [1], stable rank one was established in the case $d = 1$.)

2 Notation and preliminaries

2.1 Topological dynamical systems

In this paper, we shall only consider $\mathbb{Z}^d$-actions on a metrizable compact Hausdorff space X.

Definition 2.1. Consider a topological dynamical system $(X, T, \mathbb{Z}^d)$. A closed set $Y \subseteq X$ is said to be invariant if $T^n(Y) = Y$, $n \in \mathbb{Z}^d$, and $(X, T, \mathbb{Z}^d)$ is said to be minimal if $\varnothing$ and X are the only invariant closed subsets. The dynamical system $(X, T, \mathbb{Z}^d)$ is free if for any $x \in X$, $\{n \in \mathbb{Z}^d : T^n(x) = x\} = \{0\}$.

Remark 2.2. The dynamical system $(X, T, \mathbb{Z}^d)$ is induced by d commuting homeomorphisms of X, and vice versa.

Definition 2.3. A Borel measure μ on X is invariant under the action σ if $\mu(E) = \mu(T^n(E))$, for any $n \in \mathbb{Z}^d$ and any Borel set $E \subseteq X$. Denote by $\mathcal{M}_1(X, T, \mathbb{Z}^d)$, the collection of all invariant Borel probability measures on X. It is a Choquet simplex under the weak* topology.

Definition 2.4. (See [11, 26]) Consider a topological dynamical system $(X, T, \mathbb{Z}^d)$, and let E be a subset of X. The orbit capacity of E is defined by

$$\operatorname{ocap}(E) := \lim_{N\to\infty} \frac{1}{N^d} \sup_{x\in X} \sum_{n\in\{0,1,\ldots,N-1\}^d} \chi_E(T^n(x)),$$

where χ_E is the characteristic function of E. The limit always exists.

Definition 2.5. (See [26]) Let $\mathcal{U}$ be an open cover of X. Define

$$D(\mathcal{U}) = \min\{\operatorname{ord}(\mathcal{V}) : \mathcal{V} \leq U\},$$

where $\operatorname{ord}(\mathcal{V}) := -1 + \sup_{x\in X} \sum_{V\in\mathcal{V}} \chi_V(x)$.

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Consider a topological dynamical system $(X, T, \mathbb{Z}^d)$. Then the topological mean dimension of $(X, T, \mathbb{Z}^d)$ is defined by

$$\mathrm{mdim}(X, T, \mathbb{Z}^d) := \sup_{\mathcal{U}} \lim_{N\to\infty} \frac{1}{N^d} D\Big(\bigvee_{n\in\{0,1,\ldots,N-1\}^d} T^{-n}(\mathcal{U})\Big),$$

where $\mathcal{U}$ runs over all finite open covers of X.

Remark 2.6. It follows from the definition that if $\dim(X) < \infty$, then $\mathrm{mdim}(X, T, \mathbb{Z}^d) = 0$. By [26], if $(X, T, \mathbb{Z}^d)$ has at most countably many ergodic measures, then $\mathrm{mdim}(X, T, \mathbb{Z}^d) = 0$. Also by [24], if $(X, T, \mathbb{Z}^d)$ has finite topological entropy, then $\mathrm{mdim}(X, T, \mathbb{Z}^d) = 0$.

Definition 2.7. (Definition 3.1 of [29] and Remark 1.2) A topological dynamical system $(X, T, \mathbb{Z}^d)$ is said to have the URP if for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists a closed set $B \subseteq X$ such that the sets

$$T^n(B), \quad n \in \{0, 1, \ldots, N-1\}^d,$$

are mutually disjoint and

$$\mathrm{ocap}\Big(X \setminus \bigsqcup_{n\in\{0,1,\ldots,N-1\}^d} T^n(B)\Big) < \varepsilon.$$

In this paper, we shall show that all minimal and free dynamical systems $(X, T, \mathbb{Z}^d)$ have the property URP (Theorem 4.2).

2.2 Crossed product C*-algebras

Let $(X, T, \mathbb{Z}^d)$ be a topological dynamical system. Then the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is the universal C*-algebra

$$A = \mathrm{C}^*\{f, u_n;\ u_n f u_n^* = f \circ T^n,\ u_m u_n^* = u_{m-n},\ u_0 = 1,\ f \in \mathrm{C}(X),\ m, n \in \mathbb{Z}^d\}.$$

The C*-algebra A is nuclear, and if T is minimal and free, the C*-algebra A is simple. Moreover, the simplex of tracial states of $\mathrm{C}(X) \rtimes_\sigma \Gamma$ is canonically homeomorphic to the simplex of invariant probability measures of $(X, T, \mathbb{Z}^d)$.

2.3 Cuntz comparison of positive elements of a C*-algebra

Definition 2.8. Let A be a C*-algebra, and let $a, b \in A^+$. Then we say that a is Cuntz subequivalent to b, denoted by $a \precsim b$, if there are x_i, y_i, $i = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} x_i b y_i = a.$$

We say that a is Cuntz equivalent to b if $a \precsim b$ and $b \precsim a$. This is an equivalence relation; let us denote the equivalence class of a by $[a]$.

Let $\mathcal{K}$ denote the algebra of compact operators on a separable Hilbert space. Then the equivalence classes of positive elements of $A \otimes \mathcal{K}$ form an ordered semigroup under the addition

$$[a] + [b] := \left[\begin{pmatrix} a & \\ & b \end{pmatrix}\right], \quad a, b \in (A \otimes \mathcal{K})^+$$

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(independent of the isomorphism $\mathcal{K} \cong \mathcal{K} \otimes \mathrm{M}_2$). It is called the Cuntz semigroup of A.

Let $\tau : A \to \mathbb{C}$ be a trace. Define the rank function

$$\mathrm{d}_\tau(a) := \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) = \mu_\tau(\mathrm{sp}(a) \cap (0, +\infty)),$$

where μ_τ is the Borel measure induced by τ on the spectrum of a. It is well known that

$$\mathrm{d}_\tau(a) \leq \mathrm{d}_\tau(b), \quad \text{if } a \precsim b. \tag{2.1}$$

Example 2.1. Consider $h \in \mathrm{C}(X)^+$ and let μ be a probability measure on X. Then

$$\mathrm{d}_{\tau_\mu}(h) = \mu(h^{-1}(0, +\infty)),$$

where τ_μ is the trace of $\mathrm{C}(X)$ induced by μ.

Let $f, g \in \mathrm{C}(X)$ be positive elements. Then f and g are Cuntz equivalent if and only if $f^{-1}(0, +\infty) = g^{-1}(0, +\infty)$. That is, their equivalence classes are determined by their open supports.

With this example, let us introduce the following notation.

Definition 2.9. For each open set $E \subseteq X$, pick a continuous function

$$\varphi_E : X \to [0, +\infty) \quad \text{such that} \quad E = \varphi_E^{-1}(0, +\infty).$$

For instance, one can pick $\varphi_E(x) = d(x, X \backslash E)$, where d is a compatible metric on X. This notation will be used throughout this paper. Note that the Cuntz equivalence class of φ_E is independent of the choice of the individual function φ_E.

In general, the converse of (2.1) fails (e.g., $A = \mathrm{M}_4(\mathrm{C}(S^2 \times S^2))$). But one can measure it by the radius of comparison.

Definition 2.10. (Definition 6.1 of [36]) Let A be a C*-algebra. Denote by $\mathrm{M}_n(A)$, the C*-algebra of $n \times n$ matrices over A. Regard $\mathrm{M}_n(A)$ as the upper-left corner of $\mathrm{M}_{n+1}(A)$, and consider the union

$$\mathrm{M}_\infty(A) = \bigcup_{n=1}^{\infty} \mathrm{M}_n(A),$$

the algebra of all finite matrices over A.

The radius of comparison of a unital C*-algebra A, denoted by $\mathrm{rc}(A)$, is the infimum of the set of real numbers $r > 0$ such that if $a, b \in (\mathrm{M}_\infty(A))^+$ satisfy

$$\mathrm{d}_\tau(a) + r < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

then $a \precsim b$, where $\mathrm{T}(A)$ is the simplex of tracial states. (In [36], the radius of comparison is defined in terms of quasitraces instead of traces; but since all the algebras considered in this note are nuclear, by [17], any quasitrace actually is a trace.)

Example 2.2. Let X be a compact Hausdorff space. Then (see [37])

$$\mathrm{rc}(\mathrm{M}_n(\mathrm{C}(X))) \leq \frac{1}{2}\frac{\dim(X) - 1}{n}, \tag{2.2}$$

where $\dim(\mathrm{X})$ is the topological covering dimension of X (a lower bound for $\mathrm{rc}(\mathrm{C}(X))$ in terms of cohomological dimension is given in [5]).

The C*-algebra $C(X)$ is canonically contained in the crossed product C*-algebra $C(X) \rtimes \mathbb{Z}^d$. Thus, one can compare positive elements of $C(X)$ not only inside $C(X)$, but also inside the larger C*-algebra $C(X) \rtimes \mathbb{Z}^d$.

Definition 2.11. (Definition 4.1 of [29]) A dynamical system $(X, T, \mathbb{Z}^d)$ is said to have (λ, m)-COS, where $\lambda \in (0,1]$ and $m \in \mathbb{N}$, if for any open sets $E, F \subseteq X$ with

$$\mu(E) < \lambda\mu(F), \quad \mu \in \mathcal{M}_1(X, T, \mathbb{Z}^d),$$

then

$$\varphi_E \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{m} \quad \text{in } (C(X) \rtimes \mathbb{Z}^d) \otimes \mathcal{K},$$

where φ_E and φ_F are continuous functions given by Definition 2.9.

The dynamical system (X, Γ) is said to have COS if it has (λ, m)-COS for some λ and m.

In this paper, we shall show that all minimal and free dynamical systems $(X, T, \mathbb{Z}^d)$ have the property COS (Theorem 5.5). Together with the property URP, we obtain a dynamical version of (2.2); that is,

$$\mathrm{rc}(C(X) \rtimes \mathbb{Z}^d) \leq \frac{1}{2}\mathrm{mdim}(X, T, \mathbb{Z}^d)$$

if $(X, T, \mathbb{Z}^d)$ is minimal and free (Corollary 5.6).

The following notation and lemma will be frequently used.

Definition 2.12. Let $a \in A^+$, where A is a C*-algebra, and let $\varepsilon > 0$. Define

$$(a - \varepsilon)_+ = f(a) \in A,$$

where $f(t) = \max\{t - \varepsilon, 0\}$.

Lemma 2.13. (Section 2 of [31]) *Let a, b be positive elements of a C*-algebra A. Then $a \precsim b$ if and only if $(a - \varepsilon)_+ \precsim b$ for all $\varepsilon > 0$.*

3 Adding-one-dimension, going-down arguments, *R*-boundary points, and *R*-interior points

The adding-one-dimension and going-down arguments are introduced in [14], and they play a crucial role in the paper. Let us first make a brief review.

Definition 3.1. (Definition 1.2 of [14]) A topological dynamical system (X, T, Γ) is said to have the marker property if, for any finite set $F \subseteq \Gamma$, there is an open set $U \subseteq X$ such that $X = \bigcup_{n\in\Gamma} T^n(U)$ and $U \cap T^n(U) = \varnothing$ for all nonidentity $n \in F$.

Any topological dynamical system with the marker property is free; any minimal and free dynamical system has the marker property.

Let $(X, T, \mathbb{Z}^d)$ be a dynamical system with the marker property. Let M be an arbitrary natural number. Then there exist open set $U \subseteq X$, a compact set $K \subseteq U$, an integer $L > M$, and a continuous function $\varphi : X \to [0,1]$ with

$$\varphi|_K = 1 \quad \text{and} \quad \varphi|_{X\setminus U} = 0,$$

such that:

(1) if $\varphi(x) > 0$ for some $x \in X$, then $\varphi(T^n(x)) = 0$ for all nonzero $n \in \mathbb{Z}^d$ with $|n| \leq M$, and
(2) for any $x \in X$, there is $n \in \mathbb{Z}^d$ with $|n| \leq L$ such that $\varphi(T^n(x)) = 1$.

See Section 4 of [14] for the proof.

Let $x \in X$ be arbitrary. Following [14], one considers the set

$$\{(n, \varphi(T^n(x))^{-1}) : n \in \mathbb{Z}^d,\ \varphi(T^n(x)) \neq 0\} \subseteq \mathbb{R}^{d+1},$$

and defines the Voronoi cell $V(x,n) \subseteq \mathbb{R}^{d+1}$ with center $(n, \varphi(T^n(x))^{-1})$ by

$$V(x,n) = \left\{\xi \in \mathbb{R}^{k+1} : \left\|\xi - (n, \varphi(T^n(x))^{-1})\right\| \leq \left\|\xi - (m, \varphi(T^m(x))^{-1})\right\|, \quad m \in \mathbb{Z}^d\right\},$$

where $\|\cdot\|$ is the ℓ^2-norm on $\mathbb{R}^{d+1}$. If $\varphi(T^n(x)) = 0$, then put

$$V(x,n) = \varnothing.$$

One then has a tiling

$$\mathbb{R}^{d+1} = \bigcup_{n \in \mathbb{Z}^d} V(x,n).$$

Pick $H > (L + \sqrt{d})^2$. For each $n \in \mathbb{Z}^d$, define

$$W_H(x,n) = V(x,n) \cap (\mathbb{R}^d \times \{-H\}).$$

One then has a tiling

$$\mathcal{W}_H : \mathbb{R}^d = \bigcup_{n \in \mathbb{Z}^d} W(x,n).$$

The following are some basic properties of this construction, and the proofs can be found in [14].

Lemma 3.2. (Lemma 4.1 of [14]) *With the construction above, one has:*

(1) *$\mathcal{W}_H$ is continuous on x in the following sense: Suppose that $W(x,n)$ has nonempty interior. For any $\varepsilon > 0$, if $y \in X$ is sufficiently close to x, then the Hausdorff distance between $W_H(x,n)$ and $W_H(y,n)$ is less than ε.*
(2) *$\mathcal{W}_H$ is $\mathbb{Z}^d$-equivariant: $W_H(T^m(x), n-m) = -m + W_H(x,n)$.*
(3) *If $\varphi(T^n(x)) > 0$, then*

$$B_{\frac{M}{2}}(n, \varphi(T^n(x))^{-1}) \subseteq V(x,n).$$

(4) *If $W_H(x,n)$ is nonempty, then*

$$1 \leq \varphi(T^n(x))^{-1} \leq 2.$$

(5) *If $(a, -H) \in V(x,n)$, then*

$$\|a - n\| < L + \sqrt{d}.$$

Moreover, if one considers different horizontal cuts, at levels $-sH$ and $-H$ for some $s > 1$, then one has the following lemma.

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Lemma 3.3. (Lemma 4.1(4) of [14] and its proof) *Let $s > 1$ and $r > 0$. One can choose M sufficiently large such that if $(a, -sH) \in V(x, n)$, then*

$$B_r\left(\frac{a}{s} + \left(1 - \frac{1}{s}\right) n\right) \subseteq W_H(x, n)$$

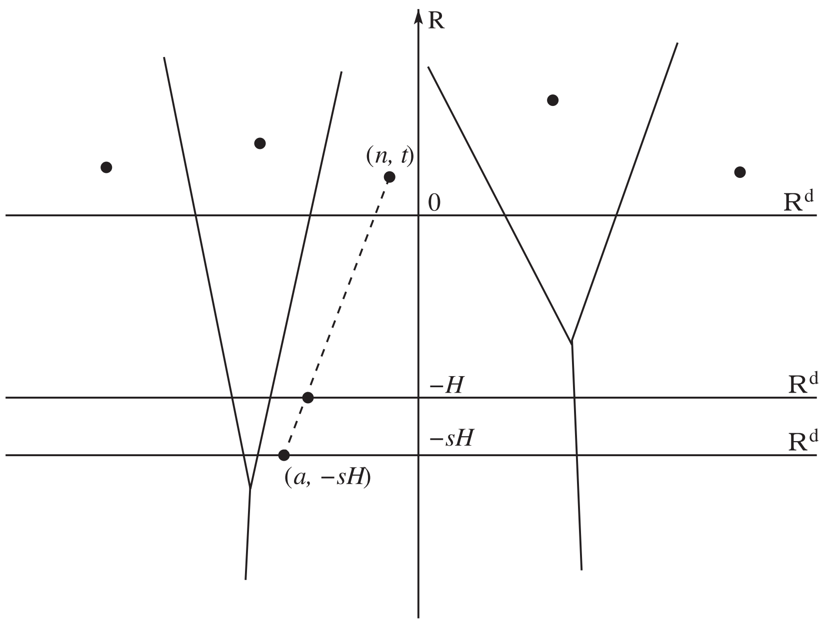
and

$$\left\| \frac{a}{s} + \left(1 - \frac{1}{s}\right) n - \left(a + \frac{(s-1)H}{sH + t}(n - a)\right) \right\| \leq \frac{4}{L + \sqrt{d}},$$

where $t = \varphi(T^n(x))^{-1}$ and $\|\cdot\|$ is the l^2-norm on $\mathbb{R}^d$.

Definition 3.4. Note that the point $\left(a + \frac{(s-1)H}{sH+t}(n - a), -H\right)$ is the image of $(a, -sH)$ in the plane $\mathbb{R}^d \times \{-H\}$ under the projection toward the center (n, t). Let us call $a + \frac{(s-1)H}{sH+t}(n - a)$ the H-projective image of a (with the center (n, t)).

This construction can be illustrated by the following picture.



The following result concerns convex bodies in $\mathbb{R}^d$; the author is indebted to Tyrrell McAllister for discussions related to this.

Lemma 3.5. *Consider the Euclidean space $\mathbb{R}^d$. For any $\varepsilon > 0$ and any $r > 0$, there is $N_0 > 0$ such that if $N \geq N_0$, then, for any convex body $V \subseteq \mathbb{R}^d$, one has*

$$\frac{1}{N^d} \left| \{ n \in \mathbb{Z}^d : \operatorname{dist}(n, \partial V) \leq r, \; n \in I_N \} \right| < \varepsilon,$$

where $I_N = [0, N]^d$.

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Proof Pick N_0 sufficiently large that

$$2\frac{\mathrm{vol}(\partial_{r+\sqrt{d}}(I_N^r))}{\mathrm{vol}(I_N)} < \varepsilon, \quad N > N_0,$$

where $\partial_E(K)$ denotes the E-neighborhood of the boundary of a convex body K and I_N^r denotes the r-neighborhood of I_N. Then N_0 verifies the conclusion of the lemma.

Indeed, for any $N \geq N_0$, denote by $\partial^+_{r+\sqrt{d}}(V \cap I_N^r)$, the outer $(r+\sqrt{d})$-neighborhood of the convex body $V \cap I_N^r$ (i.e., $\partial^+_{r+\sqrt{d}}(V \cap I_N^r) = (V \cap I_N^r)_{r+\sqrt{d}} \setminus V \cap I_N^r$, where $(V \cap I_N^r)_{r+\sqrt{d}}$ denotes the $(r+\sqrt{d})$-neighborhood of $V \cap I_N^r$). It follows from Steiner's formula (see, for instance, (4.1) of [32]) and the fact that $W_0(V \cap I_N^r) = \mathrm{vol}(V \cap I_N^r)$ that

$$\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N^r)) = \sum_{j=1}^{d} C_d^j W_j(V \cap I_N^r)(r+\sqrt{d})^j,$$

where $C_d^j = \frac{d!}{j!(d-j)!}$ and $W_j(\cdot)$ is the jth quermassintegral which can be written as

$$W_j(K) = V(\underbrace{K, \dots, K}_{d-j}, \underbrace{B, \dots, B}_{j})$$

for a convex body K with $V(\cdot)$ the mixed volume and B the unit ball of $\mathbb{R}^d$ (see (5.31) of [32]). Since the mixed volume is monotonic (see (5.25) of [32]), the quermassintegrals W_j, $j = 1, \dots, d$, are monotonic. Hence,

$$W_j(V \cap I_N^r) \leq W_j(I_N^r), \quad j = 1, 2, \dots, d,$$

and

$$\begin{aligned}\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N^r)) &= \sum_{j=1}^{d} C_d^j W_j(V \cap I_N^r)(r+\sqrt{d})^j \\ &\leq \sum_{j=1}^{d} C_d^j W_j(I_N^r)(r+\sqrt{d})^j \\ &= \mathrm{vol}(\partial^+_{r+\sqrt{d}}(I_N^r)).\end{aligned}$$

Since $\mathrm{vol}(\partial_{r+\sqrt{d}}(V \cap I_N^r)) \leq 2\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N^r))$, one has

$$\frac{\mathrm{vol}(\partial_{r+\sqrt{d}}(V \cap I_N^r))}{\mathrm{vol}(I_N)} \leq 2\frac{\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N^r))}{\mathrm{vol}(I_N)} \leq 2\frac{\mathrm{vol}(\partial_{r+\sqrt{d}}(I_N^r))}{\mathrm{vol}(I_N)} < \varepsilon.$$

On the other hand, note that,

$$(\partial V) \cap I_N^r \subseteq \partial(V \cap I_N^r),$$

and $(\partial_r V) \cap I_N$ is contained in the closed r-neighborhood of $(\partial V) \cap I_N^r$, one has that $(\partial_r V) \cap I_N$ is contained in the the closed r-neighborhood of $\partial(V \cap I_N^r)$; that is,

$$(\partial_r V) \cap I_N \subseteq \partial_r(V \cap I_N^r).$$

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Hence,

$$\begin{aligned} &\left|\{n \in \mathbb{Z}^d : \operatorname{dist}(n, \partial V) \le r,\ n \in I_N\}\right| \\ &\le \left|\partial_r(V \cap I_N^r) \cap \mathbb{Z}^d\right| \\ &\le \operatorname{vol}(\partial_{r+\sqrt{d}}(V \cap I_N^r)), \end{aligned}$$

so that

$$\frac{1}{N^d}\left|\{n \in \mathbb{Z}^d : \operatorname{dist}(n, \partial V) \le r,\ n \in I_N\}\right| \le \frac{\operatorname{vol}(\partial_{r+\sqrt{d}}(V \cap I_N^r))}{\operatorname{vol}(I_N)} < \varepsilon,$$

as desired. ∎

Definition 3.6. Consider a continuous function $X \ni x \mapsto \mathcal{W}(x)$ with $\mathcal{W}(x)$ an $\mathbb{R}^d$-tiling. For each $R \ge 0$, a point $x \in X$ is said to be an R-interior point if $\operatorname{dist}(0, \partial\mathcal{W}(x)) > R$, where $\partial\mathcal{W}(x)$ denotes the union of the boundaries of the tiles of $\mathcal{W}$. Note that, in this case, the origin $0 \in \mathbb{R}^d$ is an interior point of a (unique) tile of $\mathcal{W}(x)$. Denote this tile by $\mathcal{W}(x)_0$, and denote the set of R-interior points by $\iota_R(\mathcal{W})$.

Otherwise (if $\operatorname{dist}(0, \partial\mathcal{W}(x)) \le R$), the point x is said to be an R-boundary point. Denote by $\beta_R(\mathcal{W})$ the set of R-boundary points.

Note that $\beta_R(\mathcal{W})$ is closed and $\iota_R(\mathcal{W})$ is open.

Lemma 3.7. *Let $(X, T, \mathbb{Z}^d)$ be a dynamical system with the marker property.*

Fix $1 < s < 2$. Let $R_0 > 0$ and $\varepsilon > 0$ be arbitrary. Let $N > N_0$, where N_0 is the constant of Lemma 3.5 with respect to ε and $2R_0 + 4 + \sqrt{d}/2$, and let $R_1 > \max\{R_0, N\sqrt{d}\}$.

Then the open sets U' and U in the construction can be chosen to be sufficiently small that M is large enough that there exist a finite open cover

$$U_1 \cup U_2 \cup \cdots \cup U_K \supseteq \beta_{R_0}(\mathcal{W}_{sH}),$$

and $n_1, n_2, \ldots, n_K \in \mathbb{Z}^d$, such that:

(1) *$T^{n_i}(U_i) \subseteq \iota_{R_1}(\mathcal{W}_H) \subseteq \iota_0(\mathcal{W}_H)$, $i = 1, 2, \ldots, K$,*

(2) *the open sets*

$$T^{n_i}(U_i), \quad i = 1, 2, \ldots, K,$$

can be rearranged into m families as

$$\begin{cases} T^{n_1}(U_1), \ldots, T^{n_{s_1}}(U_{s_1}), \\ T^{n_{s_1+1}}(U_{s_1+1}), \ldots, T^{n_{s_2}}(U_{s_2}), \\ \cdots \\ T^{n_{s_{m-1}+1}}(U_{s_{m-1}+1}), \ldots, T^{n_{s_m}}(U_{s_m}), \end{cases}$$

with $m \le (\lfloor 2\sqrt{d} \rfloor + 1)^d$, in such a way that the open sets in each family are mutually disjoint, and

(3) *for each $x \in \iota_0(\mathcal{W}_H)$ and each $c \in \operatorname{int}(\mathcal{W}_H(x)_0) \cap \mathbb{Z}^d$ with $\operatorname{dist}(c, \partial\mathcal{W}_H) > N\sqrt{d}$, where $\operatorname{int}(\mathcal{W}_H(x)_0)$ denotes the set of interior points of the cell $\mathcal{W}_H(x)_0$, one has*

$$\frac{1}{N^d}\left|\left\{n \in \{0, 1, \ldots, N-1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} T^{n_i}(U_i)\right\}\right| < \varepsilon.$$

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Proof By Lemma 4.1(4) of [14] (see Lemma 3.3), one can choose M to be sufficiently large that for a fixed $H > (L + \sqrt{d})^2$, if $(a, -sH) \in V(x, n)$ for some $a \in \mathbb{R}^d$, then

$$B_{R_1+2R_0+1+\frac{\sqrt{d}}{2}}(as^{-1} + (1-s^{-1})n) \times \{-H\} \in V(x, n)$$

and

$$\left\| \frac{a}{s} + \left(1 - \frac{1}{s}\right) n - \left(a + \frac{(s-1)H}{sH+t}(n-a)\right) \right\| \leq \frac{4}{L+\sqrt{d}} < 4, \tag{3.1}$$

where $t = \varphi(T^n(x))^{-1}$ (so that $a + \frac{(s-1)H}{sH+t}(n-a)$ is the H-projective image of a).

For each $n \in \mathbb{Z}^d$, define

$$U_n = \{x \in X : \operatorname{dist}(0, \partial W_{sH}(x, n)) < 2R_0,\ \operatorname{int} W_{sH}(x, n) \neq \varnothing\}.$$

Note that U_n is open. For the same n, pick $h_n \in \mathbb{Z}^d$ such that

$$\|(1-s^{-1})n - h_n\| \leq \sqrt{d}/2. \tag{3.2}$$

For each $x \in U_n$, there is $a \in \partial W_{sH}(x, n) \subseteq \mathbb{R}^d$ with

$$\|a\| < 2R_0.$$

By the choice of M (and hence H), one has

$$B_{R_1+2R_0+1+\frac{\sqrt{d}}{2}}(as^{-1} + (1-s^{-1})n) \subseteq W_H(x, n). \tag{3.3}$$

Since

$$\|h_n - (as^{-1} + (1-s^{-1})n)\| \leq \|as^{-1}\| + \|(1-s^{-1})n - h_n\| < 2R_0 + \sqrt{d}/2, \tag{3.4}$$

by (3.3), one has

$$B_{R_1+1}(h_n) \subseteq W_H(x, n),$$

which implies

$$B_{R_1}(0) \subset B_{R_1+1}(0) \subseteq -h_n + W_H(x, n) = W_H(T^{h_n}(x), n - h_n). \tag{3.5}$$

In particular, $T^{h_n}(x) \in \iota_{R_1}(\mathcal{W}_H)$, which implies

$$T^{h_n}(U_n) \subseteq \iota_{R_1}(\mathcal{W}_H),$$

and this shows the property (1).

Note that by (3.1) and (3.4),

$$\left\| h_n - \left(a + \frac{(s-1)H}{sH+t}(n-a)\right) \right\| < 2R_0 + 4 + \sqrt{d}/2. \tag{3.6}$$

Since $a \in \partial W_{sH}(x, n)$, this implies that h_n is in the $(2R_0 + 4 + \sqrt{d}/2)$-neighborhood of the the H-projective image of $\partial W_{sH}(x, n)$ (with respect to (n, t)).

On the other hand, if $x \in \beta_{R_0}(\mathcal{W}_{sH})$, then $\operatorname{dist}(0, \partial W_{sH}(x, n)) \leq R_0$ for some $n \in \mathbb{Z}^d$ with $\operatorname{int}(W_{sH}(x, n)) \neq \varnothing$, which implies that $x \in U_n$. Therefore, the collection of

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sets $\{U_n : n \in \mathbb{Z}^d\}$ forms an open cover of $\beta_{R_0}(\mathcal{W}_{sH})$. Since $\beta_{R_0}(\mathcal{W}_{sH})$ is a compact set, there is a finite subcover

$$U_{n_1}, U_{n_2}, \dots, U_{n_K}.$$

(In fact, $\{U_n : \|n\| < L + \sqrt{d} + 2R_0\}$ already covers $\beta_{R_0}(\mathcal{W}_{sH})$ by (5) of Lemma 3.2.)

Assume that n_i and n_j satisfy

$$T^{h_{n_i}}(U_{n_i}) \cap T^{h_{n_j}}(U_{n_j}) \neq \varnothing.$$

Then there are $x_i \in U_{n_i}$ and $x_j \in U_{n_j}$ with

$$T^{h_{n_i}}(x_i) = T^{h_{n_j}}(x_j).$$

Since $x_i \in U_{n_i}$ and $x_j \in U_{n_j}$, by (3.5), one has

$$B_{R_1}(0) \subseteq W_H(T^{h_{n_i}}(x_i), n_i - h_{n_i})$$

and

$$\begin{aligned} B_{R_1}(0) &\subseteq W_H(T^{h_{n_j}}(x_j), n_j - h_{n_j}) \\ &= W_H(T^{h_{n_i}}(x_i), n_j - h_{n_j}). \end{aligned}$$

Therefore, $n_i - h_{n_i} = n_j - h_{n_j}$, and so

$$n_i - n_j = h_{n_i} - h_{n_j}.$$

Together with (3.2) and noting that $s < 2$, one has

$$\begin{aligned} \|n_i - n_j\| &= \|h_{n_j} - h_{n_j}\| \\ &\leq (1 - s^{-1}) \|n_i - n_j\| + \sqrt{d} \\ &< \|n_i - n_j\| /2 + \sqrt{d}, \end{aligned}$$

i.e.,

$$\|n_i - n_j\| < 2\sqrt{d}. \tag{3.7}$$

Note that the set $\mathbb{Z}^d$ can be divided into $(\lfloor 2\sqrt{d} \rfloor + 1)^d$ families $(\mathbb{Z}^d)_1, \dots, (\mathbb{Z}^d)_{(\lfloor 2\sqrt{d}\rfloor+1)^d}$ such that any pair of elements inside each family has distance between them at least $2\sqrt{d}$, and therefore, by (3.7),

$$T^{h_n}(U_n) \cap T^{h_{n'}}(U_{n'}) = \varnothing, \quad n, n' \in (\mathbb{Z}^d)_m,\ m = 1, \dots, (\lfloor 2\sqrt{d} \rfloor + 1)^d.$$

Then the rearrangement of $U_{n_1}, \dots, U_{n_K}$ as

$$\{U_{n_i} : i = 1, \dots, K,\ n_i \in (\mathbb{Z}^d)_1\}, \dots, \{U_{n_i} : i = 1, \dots, K,\ n_i \in (\mathbb{Z}^d)_{(\lfloor 2\sqrt{d}\rfloor+1)^d}\}$$

possesses the property (2).

Let $x \in \iota_0(\mathcal{W}_H)$ (then $\mathcal{W}_H(x)_0$ is well defined). Write

$$\mathcal{W}_H(x)_0 = W_H(x, n(x)) = V(x, n(x)) \cap (\mathbb{R}^d \times \{-H\}), \quad \text{where } n(x) \in \mathbb{Z}^d.$$

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Assume, there is $m \in \operatorname{int}(\mathcal{W}_H(x)_0) \cap \mathbb{Z}^d$ such that

$$T^m(x) \in T^{h_{n_k}}(U_{n_k}) \tag{3.8}$$

for some n_k, $k = 1, 2, \ldots, K$.

Since $m \in \operatorname{int}(\mathcal{W}_H(x)_0) \cap \mathbb{Z}^d$, one has

$$0 \in \operatorname{int}(-m + W_H(x, n(x))) = \operatorname{int} W_H(T^m(x), n(x) - m).$$

Hence, $T^m(x) \in \iota_0(\mathcal{W}_H)$ and

$$\mathcal{W}_H(T^m(x))_0 = W_H(T^m(x), n(x) - m). \tag{3.9}$$

By the assumption (3.8), there is $x_{n_k} \in U_{n_k}$ such that

$$T^m(x) = T^{h_{n_k}}(x_{n_k}).$$

Then, with (3.5), one has

$$B_{R_1}(0) \subseteq W_H(T^{h_{n_k}}(x_{n_k}), n_k - h_{n_k}) = W_H(T^m(x), n_k - h_{n_k});$$

that is, the tile $W_H(T^m(x), n_k - h_{n_k})$ contains 0 as an interior point. By (3.9), the tile $W_H(T^m(x), n(x) - m)$ contains 0 as an interior point, and therefore, these two tiles are same, i.e.,

$$W_H(T^m(x), n(x) - m) = W_H(T^{h_{n_k}}(x_{n_k}), n_k - h_{n_k}),$$

and hence,

$$V(T^m(x), n(x) - m) = V(T^{h_{n_k}}(x_{n_k}), n_k - h_{n_k}).$$

Therefore, at the $-sH$ level, one also has

$$W_{sH}(T^m(x), n(x) - m) = W_{sH}(T^{h_{n_k}}(x_{n_k}), n_k - h_{n_k}) = -h_{n_k} + W_{sH}(x_{n_k}, n_k). \tag{3.10}$$

By (3.6), h_{n_k} is in the $(2R_0 + 4 + \sqrt{d}/2)$-neighborhood of the H-projective image of $\partial W_{sH}(x_{n_k}, n_k)$ (see Definition 3.4), and therefore, 0 is in the $(2R_0 + 4 + \sqrt{d}/2)$-neighborhood of the H-projective image of

$$-h_{n_k} + \partial W_{sH}(x_{n_k}, n_k) = \partial W_{sH}(T^{h_{n_k}}(x_{n_k}), n_k - h_{n_k}).$$

Thus, by (3.10), the origin 0 is in the $(2R_0 + 4 + \sqrt{d}/2)$-neighborhood of the H-projective image of $\partial W_{sH}(T^m(x), n(x) - m)$, and hence, m is in the $(2R_0 + 4 + \sqrt{d}/2)$-neighborhood of the H-projective image of $\partial W_{sH}(x, n(x))$, which is denoted by $\partial W_{sH}^H(x, n(x))$, i.e.,

$$\partial W_{sH}^H(x, n(x)) := \left\{ a + \frac{(s-1)H}{sH + t}(n(x) - a) : a \in \partial W_{sH}(x, n(x)) \right\},$$

where $t = \varphi(T^n(x))^{-1}$ (see Definition 3.4).

Therefore, for any $c \in \operatorname{int}(\mathcal{W}_H(x)_0) \cap \mathbb{Z}^d$ with $\operatorname{dist}(c, \partial \mathcal{W}_H) > N\sqrt{d}$, since

$$c + n \in \operatorname{int}(\mathcal{W}_H(x)_0), \quad n \in \{0, 1, \ldots, N-1\}^d,$$

one has

$$\left\{n \in \{0,1,\ldots,N-1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} h_i(U_i)\right\}$$
$$\subseteq \left\{n \in \{0,1,\ldots,N-1\}^d : \operatorname{dist}(c+n, \partial W_{sH}^H(x, n(x))) < 2R_0 + 4 + \sqrt{d}/2\right\}.$$

Hence, by the choice of N and Lemma 3.5 (applied to the H-projective image of $W_{sH}(x, n(x))$),

$$\frac{1}{N^d}\left|\left\{n \in \{0,1,\ldots,N-1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} h_i(U_i)\right\}\right|$$
$$\leq \frac{1}{N^d}\left|\left\{n \in c + \{0,1,\ldots,N-1\}^d : \operatorname{dist}(n, \partial W_{sH}^H(x, n(x))) < 2R_0 + 4 + \sqrt{d}/2\right\}\right|$$
$$< \varepsilon.$$

This proves the property (3). ∎

4 Two towers

4.1 Rokhlin towers

Let $x \mapsto \mathcal{W}(x) = \bigcup_{n\in\mathbb{Z}^d} W(x,n)$ be a map with $\mathcal{W}(x)$ a tiling of $\mathbb{R}^d$, where $W(x,n)$ is the cell with label n. Assume that the map $x \mapsto \mathcal{W}(x)$ is continuous in the sense that for any $\varepsilon > 0$ and any $W(x,n)$ with nonempty interior, if $y \in X$ is sufficiently close to x, then the Hausdorff distance between $W(x,n)$ and $W(y,n)$ is less than ε. One also assumes that the map $x \mapsto \mathcal{W}(x)$ is equivariant in the sense that

$$W(T^{-m}(x), n+m) = m + W(x,n), \quad x \in X,\ m,n \in \mathbb{Z}^d.$$

The tiling functions $\mathcal{W}_H$ and $\mathcal{W}_{sH}$ constructed in the previous section clearly satisfy the assumptions above. With such a tiling function, one actually can build a Rokhlin tower as follows:

Let $N \in \mathbb{N}$ be arbitrary. Put

$$\Omega = \{x \in X : \operatorname{dist}(0, \partial\mathcal{W}(x)) > N\sqrt{d} \text{ and } \mathcal{W}(x)_0 = W(x,n) \text{ for some } n \equiv 0 \pmod N\},$$

where by $n \equiv 0 \pmod N$, one means $n_i \equiv 0 \pmod N$, $i = 1,2,\ldots,d$, if $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. Note that Ω is open.

Let $m \in \{0,1,\ldots,N-1\}^d$. Pick an arbitrary $x \in \Omega$ and consider $T^{-m}(x)$. Note that $0 \in W(x,n)$ for some $n \equiv 0 \pmod N$ and $\operatorname{dist}(0, \partial W(x,n)) > N\sqrt{d}$. Since

$$W(T^{-m}(x), n+m) = m + W(x,n),$$

one has

$$0 \in \operatorname{int} W(T^{-m}(x), n+m) \quad \text{and} \quad n+m \equiv m \pmod N.$$

Hence,

$$T^{-m}(\Omega) \subseteq \Omega'_m, \tag{4.1}$$

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where

$$\Omega'_m := \{x \in X : 0 \notin \partial\mathcal{W}(x) \text{ and } \mathcal{W}(x)_0 = W(x,n),\ n \equiv m \pmod N\}.$$

For the same reason, if one defines

$$\Omega''_m := \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) > 2N\sqrt{d} \text{ and } \mathcal{W}(x)_0 = W(x,n),\ n \equiv m \pmod N\},$$

then

$$\Omega''_m \subseteq T^{-m}(\Omega). \tag{4.2}$$

Since the sets

$$\Omega'_m, \quad m \in \{0, 1, \dots, N-1\}^d,$$

are mutually disjoint, it follows from (4.1) that the sets

$$T^{-m}(\Omega), \quad m \in \{0, 1, \dots, N-1\}^d,$$

are mutually disjoint. That is, they form a Rokhlin tower for $(X, T, \mathbb{Z}^d)$.

On the other hand, by (4.2) and the construction of Ω''_m, one has

$$\bigsqcup_{m \in \{0,1,\dots,N-1\}^d} T^{-m}(\Omega) \supseteq \bigsqcup_{m \in \{0,1,\dots,N-1\}^d} \Omega''_m = \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) > 2N\sqrt{d}\}. \tag{4.3}$$

In particular, one has

$$\mathrm{ocap}\left(X \setminus \bigsqcup_{m \in \{0,1,\dots,N-1\}^d} T^{-m}(\Omega)\right) \leq \mathrm{ocap}(\{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) \leq 2N\sqrt{d}\}). \tag{4.4}$$

Lemma 4.1. *For any $E > 0$, one has*

$$\mathrm{ocap}(\{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) \leq E\}) \leq \limsup_{R \to \infty} \frac{1}{\mathrm{vol}(B_R)} \sup_{x \in X} \mathrm{vol}(\partial_{E+\sqrt{d}}\mathcal{W}(x) \cap B_{R+\sqrt{d}}),$$

where $\partial_{E+\sqrt{d}}\mathcal{W}(x) = \{\xi \in \mathbb{R}^d : \mathrm{dist}(\xi, \partial W(x)) \leq E + \sqrt{d}\}$.

Proof Pick an arbitrary $x \in X$ and an arbitrary strictly positive number R, and consider the partial orbit

$$T^m(x), \quad \|m\| < R.$$

Note that if $\mathrm{dist}(0, \partial\mathcal{W}(T^m(x))) \leq E$ (i.e., $0 \in \partial_E\mathcal{W}(T^m(x))$) for some m, then

$$-m \in \partial_E\mathcal{W}(x).$$

Therefore,

$$\{m : \|m\| < R \text{ and } 0 \in \partial_E\mathcal{W}(T^m(x))\} \subseteq \{m : \|m\| < R \text{ and } m \in \partial_E\mathcal{W}(x)\}.$$

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One has

$$\begin{aligned}
&\frac{1}{|B_R \cap \mathbb{Z}^d|} |\{\|m\| < R : 0 \in \partial_E \mathcal{W}(T^m(x))\}| \\
&\leq \frac{1}{|B_R \cap \mathbb{Z}^d|} |\{\|m\| < R : m \in \partial_E \mathcal{W}(x)\}| \\
&\leq \frac{1}{|B_R \cap \mathbb{Z}^d|} \mathrm{vol}(\partial_{E+\sqrt{d}} \mathcal{W}(x) \cap B_{r+\sqrt{d}}).
\end{aligned}$$

Since

$$\lim_{R\to\infty} \frac{|B_R \cap \mathbb{Z}^d|}{\mathrm{vol}(B_R)} = 1 \quad \text{and} \quad \lim_{R\to\infty} \frac{\mathrm{vol}(B_{R+\sqrt{d}})}{\mathrm{vol}(B_R)} = 1,$$

$$\limsup_{R\to\infty} \frac{1}{|B_R \cap \mathbb{Z}^d|} |\{|m| < R : 0 \in \partial_E \mathcal{W}(T^m(x))\}| \leq \limsup_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \mathrm{vol}(\partial_{E+\sqrt{d}} \mathcal{W}(x) \cap B_R).$$

Since x is arbitrary, this proves the desired conclusion. ∎

Theorem 4.2. *Let $(X, T, \mathbb{Z}^d)$ be a dynamical system with the marker property. Then, for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there is an open set $\Omega \subseteq X$ such that the sets*

$$T^{-n}(\Omega), \quad n \in \{0, 1, \ldots, N-1\}^d,$$

are mutually disjoint (and hence form a Rokhlin tower), and

$$\mathrm{ocap}\left(X \setminus \bigcup_{n\in\{0,1,\ldots,N-1\}^d} T^{-n}(\Omega) \right) < \varepsilon.$$

In other words, the system $(X, T, \mathbb{Z}^d)$ has the Uniform Rohklin Property (see Definition 2.7 and Lemma 3.2 of [29]).

Proof By Lemma 4.2 of [14], there is an equivariant $\mathbb{R}^d$-tiling $x \mapsto \mathcal{W}(x)$ such that

$$\limsup_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \sup_{x\in X} \mathrm{vol}(\partial_{(2N+1)\sqrt{d}} \mathcal{W}(x) \cap B_R) < \varepsilon.$$

The conclusion of the theorem follows by (4.4) and Lemma 4.1 (with $E = 2N\sqrt{d}$). ∎

4.2 The two towers

The Rokhlin tower constructed above, in general, does not cover the whole space X. Now, consider the two continuous tiling functions $\mathcal{W}_{sH}$ and $\mathcal{W}_H$ of Section 3, and the Rokhlin towers $\mathcal{T}_0$ and $\mathcal{T}_1$ constructed from them, respectively. It is still possible that $\mathcal{T}_0$ and $\mathcal{T}_1$ together do not cover the whole space X. However, in the following theorem, one can show that the complement of the tower $\mathcal{T}_0$ can be cut into pieces and then each piece can be translated into the tower $\mathcal{T}_1$ in such a way that the orders of the overlaps of the translations are bounded by $(\lfloor 2\sqrt{d} \rfloor + 1)^d$, and the intersections of these translations with each $\mathcal{T}_1$-orbit are small. This eventually leads to Cuntz comparison of open sets for minimal-free $\mathbb{Z}^d$-actions (Theorem 5.5).

Theorem 4.3. *Consider a dynamical system $(X, T, \mathbb{Z}^d)$ with the marker property. Let $N \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. There exist two Rokhlin towers*

$$\mathcal{T}_0 := \{T^{-m}(\Omega_0) : m \in \{0, 1, \ldots, N_0 - 1\}^d\} \quad \text{and} \quad \mathcal{T}_1 := \{T^{-m}(\Omega_1) : m \in \{0, 1, \ldots, N_1 - 1\}^d\},$$

with $N_0, N_1 \geq N$ and $\Omega_0, \Omega_1 \subseteq X$ open, an open cover $\{U_1, U_2, \ldots, U_K\}$ of $X \setminus \bigcup_m T^{-m}(\Omega_0)$, and $h_1, h_2, \ldots, h_K \in \mathbb{Z}^d$ such that:

(1) *$T^{h_k}(U_k) \subseteq \bigcup_m T^{-m}(\Omega_1)$, $k = 1, 2, \ldots, K$;*

(2) *the open sets*

$$T^{h_k}(U_k), \quad k = 1, 2, \ldots, K,$$

can be rearranged into m families as

$$\begin{cases} T^{h_1}(U_1), \ldots, T^{h_{s_1}}(U_{s_1}), \\ T^{h_{s_1+1}}(U_{s_1+1}), \ldots, T^{h_{s_2}}(U_{s_2}), \\ \cdots \\ T^{h_{s_{m-1}+1}}(U_{s_{m-1}+1}), \ldots, T^{h_{s_m}}(U_{s_m}), \end{cases}$$

for some $m \leq (\lfloor 2\sqrt{d} \rfloor + 1)^d$, such that the open sets in each family are mutually disjoint;

(3) *for each $x \in \Omega_1$, one has*

$$\frac{1}{N_1^d} \left| \left\{ m \in \{0, 1, \ldots, N_1 - 1\}^d : T^m(x) \in \bigcup_{k=1}^{K} T^{n_k}(U_k) \right\} \right| < \varepsilon.$$

Proof Applying Lemma 3.7 with $R_0 = 2N\sqrt{d}$, ε, and some $1 < s < 2$, together with some $N_1 > \max\{N(R_0, \varepsilon), N\}$ (in place of N) and $R_1 > \max\{R_0, 2N_1\sqrt{d}\}$, where $N(R_0, \varepsilon)$ is the constant of Lemma 3.5 with respect to ε and $2R_0 + 4 + \sqrt{d}/2$ (so $N(R_0, \varepsilon)$ is the constant N_0 of Lemma 3.7; but N_0 will be reserved for the height of the first Rokhlin tower in this proof), we obtain two continuous equivariant $\mathbb{R}^d$-tilings $\mathcal{W}_{sH}$ and $\mathcal{W}_H$ for some (sufficiently large) $H > 0$, a finite open cover

$$U_1 \cup U_2 \cup \cdots \cup U_K \supseteq \beta_{R_0}(\mathcal{W}_{sH}),$$

and $n_1, n_2, \ldots, n_K \in \mathbb{Z}^d$ such that:

(1) $T^{n_i}(U_i) \subseteq \iota_{R_1}(\mathcal{W}_H) \subseteq \iota_0(\mathcal{W}_H)$, $i = 1, 2, \ldots, K$;

(2) the open sets

$$T^{n_i}(U_i), \quad i = 1, 2, \ldots, K,$$

can be rearranged as

$$\begin{cases} T^{n_1}(U_1), \ldots, T^{n_{s_1}}(U_{s_1}), \\ T^{n_{s_1+1}}(U_{s_1+1}), \ldots, T^{n_{s_2}}(U_{s_2}), \\ \cdots \\ T^{n_{s_{m-1}+1}}(U_{s_{m-1}+1}), \ldots, T^{n_{s_m}}(U_{s_m}), \end{cases}$$

with $m \leq (\lfloor 2\sqrt{d} \rfloor + 1)^d$, such that the open sets in each family are mutually disjoint;

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(3) for each $x \in \iota_0(\mathcal{W}_H)$ and each $c \in \mathrm{int}(\mathcal{W}_H(x)_0) \cap \mathbb{Z}^d$ with $\mathrm{dist}(c, \partial \mathcal{W}_H) > N_1\sqrt{d}$, one has

$$\frac{1}{N_1^d}\left|\left\{n \in \{0, 1, \ldots, N_1 - 1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} T^{n_i}(U_i)\right\}\right| < \varepsilon.$$

Put

$$\Omega_0 = \{x \in X : \mathrm{dist}(0, \partial \mathcal{W}_{sH}(x)) > N\sqrt{d} \text{ and } \mathcal{W}_{sH}(x)_0 = W_{sH}(x, n),\ n \equiv 0 \ (\mathrm{mod}\ N)\}.$$

Then the sets

$$T^{-m}(\Omega_0), \quad m \in \{0, 1, \ldots, N_0 - 1\}^d,$$

form a Rokhlin tower with $N_0 = N$, and by (4.3),

(4.5)
$$X \setminus \bigsqcup_{m \in \{0,1,\ldots,N_0-1\}^d} T^{-m}(\Omega_0) \subseteq \{x \in X : \mathrm{dist}(0, \partial \mathcal{W}_{sH}(x)) \leq 2N\sqrt{d}\} = \beta_{2N\sqrt{d}}(\mathcal{W}_{sH}).$$

Thus, $U_1, U_2, \ldots, U_K$ form an open cover of $X \setminus \bigsqcup_{m \in \{0,1,\ldots,N_0-1\}^d} T^{-m}(\Omega_0)$.

Put

$$\Omega_1 = \{x \in X : \mathrm{dist}(0, \partial \mathcal{W}_H(x)) > N_1\sqrt{d} \text{ and } \mathcal{W}_H(x)_0 = W_H(x, n),\ n \equiv 0 \ (\mathrm{mod}\ N_1)\}.$$

Then the sets

$$T^{-m}(\Omega_1), \quad m \in \{0, 1, \ldots, N_1 - 1\}^d,$$

form a Rokhlin tower, and by (4.3) (and the assumption that $R_1 > 2N_1\sqrt{d}$),

(4.6)
$$\bigsqcup_{m \in \{0,1,\ldots,N_1-1\}^d} T^{-m}(\Omega_1) \supseteq \{x \in X : \mathrm{dist}(0, \partial \mathcal{W}_H) > 2N_1\sqrt{d}\} \supseteq \iota_{R_1}(\mathcal{W}_H).$$

Thus, $T^{-h_i}(U_i) \subseteq \bigsqcup_{m \in \{0,1,\ldots,N_1-1\}^d} T^{-m}(\Omega_1)$.

If $x \in \Omega_1$ (whence $x \in \iota_0(\mathcal{W}_H)$ and $\mathrm{dist}(0, \partial \mathcal{W}_H) > N_1\sqrt{d}$), it then follows from (3) (with $c = 0$) that

$$\frac{1}{N_1^d}\left|\left\{m \in \{0, 1, \ldots, N_1 - 1\}^d : T^{m}(x) \in \bigcup_{k=1}^{K} T^{n_k}(U_k)\right\}\right| < \varepsilon,$$

as desired. ■

5 Cuntz comparison of open sets, radius of comparison, and the mean topological dimension

With the two-tower construction in the previous section, let us show that the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ has Cuntz comparison of open sets (Theorem 5.5), and therefore, the radius of comparison of $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is at most half of the mean dimension of $(X, T, \mathbb{Z}^d)$.

As a preparation, one has the following two simple observations on the Cuntz semigroup of a C*-algebra.

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Lemma 5.1. *Let A be a C^*-algebra, and let $a_1, a_2, \dots, a_m \in A$ be positive elements. Then*

$$[a_1] + [a_2] + \cdots + [a_m] \le m[a_1 + a_2 + \cdots + a_m].$$

Proof The lemma follows from the observation:

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} \le \begin{pmatrix} a_1 + \cdots + a_m & & & \\ & a_1 + \cdots + a_m & & \\ & & \ddots & \\ & & & a_1 + \cdots + a_m \end{pmatrix}.$$

∎

For any open set $U \subseteq X$, recall that φ_U is a positive continuous function on X such that $U = \varphi_U^{-1}(0, +\infty)$ (see Example 2.9).

Lemma 5.2. *Let $U_1, U_2, \dots, U_K \subseteq X$ be open sets which can be divided into M families in such a way that each family consists of mutually disjoint sets. Then*

$$[\varphi_{U_1}] + \cdots + [\varphi_{U_K}] \le M[\varphi_{U_1 \cup \cdots \cup U_K}] = M[\varphi_{U_1} + \cdots + \varphi_{U_K}].$$

Proof Write $U_1, U_2, \dots, U_K$ as

$$\{U_1, \dots, U_{s_1}\}, \{U_{s_1+1}, \dots, U_{s_2}\}, \dots, \{U_{s_{m-1}+1}, \dots, U_{s_M}\},$$

so that the open sets in each family are mutually disjoint. Then

$$[\varphi_{U_{s_i+1}}] + \cdots + [\varphi_{U_{s_{i+1}}}] = [\varphi_{U_{s_i+1}} + \cdots + \varphi_{U_{s_{i+1}}}] = [\varphi_{U_{s_i+1} \cup \cdots \cup U_{s_{i+1}}}], \quad i = 0, 1, \dots, M-1,$$

and together with the lemma above, one has

$$\begin{aligned} [\varphi_{U_1}] + \cdots + [\varphi_{U_K}] &= [\varphi_{U_1} + \cdots + \varphi_{U_{s_1}}] + \cdots + [\varphi_{U_{s_{m-1}+1}} + \cdots + \varphi_{U_{s_M}}] \\ &= [\varphi_{U_1 \cup \cdots \cup U_{s_1}}] + \cdots + [\varphi_{U_{s_{m-1}+1} \cup \cdots \cup U_{s_M}}] \\ &\le M[\varphi_{U_1 \cup \cdots \cup U_K}], \end{aligned}$$

as desired. ∎

Definition 5.3. Consider a topological dynamical system (X, Γ), where X is a compact metrizable space and Γ is a discrete group acting on X on the right, and consider a Rokhlin tower

$$\mathcal{T} = \{\Omega\gamma, \ \gamma \in \Gamma_0\},$$

where $\Omega \subseteq X$ is open and $\Gamma_0 \subseteq \Gamma$ is a finite set containing the unit e of the discrete group Γ. Define the C*-algebra

$$\mathrm{C}^*(\mathcal{T}) := \mathrm{C}^*\{u_\gamma \mathrm{C}_0(\Omega), \ \gamma \in \Gamma_0\} \subseteq \mathrm{C}(X) \rtimes \Gamma.$$

By Lemma 3.11 of [29], $\mathrm{C}^*(\mathcal{T})$ is canonically isomorphic to $\mathrm{M}_{|\Gamma_0|}(\mathrm{C}_0(\Omega))$, and

$$\mathrm{C}_0(\bigcup_{\gamma \in \Gamma_0} \Omega\gamma) \ni \phi \mapsto \mathrm{diag}\{\phi|_{\Omega\gamma_1}, \phi|_{\Omega\gamma_2}, \dots, \phi|_{\Omega\gamma_{|\Gamma_0|}}\} \in \mathrm{M}_{|\Gamma_0|}(\mathrm{C}_0(\Omega))\}$$

under this isomorphism.

The following comparison result is essentially a special case of Theorem 7.8 of [29].

Lemma 5.4. (Theorem 7.8 of [29]) *Let Z be a locally compact metrizable space, and consider the C^*-algebra $\mathrm{M}_n(\mathrm{C}_0(Z))$. Let $a, b \in \mathrm{M}_n(\mathrm{C}_0(Z))$ be two positive diagonal elements, i.e.,*

$$a(t) = \mathrm{diag}\{a_1(t), a_2(t), \ldots, a_n(t)\} \quad \text{and} \quad b(t) = \mathrm{diag}\{b_1(t), b_2(t), \ldots, b_n(t)\}$$

for continuous functions $a_1, \ldots, a_n, b_1, \ldots, b_n : Z \to [0, +\infty)$. If

$$\mathrm{rank}(a(t)) \leq \frac{1}{4}\mathrm{rank}(b(t)), \quad t \in Z,$$

and

$$4 < \mathrm{rank}(b(t)), \quad t \in Z,$$

then $a \precsim b$ in $\mathrm{M}_n(\mathrm{C}_0(Z))$.

Proof It is enough to show that $(a - \varepsilon)_+ \precsim b$ for arbitrary $\varepsilon > 0$. For a given $\varepsilon > 0$, there is a compact subset $D \subseteq Z$ such that $(a - \varepsilon)_+$ is supported inside D. Denote by $\pi : \mathrm{M}_n(\mathrm{C}_0(Z)) \to \mathrm{M}_n(\mathrm{C}(D))$ the restriction map. One then has

$$\mathrm{rank}(\pi((a - \varepsilon)_+)(t)) \leq \frac{1}{4}\mathrm{rank}(\pi(b)(t)), \quad t \in D,$$

and

$$\frac{1}{n} < \frac{1}{4n}\mathrm{rank}(b(t)), \quad t \in D.$$

By Theorem 7.8 of [29], one has $\pi((a - \varepsilon)_+) \precsim \pi(b)$ in $\mathrm{M}_n(\mathrm{C}(D))$, i.e., there is a sequence $(v_k) \subseteq \mathrm{M}_n(\mathrm{C}(D))$ such that $v_k(\pi(b))v_k^* \to \pi((a - \varepsilon)_+)$ as $k \to \infty$. Extend each v_k to a function in $\mathrm{M}_n(\mathrm{C}_0(Z))$, and still denote it by v_k. It is clear that the new sequence (v_k) satisfies $v_k b v_k^* \to (a - \varepsilon)_+$ as $k \to \infty$, and hence $(a - \varepsilon)_+ \precsim b$, as desired. ■

Theorem 5.5. *Let $(X, T, \mathbb{Z}^d)$ be a dynamical system with the marker property, and let $E, F \subseteq X$ be open sets such that*

$$\mu(E) < \frac{1}{4}\nu(F), \quad \mu \in \mathcal{M}_1(X, T, \mathbb{Z}^d).$$

Then,

$$[\varphi_E] \leq ((2\lfloor\sqrt{d}\rfloor + 1)^d + 1)[\varphi_F]$$

in the Cuntz semigroup of $\mathrm{C}(X) \rtimes \mathbb{Z}^d$, where φ_E and φ_F are continuous functions supported on E and F, respectively (see Example 2.9). In other words, the C^*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ has $(\frac{1}{4}, (2\lfloor\sqrt{d}\rfloor + 1)^d + 1)$-COS (see Definition 2.11).

Proof Let E and F be open sets satisfying the condition of the theorem. Let $\varepsilon > 0$ be arbitrary. In order to prove the statement of the theorem, it is enough to show that

$$(\varphi_E - \varepsilon)_+ \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{(2\lfloor\sqrt{d}\rfloor+1)^d+1}.$$

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For the given ε, pick a compact set $E' \subseteq E$ such that

$$(\varphi_E - \varepsilon)_+(x) = 0, \quad x \notin E'. \tag{5.1}$$

By the assumption of the theorem, one has

$$\mu(E') < \frac{1}{4}\mu(F), \quad \mu \in \mathcal{M}_1(X, T, \mathbb{Z}^n). \tag{5.2}$$

It follows that there is $N \in \mathbb{N}$ such that for any $M > N$ and any $x \in X$,

(5.3)
$$\frac{1}{M^d}\{m \in \{0,1,\dots,M-1\}^d : T^{-m}(x) \in E'\} < \frac{1}{4}\frac{1}{M^d}\{m \in \{0,1,\dots,M-1\}^d : T^{-m}(x) \in F\}.$$

Otherwise, there are sequences $N_k \in \mathbb{N}$, $x_k \in X$, $k = 1, 2, \dots$, such that $N_k \to \infty$ as $k \to \infty$, and for any k,

$$\frac{1}{N_k^d}\{m \in \{0,1,\dots,N_k-1\}^d : T^{-m}(x_k) \in E'\} \geq \frac{1}{4}\frac{1}{N_k^d}\{m \in \{0,1,\dots,N_k-1\}^d : T^{-m}(x_k) \in F\}.$$

That is,

$$4\delta_{N_k,x_k}(E') \geq \delta_{N_k,x_k}(F), \quad k = 1, 2, \dots, \tag{5.4}$$

where $\delta_{N_k,x_k} = \frac{1}{N_k^d}\sum_{m\in\{0,1,\dots,N_k-1\}^d}\delta_{T^{-m}(x_k)}$ and δ_y is the Dirac measure concentrated at y. Let δ_∞ be a limit point of $\{\delta_{N_k,x_k}, k = 1, 2, \dots\}$. It is clear that $\delta_\infty \in \mathcal{M}_1(X, T, \mathbb{Z}^d)$. Passing to a subsequence of k, one has

$$\begin{aligned}
\delta_\infty(F) &\leq \liminf_{k\to\infty}\delta_{N_k,x_k}(F) && (F \text{ is open})\\
&\leq 4\liminf_{k\to\infty}\delta_{N_k,x_k}(E') && (\text{by } (5.4))\\
&\leq 4\limsup_{k\to\infty}\delta_{N_k,x_k}(E')\\
&\leq 4\delta_\infty(E') && (E' \text{ is closed}),
\end{aligned}$$

which contradicts (5.2).

By (5.1) and (5.3), for any $M > N$ and any $x \in X$,

$$\begin{aligned}
&\frac{1}{M^d}\{m \in \{0,1,\dots,M-1\}^d : (\varphi_E - \varepsilon)_+(T^{-m}(x)) > 0\}\\
&\leq \frac{1}{M^d}\{m \in \{0,1,\dots,M-1\}^d : T^{-m}(x) \in E'\}\\
&< \frac{1}{4}\frac{1}{M^d}\{m \in \{0,1,\dots,M-1\}^d : T^{-m}(x) \in F\}\\
&= \frac{1}{4}\frac{1}{M^d}\{m \in \{0,1,\dots,M-1\}^d : \varphi_F(T^{-m}(x)) > 0\}.
\end{aligned} \tag{5.5}$$

Note that, by the assumption,

$$\mu(F) > 0, \quad \mu \in \mathcal{M}_1(X, T, \mathbb{Z}^d).$$

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The compactness argument same as above shows that N can be chosen sufficiently large so that there is $\delta > 0$ such that for any $M > N$,

$$\frac{1}{4M^d}\left|\{m \in \{0,1,\dots,M-1\}^d : \varphi_F(T^{-m}(x)) > 0\}\right| > \delta, \quad x \in X. \tag{5.6}$$

Let

$$\mathcal{T}_0 = \{T^{-m}(\Omega_0), \quad m \in \{0,1,\dots,N_0-1\}^d\}$$

and

$$\mathcal{T}_1 = \{T^{-m}(\Omega_1), \quad m \in \{0,1,\dots,N_1-1\}^d\}$$

denote the two towers obtained from Theorem 4.3 with respect to $\max\{N, \sqrt[d]{\frac{1}{\delta}}\}$ and δ. Denote by $U_1, U_2, \dots, U_K$ and $n_1, n_2, \dots, n_K \in \mathbb{Z}^d$, the open sets and group elements, respectively, obtained from Theorem 4.3.

Pick $\chi_0 \in \mathrm{C}(X)^+$ such that

$$\begin{cases} \chi_0(x) = 1, & x \notin \bigcup_{k=1}^K U_k, \\ \chi_0(x) > 0, & x \in \bigsqcup_{m\in\{0,1,\dots,N_0-1\}^d} T^{-m}(\Omega_0), \\ \chi_0(x) = 0, & x \notin \bigsqcup_{m\in\{0,1,\dots,N_0-1\}^d} T^{-m}(\Omega_0). \end{cases} \tag{5.7}$$

Note that then $(1-\chi_0)$ is supported in the set $U_1 \cup U_2 \cup \cdots \cup U_K$. Consider

$$(\varphi_E - \varepsilon)_+ = (\varphi_E - \varepsilon)_+(1-\chi_0) + (\varphi_E - \varepsilon)_+\chi_0.$$

Then, for any $x \in \Omega_0$, it follows from (5.5) and (5.7) that

$$\begin{aligned} &\left|\{m \in \{0,1,\dots,N_0-1\}^d : ((\varphi_E - \varepsilon)_+\chi_0)(T^{-m}(x)) > 0\}\right| \\ &= \left|\{m \in \{0,1,\dots,N_0-1\}^d : (\varphi_E - \varepsilon)_+(T^{-m}(x)) > 0\}\right| \\ &< \frac{1}{4}\left|\{m \in \{0,1,\dots,N_0-1\}^d : \varphi_F(T^{-m}(x)) > 0\}\right| \\ &= \frac{1}{4}\left|\{m \in \{0,1,\dots,N_0-1\}^d : (\varphi_F\chi_0)(T^{-m}(x)) > 0\}\right|. \end{aligned}$$

Therefore, with respect to the isomorphism $\mathrm{C}^*(\mathcal{T}_0) \cong \mathrm{M}_{N_0^d}(\mathrm{C}_0(\Omega_0))$ (see Definition 5.3), one has

$$\mathrm{rank}(((\varphi_E - \varepsilon)_+\chi_0)(x)) \leq \frac{1}{4}\mathrm{rank}((\varphi_F\chi_0)(x)), \quad x \in \Omega_0.$$

Moreover, it follows from (5.6) and the fact that $N_0 > \sqrt[d]{\frac{1}{\delta}}$ that, for any $x \in \Omega_0$,

$$\frac{1}{4N_0^d}\mathrm{rank}((\varphi_F\chi_0)(x)) = \frac{1}{4N_0^d}\left|\{m \in \{0,1,\dots,N_0-1\}^d : \varphi_F(T^{-m}(x)) > 0\}\right| > \delta > \frac{1}{N_0^d}.$$

Then, by Lemma 5.4,

$$(\varphi_E - \varepsilon)_+\chi_0 \precsim \varphi_F\chi_0 \precsim \varphi_F. \tag{5.8}$$

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Consider the product $(\varphi_E - \varepsilon)_+(1-\chi_0)$. Since $(1-\chi_0)$ is supported in $U_1 \cup U_2 \cup \cdots \cup U_K$, one has

$$(\varphi_E - \varepsilon)_+(1-\chi_0) \precsim (1-\chi_0) \precsim \varphi_{U_1\cup\cdots\cup U_K} \sim \varphi_{U_1} + \cdots + \varphi_{U_K} \precsim \varphi_{U_1} \oplus \cdots \oplus \varphi_{U_K}.$$

On the other hand, by Lemma 5.2,

$$\varphi_{T^{n_1}(U_1)} \oplus \cdots \oplus \varphi_{T^{n_K}(U_K)} \precsim \bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d} (\varphi_{T^{n_1}(U_1)} + \cdots + \varphi_{T^{n_K}(U_K)}).$$

Note that $\varphi_{U_i} \sim \varphi_{T^{n_i}(U_i)}$, $i = 1, 2, \ldots, K$, and so one has

$$(\varphi_E - \varepsilon)_+(1-\chi_0) \precsim \bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d} \varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)}. \tag{5.9}$$

By Theorem 4.3,

$$\frac{1}{N_1^d}\left|\left\{m \in \{0,1,\ldots,N_1-1\}^d : T^{-m}(x) \in \bigcup_{k=1}^{K} T^{n_k}(U_k)\right\}\right| < \delta, \quad x \in \Omega_1. \tag{5.10}$$

Let $\chi_1 : X \to [0,1]$ be a continuous function such that

$$\begin{cases} \chi_1(x) > 0, & x \in \bigsqcup_{m\in\{0,1,\ldots,N_1-1\}^d} T^{-m}(\Omega_1), \\ \chi_1(x) = 0, & x \notin \bigsqcup_{m\in\{0,1,\ldots,N_1-1\}^d} T^{-m}(\Omega_1). \end{cases}$$

Then

$$\frac{1}{4N_1^d}\operatorname{rank}((\varphi_F\chi_1)(x)) = \frac{1}{4N_1^d}\left|\{m \in \{0,1,\ldots,N_1-1\}^d : \varphi_F(T^{-m}(x)) > 0\}\right| > \delta > \frac{1}{N_1^d}$$

for all $x \in \Omega_1$, and hence, by (5.10), one has

$$\begin{aligned}\operatorname{rank}(\varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)}(x)) &= \left|\left\{m \in \{0,1,\ldots,N_1-1\}^d : T^{-m}(x) \in \bigcup_{k=1}^{K} T^{n_k}(U_k)\right\}\right| \\ &< N_1^d\delta < \frac{1}{4}\operatorname{rank}((\varphi_F\chi_1)(x))\end{aligned}$$

for any $x \in \Omega_1$.

By Lemma 5.4,

$$\varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)} \precsim \varphi_F\chi_1 \precsim \varphi_F,$$

and together with (5.9) and (5.8), this implies

$$\begin{aligned}(\varphi_E - \varepsilon)_+ &\precsim (\varphi_E - \varepsilon)_+(1-\chi_0) \oplus (\varphi_E - \varepsilon)_+\chi_0 \\ &\precsim \Big(\bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d} (\varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)})\Big) \oplus \varphi_F \\ &\precsim \Big(\bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d} \varphi_F\Big) \oplus \varphi_F,\end{aligned}$$

as desired. ∎

Theorem 5.6. *Let $(X, T, \mathbb{Z}^d)$ be a minimal-free dynamical system. Then*

$$\operatorname{rc}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) \le \frac{1}{2}\operatorname{mdim}(X, T, \mathbb{Z}^d).$$

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Proof Note that any minimal-free dynamical system has the marker property. By Theorem 5.5, the C*-algebra $C(X) \rtimes \mathbb{Z}^d$ has the property COS. By Theorem 4.2, the dynamical system $(X, T, \mathbb{Z}^d)$ has the property URP. The statement follows directly from Theorem 4.8 of [29]. ∎

Remark 5.7. The proof of Theorem 4.8 of [29] used the simplicity of the C*-algebra $C(X) \rtimes \mathbb{Z}^d$.

The following corollary generalizes Corollary 4.9 of [7] (where $d = 1$) and generalizes the classifiability result of [33] (where $\dim(X) < \infty$).

Theorem 5.8 *Let $(X, T, \mathbb{Z}^d)$ be a minimal-free dynamical system with mean dimension zero. Then $C(X) \rtimes \mathbb{Z}^d$ absorbs the Jiang–Su algebra tensorially, and hence is classified by its Elliott invariant (i.e., belongs to the classifiable class of Theorem 2.7 of [4]). In particular, if $\dim(X) < \infty$, or $(X, T, \mathbb{Z}^d)$ has at most countably many ergodic measures, or $(X, T, \mathbb{Z}^d)$ has finite topological entropy, then $C(X) \rtimes \mathbb{Z}^d$ is classified by its Elliott invariant.*

Proof By Theorems 4.2 and 5.5, the dynamical system $(X, \mathbb{Z}^d)$ has the URP and COS. The statement then follows from Theorem 4.8 of [28]. ∎

The following is a generalization of Corollary 5.7 of [7].

Corollary 5.9. *Let $(X_1, T_1, \mathbb{Z}^{d_1})$ and $(X_2, T_2, \mathbb{Z}^{d_2})$ be arbitrary minimal-free dynamical systems, where $d_1, d_2 \in \mathbb{N}$. Then the tensor product C*-algebra $(C(X_1) \rtimes \mathbb{Z}^{d_1}) \otimes (C(X_2) \rtimes \mathbb{Z}^{d_2})$ absorbs the Jiang–Su algebra tensorially, and hence is classified by its Elliott invariant.*

Proof Note that

$$(C(X_1) \rtimes \mathbb{Z}^{d_1}) \otimes (C(X_2) \rtimes \mathbb{Z}^{d_2}) \cong C(X_1 \times X_2) \rtimes (\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}),$$

where $\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$ acting on $X_1 \times X_2$ by

$$(T_1 \times T_2)^{(n_1, n_2)}((x_1, x_2)) = (T_1^{n_1}(x_1), T_2^{n_2}(x_2)), \quad n_1 \in \mathbb{Z}^{d_1},\ n_2 \in \mathbb{Z}^{d_2}.$$

By the argument of Remark 5.8 of [7], one has

$$\mathrm{mdim}(X_1 \times X_2, T_1 \times T_2, \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}) = 0,$$

and the statement then follows from Theorem 5.8. ∎

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