ON A DISCRIMINANT INEQUALITY

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The following result has been conjectured by Dr. Birch. *Let* $z_1, z_2, \ldots, z_n$ *be any n complex numbers such that*

$$(1) \qquad |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 = n.$$

Then

$$(2) \qquad \Delta = \prod_{r>s\geqslant 1}^{n} |z_r - z_s|^2$$

attains its greatest value when the z are at the vertices of a regular n-sided polygon inscribed in the circle $|z| = 1$.

It seems to be difficult to prove this but Dr. Birch informs me that some work by Mullholland[1] shows that the result is false for large n. I can, however, prove that the result is true for $n = 3$, and then $\Delta \leqslant 27$. The suggested general result would be $\Delta \leqslant n^n$.

I show first that the maximum value of Δ arises from values of z satisfying either the equation

$$(3) \qquad \sum_{\substack{s=1\\ s\neq r}}^{n} \frac{1}{z_r - z_s} = \frac{1}{2}(n-1)\bar{z}_r, \qquad (r = 1, 2, \ldots, n),$$

where $\bar{z}_r$ denotes the conjugate of z_r; or the equations typified by $z_n = 0$ and

$$(4) \qquad \frac{1}{z_r} + \sum_{\substack{s=1\\ s\neq r}}^{n-1} \frac{1}{z_r - z_s} = \frac{1}{2}(n-1)\bar{z}_r \qquad (r = 1, 2, \ldots, n-1).$$

The conjectured result is then proved for $n = 3$. It is also proved that the result is true if we impose the condition that the z's lie on the circle $|z| = 1$. In the original version of this paper, this result was used to prove the result for $n = 3$. This led to a very interesting maximum problem in two variables, namely,

Problem. To find the maximum value when $x^2 + y^2 \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$, *of*

$$(5) \qquad f(x, y) = (x^2 + y^2 - kxy)(1 - x^2)(1 - y^2).$$

Though the deduction of the conjectured result for $n = 3$ is not short, it seems worth-while reproducing the original proof since the ideas involved may be of further use. I think the method may give the greatest value of Δ for $n = 4$, but this I leave to others.

The general problem was brought to my notice by Dr. J. H. H. Chalk, who after reading the original version of my paper, informed me that the conjecture was false for $n \geqslant 6$. His counter example is given by

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[1] "Inequalities between the geometric mean difference and the polar moments of a plane distribution," *Journal of the London Mathematical Society*, 33 (1958) 260–269.

$$z_n = 0 \quad \text{and} \quad z_r = \left(\frac{n}{n-1}\right)^{\frac{1}{2}} \exp\left(\frac{2\pi i r}{n-1}\right) \quad (r = 1, 2, \ldots, n-1).$$

The equation (3) (but not (4)) was communicated to me by Dr. Birch after I had written the original version of this paper. He did not give the factor $\frac{1}{2}(n-1)$ in (3).

For the general case, write

$$z_1 = r_1 e^{i\theta_1}, \ldots, z_n = r_n e^{i\theta_n}, \qquad r_1 \geqslant 0, \ldots, r_n \geqslant 0.$$

Then

(6) $$\Delta = \Pi(r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)),$$

where

$$r_1^2 + r_2^2 + \ldots + r_n^2 = n.$$

Suppose first that no r is zero. Then we can apply Lagrange's method of undetermined multipliers. Hence we have two sets of equations, one typified by the two

(7) $$\sum_{s=2}^{k} \frac{r_1 - r_s \cos(\theta_1 - \theta_s)}{r_1^2 + r_s^2 - 2r_1 r_s \cos(\theta_1 - \theta_s)} - \lambda r_1 = 0,$$

(8) $$\sum_{\substack{s=1 \\ s\neq 2}}^{n} \frac{r_2 - r_s \cos(\theta_2 - \theta_s)}{r_2^2 + r_s^2 - 2r_2 r_s \cos(\theta_2 - \theta_s)} - \lambda r_2 = 0,$$

where λ is an undetermined multiplier, and the other typified by

(9) $$\sum_{s=2}^{n} \frac{r_1 r_s \sin(\theta_1 - \theta_s)}{r_1^2 + r_s^2 - 2r_1 r_s \cos(\theta_1 - \theta_s)} = 0.$$

Multiply the equations (7), (8), ..., by $r_1, r_2 \ldots$, and add. Then $\lambda = \frac{1}{2}(n-1)$. Multiply (9) by $-i/r_1$ and add to (7). Then

$$\sum_{s=2}^{n} \frac{1}{r_1 - r_s e^{-i(\theta_1 - \theta_s)}} = \tfrac{1}{2}(n-1) r_1.$$

Hence

(10) $$\sum_{s=2}^{n} \frac{1}{z_1 - z_s} = \tfrac{1}{2}(n-1)\bar{z}_1.$$

Adding the equations typified by (10), we find

(11) $$\sum_{s=1}^{n} z_s = 0.$$

Suppose next that some r are zero. Clearly at most one can be zero, say $r_n = 0$. Then we have

$$\Delta = r_1^2 r_2^2 \ldots r_{n-1}^2 \Pi(r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)),$$

where the product is extended over $r_1, r_2, \ldots, r_{n-1}$. Lagrange's method now gives

(12) $$\frac{1}{r_1} + \sum_{s=2}^{n-1} \frac{r_1 - r_s \cos(\theta_1 - \theta_s)}{r_1^2 + r_s^2 - 2r_1 r_s \cos(\theta_1 - \theta_s)} - \mu r_1 = 0,$$

(13) $$\sum_{s=2}^{n-1} \frac{r_s \sin(\theta_1 - \theta_s)}{r_1^2 + r_s^2 - 2r_1 r_s \cos(\theta_1 - \theta_s)} = 0.$$

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On multiplying the equations typified by (12) by $r_1, r_2, \ldots$, and adding, we have

$$n - 1 + \tfrac{1}{2}(n-1)(n-2) - \mu n = 0,$$

or

$$\mu = \tfrac{1}{2}(n-1).$$

Hence proceeding as before, we have the equation (4).

We now prove the conjecture for $n = 3$. The equation (3) gives

$$\frac{1}{z_1 - z_2} + \frac{1}{z_1 - z_3} = \bar{z}. \tag{14}$$

From (11), $z_1 + z_2 + z_3 = 0$, and so (14) gives

$$3z_1 = \bar{z}_1(z_1 - z_2)(z_1 - z_3).$$

Hence

$$3\bar{z}_1 = \bar{z}_1(\bar{z}_1 - \bar{z}_2)(\bar{z}_1 - \bar{z}_3),$$

and so on multiplying these together,

$$|z_1 - z_2|\,|z_1 - z_3| = 3.$$

Clearly

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = 3^{\frac{1}{2}},$$

and so z_1, z_2, z_3 are at the vertices of an equilateral triangle whose side is of length $3^{\frac{1}{2}}$. Also the incentre of the triangle is at the origin. Clearly $\Delta = 27$.

Suppose next that $z_3 = 0$. The equation (4) gives

$$\frac{1}{z_1} + \frac{1}{z_1 - z_2} = \bar{z}_1, \frac{1}{z_2} + \frac{1}{z_2 - z_1} = \bar{z}_2' .$$

Add these and take also the conjugate equation. Then on multiplying these together, we find either

$$z_1 + z_2 = 0, \text{ or } |z_1 z_2| = 1.$$

Since

$$z_1^2 + z_2^2 = \bar{z}_1^2 + \bar{z}_2^2 = 3,$$

we find either

$$|z_1| = |z_2| = \left(\frac{3}{2}\right)^{\frac{1}{2}} \quad \text{or} \quad |z_1| = \frac{5^{\frac{1}{2}} \pm 1}{2}\, |z_2| = \frac{5^{\frac{1}{2}} \mp 1}{2} .$$

Then

$$\Delta = z_1^2 z_2^2 (z_1 - z_2)^2 = \frac{27}{2} \quad \text{or} \quad 1.$$

Hence these values of the z do not give the greatest value of Δ.

We now prove the general conjecture when we impose the condition that the z lie on the circle $|z| = 1$. The equation (7) is true independently of Lagrange's method when $\lambda = \frac{1}{2}(n-1)$ since now the r's are equal to 1. The equation (9) with the r's equal to 1 still arises by Lagrange's method applied to the θ. Hence (10) is still true and now $\bar{z}_1 = 1/z_1$; and so the z satisfy equations typified by

$$(15) \qquad \sum_{s=2}^{n} \frac{1}{z_1 - z_s} = \frac{1}{2}(n-1)\frac{1}{z_1}.$$

Let $z = z_1, z_2, \ldots, z_n$ be the roots of the polynomial equation $f(z) = 0$. Then (15) gives

$$\frac{f''(z)}{f'(z)} = \frac{n-1}{2},$$

and so

$$(16) \qquad zf''(z) - (n-1)f'(z) = 0$$

for $z = z_1, z_2, \ldots, z_n$. Since (16) is of degree $n - 1$ in z, (16) holds identically in z. Hence

$$\log(f'(z)) = (n-1)\log z + \log c_1$$
$$f'(z) = c_1 z^{n-1}$$
$$f(z) = \frac{c_1}{n} z^n + c_2,$$

where c_1, c_2 are arbitrary constants. Hence the result.

Since $|z| = 1$, the equation $f(z) = 0$ must be equivalent to $z^n - e^{2\pi i\alpha/n} = 0$ where α is real. This shows that $z_1, z_2, \ldots, z_n$ are at the vertices of a regular n-sided polygon inscribed in the circle $|z| = 1$. To find Δ, there is no loss of generality in taking $\alpha = 0$. Then the vertices are at $z_r = e^{2\pi ir/n}$, $(r = 0, 1, \ldots, n-1)$. Then for these z,

$$(17) \qquad \begin{aligned}\Delta &= \Pi_{r\neq s}(2 - 2\cos(\theta_r - \theta_s)) \\ &= \Pi_{r\neq s} 4^{\frac{1}{2}n(n-1)} \sin^2\left(\frac{\theta_r - \theta_s}{2}\right) = n^n\end{aligned}$$

follows from

$$\prod_{r=1}^{n-1} \sin\left(\frac{r\pi}{n}\right) = \frac{n}{2^{n-1}}.$$

This is deduced from

$$\cos(n\theta) - 1 = 2^{n-1} \prod_{r=0}^{n-1} \left(\cos\theta - \cos\frac{2r\pi}{n}\right)$$

on dividing by $\cos\theta - 1$ and putting $\theta = 0$. Then obviously

$$\prod \sin^2\left(\frac{\theta_r - \theta_s}{2}\right) = (n/2^{n-1})^n.$$

I now give the original proof of the general conjecture when $n = 3$ found by using the result above. Write

$$\Delta = \Pi(r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)),$$

where

$$(18) \qquad r_1^2 + r_2^2 + r_3^2 = 3.$$

The greatest value of Δ cannot arise when $r_3 = 0$, since then $r_1^2 + r_2^2 = 3$ and

$$\Delta \leqslant r_1^2 r_2^2 (r_1 + r_2)^2 \leqslant 27/2.$$

Suppose first that all the cosines are $\leqslant 0$. If $\cos\theta \leqslant 0$, then

$$x^2 + y^2 - 2xy\cos\theta \leqslant 2\sin^2(\tfrac{1}{2}\theta)(x^2 + y^2) \tag{19}$$

since

$$(x - y)^2\cos\theta \leqslant \theta.$$

Hence

$$\Delta \leqslant 8\pi(r_1^2 + r_2^2)\Pi\sin^2(\theta_1 - \theta_2) \leqslant 8\cdot 8\cdot 27/64 \leqslant 27$$

from (18) and (17), equality arising only when $r_1 = r_2 = r_3 = 1$; and this is the case of the equilateral triangle.

Suppose secondly that all the cosines are $\geqslant 0$. Then

$$\Delta \leqslant \pi(r_1^2 + r_2^2) \leqslant 8.$$

Suppose thirdly that only two of the cosines are $\geqslant 0$, say $\cos(\theta_1 - \theta_2) \leqslant 0$. Then

$$\begin{aligned}\Delta &\leqslant (r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)(r_1^2 + r_3^2)(r_2^2 + r_3^2), \\ &\leqslant 2(r_1^2 + r_2^2)(r_1^2 + r_3^2)(r_2^2 + r_3^2) \leqslant 16.\end{aligned}$$

Suppose finally that only one of the cosines $\geqslant 0$, say $\cos(\theta_1 - \theta_2) \geqslant 0$. Then from (19)

$$\begin{aligned}\Delta \leqslant 4\left((r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)\right) \\ \times (r_1^2 + r_3^2)(r_3^2 + r_2^2)\sin^2\left(\frac{\theta_2 - \theta_3}{2}\right)\sin^2\left(\frac{\theta_3 - \theta_1}{2}\right) \\ \leqslant 4(r_1 - r_2)^2(r_1^2 + r_3^2)(r_2^2 + r_3^2) + 16r_1r_2(r_1^2 + r_2^2)(r_2^2 + r_3^2)27/64\end{aligned}$$

on noting (17). Hence since $r_1^2 + r_3^2 = 3 - r_2^2$, etc.,

$$4\Delta \leqslant (3 - r_1^2)(3 - r_2^2)(16r_1^2 - 5r_1r_2 + 16r_2^2).$$

We require the maximum value of the right-hand side where $r_1^2 + r_2^2 \leqslant 3$. On putting $r_1^2 = 3x^2$, $r_2^2 = 3y^2$, we are led to the

Problem. To find the maximum value M of

$$f = f(x, y) = (x^2 + y^2 - kxy)(1 - x^2)(1 - y^2) \tag{20}$$

when

$$x^2 + y^2 \leqslant 1, x \geqslant 0, y \geqslant 0.$$

Clearly $M \geqslant \frac{1}{4}$ on taking $x^2 = \frac{1}{2}$, $y = 0$. We prove in particular that when $k = \frac{5}{16}$, then $M = \frac{1}{4}$ arising from $x^2 = y^2 = \frac{1}{3}$ or from $xy = 0$, $x^2 + y^2 = \frac{1}{2}$. These give $4\Delta \leqslant 108$. We note that $xy = 0$ does not lead to a maximum value of Δ.

Write

$$x = \sqrt{r}\cos\theta,\ y = \sqrt{r}\sin\theta,\ s = \sin\theta\cos\theta,$$

and so

$$0 \leqslant r \leqslant 1,\ 0 \leqslant s \leqslant \tfrac{1}{2}.$$

Then

(21) $$f(x, y) = g(r, s) = r(1 - ks)(1 - r + r^2 s^2).$$

Several cases must be considered, and so we denote by $M_1, M_2, \ldots$, possible values among which M must be found. We first investigate possible maximum values of $g(r, s)$ arising from the boundary values of r, s.

We begin with the boundary values of s.

First, $s = 0$. Then $g = r(1-r)$ and so $M_1 = \frac{1}{4}$ when $r = \frac{1}{2}$. Then $x = 1/\sqrt{2}$, $y = 0$, or $x = 0$, $y = 1/\sqrt{2}$.

Secondly, $s = \frac{1}{2}$. Then

$$g = \left(1 - \frac{1}{2}k\right)r\left(1 - \frac{r}{2}\right)^2.$$

We need only consider $k \leqslant 2$. Then the maximum M_2 arises when $r = 2/3$ giving $M_2 = 4(2 - k)/27$. We can reject this unless $4(2 - k)/27 \geqslant \frac{1}{4}$ or $k \leqslant 5/16$. Hence if $k \leqslant 5/16$, $M_2 = 4(2 - k)/27$ arising from $x = y = 1/\sqrt{3}$. When $k = \frac{5}{16}$, $M_2 = M_1 = \frac{1}{4}$.

We need only take the boundary value $r = 1$. Then $g = (1 - ks)s^2$ and the extremal value arises from $s = 2/3k$. This satisfies $0 \leqslant s \leqslant \frac{1}{2}$ only if $k \geqslant 4/3$. Then $g = 4/27k^2 < \frac{1}{4}$. Hence if $k < 4/3$, a possible maximum may arise from $s = \frac{1}{2}$ and then $M_3 = (2 - k)/8 \geqslant \frac{1}{4}$ only if $k \leqslant 0$. Clearly $M_3 < M_2$.

To summarize, the boundary values give possible maxima $M_1 = \frac{1}{4}$ for $k \geqslant 5/16$, and $M_2 = 4(2 - k)/27$ when $k \leqslant 5/16$.

We now consider non-boundary values of r, s. We put

$$\frac{\partial g}{\partial r} = 0, \qquad \frac{\partial g}{\partial s} = 0.$$

Hence

$$1 - 2r + 3r^2 s^2 = 0, \; -k + kr - 3kr^2s^2 + 2r^2s = 0.$$

Multiply the first equation by k and add to the second. Then $-kr + 2r^2s = 0$, and so $2rs = k$ since we need not consider $r = 0$. The solutions arising are certainly not admissible unless $0 < k \leqslant 1$ since we have excluded $s = 0$. Clearly $1 - 2r + 3k^2/4 = 0$, and so

$$r = \frac{3k^2 + 4}{8}, \qquad s = \frac{4k}{3k^2 + 4}.$$

These must satisfy $r \leqslant 1$ which is obvious, and $s \leqslant \frac{1}{2}$ which requires $3k^2 + 4 - 8k \geqslant 0$ or $k \geqslant 2/3$. But then

$$g = \left(\frac{3k^2 + 4}{8}\right)\left(\frac{4 - k^2}{3k^2 + 4}\right)\left(1 - \frac{3k^2 + 4}{8} + \frac{k^2}{8}\right)$$
$$= \frac{(4 - k^2)^2}{64} \leqslant \frac{1}{4}.$$

Hence the maximum value of g arises from the boundary values. Then $M = 4(2 - k)/27$ or $\frac{1}{4}$ according as $k \leqslant 5/16$ or $k \geqslant 5/16$.

This disposes of the problem.

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