*Glasgow Math. J.* **50** (2008) 467–481. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004345. Printed in the United Kingdom

# CROSS-CONSTRAINED VARIATIONAL PROBLEM AND THE NON-LINEAR KLEIN–GORDON EQUATIONS*

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(Received 21 June 2007; accepted 19 February 2008)

**Abstract.** In this paper, we put forward a cross-constrained variational method to study the non-linear Klein–Gordon equations with an inverse square potential in three space dimensions. By constructing a type of cross-constrained variational problem and establishing so-called cross-invariant manifolds of the evolution flow, we establish some new types of invariant sets for the equation and derive a sharp threshold of blowup and global existence for its solution. Finally, we give an answer to the question how small the initial data are for the global solution to exist.

2000 *Mathematics Subject Classification.* 35A15, 35L15.

**1. Introduction.** In this paper, we put forward a cross-constrained variational method to study the non-linear Klein–Gordon equations with an inverse square potential in three space dimensions

$$\phi_{tt} - \Delta\phi + \phi + a|x|^{-2}\phi = |\phi|^{p-1}\phi, \quad t \geq 0, \quad x \in \mathbf{R}^3, \tag{1.1}$$

which is a representative of the class of equations of interest [**17**]. Here $\phi = \phi(t, x)$ is an unknown complex-valued function of $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^3$, $\Delta$ is the Laplace operator on $\mathbf{R}^3$, $a > 0$ and $3 < p < 5$.

When $a = 0$, equation (1.1) is a classical non-linear model in field theory and there are many works on the study of it [**1**, **2**, **9**, **10**, **13**, **16**, **21**, **22**, **24**]. From the viewpoint of Physics, the following problems are very important: (i) Under what conditions, will the solutions of equation (1.1) become unstable to collapse (blowup)? (ii) Under what conditions, will the solutions of equation (1.1) exist for all time (global existence)? Especially the sharp criteria for blowup and global existence is pursued strongly.

Meanwhile, as a class of non-linear Klein–Gordon equations with an inverse square potential, the study of equation (1.1) has its special significance [**17**]. In the theory of non-linear waves, the sharp criteria for blowup and global existence is also very interesting and extensively open [**3**, **11**, **19**], especially to the equation with a real-valued potential. Moreover, it is also important to investigate the sufficient and

*This work is supported by National Natural Science Foundation of P. R. China (10771151, 10726034) and Sichuan Youth Science and Technology Foundation (07ZQ026-009) and The Institute of Mathematical Sciences at The Chinese University of Hong Kong.

necessary condition of blowup for solution to the Cauchy problem of equation (1.1) from a mathematical viewpoint.

In this paper, by introducing a cross-constrained variational method, we study the sharp threshold of global existence and blowup for (1.1) with Cauchy data,

$$\phi(0, x) = \phi_0(x), \quad \phi_t(0, x) = \phi_1(x), \quad x \in \mathbf{R}^3. \tag{1.2}$$

The approach used in this paper is motivated by the context of study of a class of non-linear Schrödinger equations and non-linear wave equations [**2**, **8**, **9**, **14**, **15**, **18**, **19**, **23**]. In [**24**], the sharp criterion for blowup and global existence of the solution to the non-linear Klein–Gordon equation without any potential was got. For the non-linear Klein–Gordon equation with a non-negative potential, Gan and Zhang [**5**] obtained a sharp threshold of blowup and global existence for its solution by using the method proposed in [**24**]. For the study of the non-linear Schrödinger equation with a harmonic potential, Zhang [**25**] derived a sharp threshold of blowup and global existence for its solution by introducing a type of cross-constrained variational method. Own to the natural relation between the Klein–Gordon equation and the Schrödinger equation, it is interesting to apply the method given in [**25**] to obtain a sharp threshold of blowup and global existence for the non-linear Klein–Gordon equation with a potential. Unfortunately, as far as our knowledge is concerned, the method in [**25**] cannot apply to the Klein–Gordon equation with a general potential except the inverse square potential $|x|^{-2}$. In this paper, we mainly clarify the idea that how to utilize the method to study (1.1), and thus obtain a sharp threshold of blowup and global existence for the solution to (1.1) in three space dimensions.

It should be pointed out that the optimal range on $p$ is $\frac{7}{3} \le p < 5$ for the sharp threshold of blowup and global existence of the solution to the non-linear Schrödinger equation with a potential, and in this paper we can only discuss the range $p \in (3, 5)$. For $p \in [\frac{7}{3}, 3]$, whether the similar result to this paper holds or not remains open. But to our knowledge, this is the first result in this direction for the Klein–Gordon equation with a potential by using the cross-constrained variational method, which seems new even for the non-linear Klein–Gordon equation without a potential.

For simplicity, throughout the paper we denote $\int_{\mathbf{R}^3} \cdot\, dx$ by $\int \cdot\, dx$ and we denote by $c$ a universal positive constant which depends only on $a$ and $p$.

**2. Preliminaries.** In this paper, as in refs. [**6**, **7**], we do not study the local well-posedness of the Cauchy problem (1.1), (1.2).

For equation (1.1), we define the energy space in the course of nature as

$$H := \left\{ \varphi \in H^1(\mathbf{R}^3); \quad \int |x|^{-2} |\varphi|^2\, dx < \infty \right\}. \tag{2.1}$$

By the definition of $H$, $H$ becomes a Hilbert space, continuously embedded in $H^1(\mathbf{R}^3)$, when endowed with the inner product

$$(\varphi, \phi)_H := \int [\nabla\varphi \nabla\bar{\phi} + \varphi\bar{\phi} + a|x|^{-2}\varphi\bar{\phi}]\, dx, \tag{2.2}$$

whose associated norm we denote by $\|\cdot\|_H$. Moreover, we define the energy functional in $H$ as follows:

$$E(\phi) := \frac{1}{2}\int |\phi_t|^2\,dx + \frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx \\ + \frac{1}{2}a\int |x|^{-2}|\phi|^2\,dx - \frac{1}{p+1}\int |\phi|^{p+1}\,dx. \tag{2.3}$$

From the view-point of Hamiltonian systems, $E$ is the generating Hamiltonian of equation (1.1). For $\forall t \in [0, T)$, $\phi(t, x)$ satisfies conservation of energy,

$$E(\phi) = E(\phi_0). \tag{2.4}$$

Furthermore, from the conservation law (2.4), the Gagliardo–Nirenberg inequality and the Sobolev inequality, for global existence of the Cauchy problem (1.1), (1.2), we have

Proposition 2.1. *Let* $(\phi_0, \phi_1) \in H \times L^2(\mathbf{R}^3)$. *Then for* $3 < p < 5$, *when* $\|\phi_0\|_H + \|\phi_1\|_{L^2(\mathbf{R}^3)}$ *is sufficiently small, the Cauchy problem* (1.1), (1.2) *has a unique bounded global solution.*

Moreover, by a direct calculation we have

Proposition 2.2. *Let* $(\phi_0, \phi_1) \in H \times L^2(\mathbf{R}^3)$ *and* $\phi$ *be a solution of the Cauchy problem* (1.1), (1.2) *on* $[0, T)$. *We put*

$$F(t) = \int |\phi|^2\,dx. \tag{2.5}$$

*Then one has*

$$F'(t) = \int (\phi_t\bar{\phi} + \phi\bar{\phi}_t)\,dx = 2Re\int \phi_t\bar{\phi}\,dx, \tag{2.6}$$

*and*

$$\begin{aligned} F''(t) &= 2\int |\phi_t|^2\,dx - 2\int |\nabla\phi|^2\,dx - 2\int |\phi|^2\,dx \\ &\quad - 2a\int |x|^{-2}|\phi|^2\,dx + 2\int |\phi|^{p+1}\,dx \\ &= (p+3)\int |\phi_t|^2\,dx + (p-1)\int (|\nabla\phi|^2 + |\phi|^2 + a|x|^{-2}|\phi|^2)\,dx \\ &\quad - 2(p+1)E. \end{aligned} \tag{2.7}$$

Thus one has

Proposition 2.3. *Let* $(\phi_0, \phi_1) \in H \times L^2(\mathbf{R}^3)$. *For* $1 \le p < 5$, *when* $E(0) < 0$, *the solution* $\phi(t, x)$ *of the Cauchy problem* (1.1), (1.2) *blows up in a finite time.*

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**3. The cross-constrained variational problem and invariant manifolds.** For $u \in H$ and $3 < p < 5$, we define the following functionals and manifolds:

$$J(u) := \frac{1}{2}\int |\nabla u|^2\,dx + \frac{a}{2}\int |x|^{-2}|u|^2\,dx + \frac{1}{2}\int |u|^2\,dx - \frac{1}{p+1}\int |u|^{p+1}\,dx, \tag{3.1}$$

$$K(u) := \frac{1}{2}\int |\nabla u|^2\,dx + \frac{a}{2}\int |x|^{-2}|u|^2\,dx + \frac{3}{2}\int |u|^2\,dx - \frac{3}{p+1}\int |u|^{p+1}\,dx, \tag{3.2}$$

$$I(u) := \frac{1}{2}\int |\nabla u|^2\,dx + \frac{a}{2}\int |x|^{-2}|u|^2\,dx - \frac{1}{2}\int |u|^2\,dx - \frac{p-2}{p+1}\int |u|^{p+1}\,dx, \tag{3.3}$$

$$H(u) := \int |\nabla u|^2\,dx + a\int |x|^{-2}|u|^2\,dx + \int |u|^2\,dx - \int |u|^{p+1}\,dx, \tag{3.4}$$

$$B := \{u \in H\backslash\{0\},\ K(u) = 0\}, \tag{3.5}$$

$$M := \{u \in H\backslash\{0\},\ K(u) < 0,\ I(u) = 0\}. \tag{3.6}$$

Thus, we define two constrained variational problems:

$$d_B := \inf_B J(u), \tag{3.7}$$

$$d_M := \inf_M J(u). \tag{3.8}$$

From (3.7), we have the result.

LEMMA 3.1. *$d_B > 0$ provided $3 < p < 5$.*

*Proof.* From (3.1), (3.2), (3.5) and (3.7), one has

$$J(u) = \frac{1}{3}\int |\nabla u|^2\,dx + \frac{1}{3}a\int |x|^{-2}|u|^2\,dx = \frac{2}{p+1}\int |u|^{p+1}\,dx - \int |u|^2\,dx, \tag{3.9}$$

that is, $J(u) > 0$ on $B$. Thus by (3.7), we get $d_B \geq 0$. Here, we prove $d_B \neq 0$ by contradiction. If $d_B = 0$, then from (3.7), there would be a sequence $u_n \subset B$ such that $K(u_n) = 0$ and $J(u_n) \to 0$ as $n \to \infty$. So (3.9) implies that as $n \to \infty$,

$$\frac{1}{3}\int |\nabla u_n|^2\,dx + \frac{1}{3}a\int |x|^{-2}|u_n|^2\,dx \to 0, \tag{3.10}$$

and

$$\frac{2}{p+1}\int |u_n|^{p+1}\,dx - \int |u_n|^2\,dx \to 0. \tag{3.11}$$

That is, as $n \to \infty$,

$$\int |x|^{-2}|u_n|^2\,dx \to 0 \ \text{ and } \int |\nabla u_n|^2\,dx \to 0. \tag{3.12}$$

From the Gagliado–Nirenberg inequality

$$\|v\|^{p+1}_{L^{p+1}(\mathbf{R}^N)} \leq c\|\nabla v\|^{\frac{3}{2}(p-1)}_{L^2(\mathbf{R}^N)}\|v\|^{p+1-\frac{3}{2}(p-1)}_{L^2(\mathbf{R}^N)}, \quad v \in H^1(\mathbf{R}^N),$$

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for $u_n$ we have

$$\begin{aligned}
&\frac{1}{3}\int |\nabla u_n|^2\,dx + \frac{1}{3}a\int |x|^{-2}|u_n|^2\,dx\\
&\quad = \frac{2}{p+1}\int |u_n|^{p+1}\,dx - \int |u_n|^2\,dx\\
&\quad \le \frac{2}{p+1}c\|\nabla u_n\|_{L^2(\mathbf{R}^3)}^{\frac{3}{2}(p-1)}\|u_n\|_{L^2(\mathbf{R}^3)}^{p+1-\frac{3}{2}(p-1)} - \int |u_n|^2\,dx\\
&\quad < c_2\|\nabla u_n\|_{L^2(\mathbf{R}^3)}^{\frac{3}{2}(p-1)}\|u_n\|_{L^2(\mathbf{R}^3)}^{p+1-\frac{3}{2}(p-1)} - \int |u_n|^2\,dx,
\end{aligned}$$

where $c_2 > \frac{1}{p+1}c$. Thus from (3.10)–(3.12), it follows that as $n \to \infty$, $0 < -\int |u_n|^2\,dx$. This is impossible. Since we have showed $d_B \neq 0$, we get $d_B > 0$ for $3 < p < 5$. $\square$

Moreover, we give the following lemmas.

LEMMA 3.2. There exists $u \in H\backslash\{0\}$ such that $K(u) = 0$ and $I(u) = 0$.

*Proof.* According to [**4**, **12**, **15**, **20**], there exists $u \in H\backslash\{0\}$ such that $u$ is a solution of the following elliptic equation:

$$-\triangle u + u + a|x|^{-2}u - |u|^{p-1}u = 0. \tag{3.13}$$

Thus $H(u) = 0$. From (3.13), we have

$$\frac{1}{2}\int |\nabla u|^2\,dx + \frac{a}{2}\int |x|^{-2}|u|^2\,dx + \frac{3}{2}\int |u|^2\,dx - \frac{3}{p+1}\int |u|^{p+1}\,dx = 0,$$

which is obtained from multiplying (3.13) by $x \cdot \nabla u$ and then integrating, that is, $K(u) = 0$. Thus $I(u) = 0$ from $H(u) = K(u) + I(u)$. $\square$

LEMMA 3.3. *$M$ is not empty provided* $3 < p < 5$.

*Proof.* From Lemma 3.2, there exists $u \in H\backslash\{0\}$ such that both $K(u) = 0$ and $I(u) = 0$. For arbitrary $\beta > 1$, put $u^*(x) = u(x/\beta)$. Then $u^* \in H$. From (3.2) and (3.3) it follows that

$$\begin{aligned}
K(u^*) &= \frac{1}{2}\int |\nabla u^*|^2\,dx + \frac{a}{2}\int |x|^{-2}|u^*|^2\,dx + \frac{3}{2}\int |u^*|^2\,dx - \frac{3}{p+1}\int |u^*|^{p+1}\,dx\\
&= \frac{\beta}{2}\int (|\nabla u|^2 + a|x|^{-2}|u|^2)\,dx + \frac{3\beta^3}{2}\int |u|^2\,dx - \frac{3\beta^3}{p+1}\int |u|^{p+1}\,dx\\
&< 0,\\
I(u^*) &= \frac{\beta}{2}\int (|\nabla u|^2 + a|x|^{-2}|u|^2)\,dx - \frac{\beta^3}{2}\int |u|^2\,dx - \frac{p-2}{p+1}\beta^3\int |u|^{p+1}\,dx\\
&< 0.
\end{aligned}$$

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Now we let $u^*_\lambda = \lambda^{\frac{2}{p-1}} u^*(\lambda x)$, $\lambda > 1$. Put $\alpha = \frac{5-p}{p-1}$, $\gamma = \frac{7-3p}{p-1}$, it follows from $3 < p < 5$ that $\alpha > 0$, $\gamma < 0$ and $\alpha = \gamma + 2$. Then

$$\begin{aligned} K(u^*_\lambda) &= \frac{1}{2}\lambda^\alpha \int |\nabla u^*|^2\,dx + \frac{a}{2}\lambda^\alpha \int |x|^{-2}|u^*|^2\,dx \\ &\quad + \frac{3}{2}\lambda^\gamma \int |u^*|^2\,dx - \frac{3}{p+1}\lambda^\alpha \int |u^*|^{p+1}\,dx, \end{aligned} \tag{3.14}$$

$$\begin{aligned} I(u^*_\lambda) &= \frac{1}{2}\lambda^\alpha \int |\nabla u^*|^2\,dx + \frac{a}{2}\lambda^\alpha \int |x|^{-2}|u^*|^2\,dx \\ &\quad - \frac{1}{2}\lambda^\gamma \int |u^*|^2\,dx - \frac{p-2}{p+1}\lambda^\alpha \int |u^*|^{p+1}\,dx. \end{aligned} \tag{3.15}$$

Thus $I(u^*) < 0$ implies that there exists $\lambda^* > 1$ such that $I(u^*_{\lambda^*}) = 0$.

On the other hand, from $\lambda^* > 1$, $K(u^*) < 0$ and (3.14), we still have $K(u^*_{\lambda^*}) < 0$. So $u^*_{\lambda^*} \in M$. This proves $M$ is not empty provided $3 < p < 5$. □

LEMMA 3.4. *$d_M > 0$ provided $3 < p < 5$.*

*Proof.* Let $u \in M$. By $K(u) < 0$, we have $u \neq 0$. From $I(u) = 0$, we have

$$J(u) = \frac{p-3}{2(p-2)} \int |\nabla u|^2\,dx + \frac{p-3}{2(p-2)} a \int |x|^{-2}|u|^2\,dx + \frac{p-1}{2(p-2)} \int |u|^2\,dx. \tag{3.16}$$

Since $3 < p < 5$, (3.16) and $u \neq 0$ imply that $J(u) > 0$ for all $u \in M$. Thus, from (3.8), we get $d_M \geq 0$. By $3 < p < 5$, it follows from the Sobolev embedding inequality that

$$\int |u|^{p+1}\,dx \leq c\left(\int |\nabla u|^2\,dx + \int |u|^2\,dx\right)^{\frac{p+1}{2}}. \tag{3.17}$$

From $K(u) < 0$ it follows that

$$\begin{aligned} &\frac{1}{2}\int |\nabla u|^2\,dx + \frac{a}{2}\int |x|^{-2}|u|^2\,dx + \frac{3}{2}\int |u|^2\,dx \\ &\qquad < \frac{3}{p+1}\int |u|^{p+1}\,dx \\ &\qquad \leq c\left(\int |\nabla u|^2\,dx + \int |u|^2\,dx + a\int |x|^{-2}|u|^2\,dx\right)^{\frac{p+1}{2}}, \end{aligned}$$

that is,

$$\left(\int |\nabla u|^2\,dx + \int |u|^2\,dx + a\int |x|^{-2}|u|^2\,dx\right)^{\frac{p-1}{2}} \geq \frac{1}{c} > 0. \tag{3.18}$$

Since $3 < p < 5$, (3.16) and (3.18) yield

$$J(u) \geq c > 0, \quad \text{for all } u \in M.$$

Thus (3.8) implies that $d_M > 0$ for $3 < p < 5$. □

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Now we define

$$d := \min\{d_M, d_B\}. \tag{3.19}$$

By Lemmas 3.1 and 3.4, the following conclusion is true.

PROPOSITION 3.1. *$d > 0$ when $3 < p < 5$.*

Now we further define

$$R := \{u \in H, J(u) < d, K(u) < 0, I(u) < 0\}. \tag{3.20}$$

Then one has

LEMMA 3.5. *$R$ is not empty provided $3 < p < 5$.*

*Proof.* By Lemma 3.2, there exists $u \in H\backslash\{0\}$ such that both $K(u) = 0$ and $I(u) = 0$. Let $u_\lambda(x) = \lambda u(x)$, it follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} K(u_\lambda) &= \frac{1}{2}\int |\nabla u_\lambda|^2\,dx + \frac{1}{2}\int |x|^{-2}|u_\lambda|^2\,dx + \frac{3}{2}\int |u_\lambda|^2\,dx - \frac{3}{p+1}\int |u_\lambda|^{p+1}\,dx \\ &= \frac{1}{2}\lambda^2\int |\nabla u|^2\,dx + \frac{1}{2}\lambda^2\int |x|^{-2}|u|^2\,dx \\ &\quad + \frac{3}{2}\lambda^2\int |u|^2\,dx - \frac{3}{p+1}\lambda^{p+1}\int |u|^{p+1}\,dx, \end{aligned}$$

$$\begin{aligned} I(u_\lambda) &= \frac{1}{2}\int |\nabla u_\lambda|^2\,dx + \frac{1}{2}\int |x|^{-2}|u_\lambda|^2\,dx - \frac{1}{2}\int |u_\lambda|^2\,dx - \frac{p-2}{p+1}\int |u_\lambda|^{p+1}\,dx \\ &= \frac{1}{2}\lambda^2\int |\nabla u|^2\,dx + \frac{1}{2}\lambda^2\int |x|^{-2}|u|^2\,dx \\ &\quad - \frac{1}{2}\lambda^2\int |u|^2\,dx - \frac{p-2}{p+1}\lambda^{p+1}\int |u|^{p+1}\,dx, \end{aligned}$$

$$\begin{aligned} J(u_\lambda) &= \frac{1}{2}\int |\nabla u_\lambda|^2\,dx + \frac{1}{2}\int |x|^{-2}|u_\lambda|^2\,dx + \frac{1}{2}\int |u_\lambda|^2\,dx - \frac{1}{p+1}\int |u_\lambda|^{p+1}\,dx \\ &= \frac{1}{2}\lambda^2\int |\nabla u|^2\,dx + \frac{1}{2}\lambda^2\int |x|^{-2}|u|^2\,dx \\ &\quad + \frac{1}{2}\lambda^2\int |u|^2\,dx - \frac{1}{p+1}\lambda^{p+1}\int |u|^{p+1}\,dx. \end{aligned}$$

Since $d > 0$, from (3.1), (3.2) and (3.3), for $\lambda > 1$ large enough, we can get $K(u_\lambda) < 0$, $I(u_\lambda) < 0$ and $J(u_\lambda) < d$. Thus $u_\lambda(x) = \lambda u(x) \in R$ and $R$ is not empty. □

Moreover, we have the following theorem.

PROPOSITION 3.2. *If $3 < p < 5$ and $E(0) < d$, then $R$ is an invariant manifold of (1.1) and (1.2). More precisely, from $\phi_0 \in R$, it follows that the solution $\phi(t, x)$ of the Cauchy problem (1.1), (1.2) satisfies $\phi(t, x) \in R$ for any $t \in [0, T)$.*

*Proof.* Let $\phi_0 \in R$. From (2.4), we have

$$J(\phi) < E(t) = E(0) < d.$$

Now we show that $K(\phi) < 0$ for $t \in [0, T)$. Otherwise, from continuity, there would be a $t_0 \in (0, T)$ such that $K(\phi(t_0, .)) = 0$ and $\phi(t_0, .) \neq 0$. From (3.7) and (3.19), it follows that $J(\phi(t_0, .)) \geq d_B \geq d$. This is contradictory with $J(\phi(t)) < d$ for $t \in [0, T)$. Therefore for all $t \in [0, T)$, $K(\phi(t, .)) < 0$.

At last, we show $I(\phi(t, .)) < 0$ for $t \in [0, T)$. Otherwise, from continuity, there would be a $t_1 \in (0, T)$ such that $I(\phi(t_1, .)) = 0$. Because we have showed $K(\phi(t_1, .)) < 0$, it follows that $\phi(t_1, .) \in M$. Thus (3.8) and (3.19) imply that $J(\phi(t_1, .)) \geq d_M \geq d$. This is contradictory with $J(\phi(t_1)) < d$ for $t \in [0, T)$. Therefore $I(\phi(t, .)) < 0$ for all $t \in [0, T)$.

So $\phi(t, .) \in R$ for any $t \in [0, T)$. Thus we complete the proof of Proposition 3.2. □

By the same argument in Proposition 3.2, we get

PROPOSITION 3.3. *Let* $3 < p < 5$ *and* $E(0) < d$. *Define*

$$\begin{aligned} R_+ &:= \{u \in H, J(u) < d, K(u) < 0, I(u) > 0\}, \\ K_- &:= \{u \in H, J(u) < d, K(u) < 0\}, \\ K_+ &:= \{u \in H, J(u) < d, K(u) > 0\}. \end{aligned}$$

*Then* $R_+$, $K_-$ *and* $K_+$ *are all invariant manifolds of* (1.1) *and* (1.2).

In the course of nature, we call $R$ and $R_+$ cross-invariant manifolds of (1.1) and (1.2).

By the definitions of $R$, $R_+$ and $K_+$ as well as by (3.7), (3.8) and (3.19), we can get the following result:

PROPOSITION 3.4. *Let* $3 < p < 5$ *and* $E(0) < d$. *Then*

$$\{u \in H\backslash\{0\}, J(u) < d\} = R \bigcup R_+ \bigcup K_+.$$

**4. Sharp threshold for blowup and global existence.** In this section, we first establish a sharp sufficient condition of blowup and global existence. Next, by using the former result, we obtain a sufficient and necessary condition of blowup and a small data criterion for the solution to the Cauchy problem (1.1), (1.2). Firstly, we have

THEOREM 4.1. *Let* $3 < p < 5$. *Assume that* $(\phi_0, \phi_1) \in H \times L^2(\mathbf{R}^3)$ *and* $E(0) < d$. *Then if* $\phi_0 \in R$, *the solution* $\phi(t, x)$ *of the Cauchy problem* (1.1), (1.2) *blows up in finite time.*

*Proof.* From $\phi_0 \in R$, Proposition 3.2 implies that $\phi(t, x) \in R$ for $t \in [0, T)$. Put

$$F(t) = \int |\phi(t, x)|^2 \, dx. \tag{4.1}$$

Proposition 2.2 implies that

$$F''(t) = 2\int |\phi_t|^2 \, dx - 2K(\phi) - 2I(\phi). \tag{4.2}$$

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By (4.2) and $\phi \in R$, $F(t)$ is a convex function of $t$. On the other hand, from (2.3) and (2.4), one has

$$F''(t) = (p+3)\int |\phi_t|^2\,dx + (p-1)\int [|\nabla\phi|^2 + (1+a|x|^{-2})|\phi|^2]\,dx - 2(p+1)E(0). \tag{4.3}$$

It follows that if there exists a time $t_1$ such that $F'(t)|_{t=t_1} > 0$, then $F(t)$ is increasing for all $t > t_1$ (within the interval of existence). In that case, the quantity

$$\begin{aligned}(p-1)&\int (1+a|x|^{-2})|\phi|^2\,dx - 2(p+1)E(0)\\ &\geq (p-1)\int |\phi|^2\,dx - 2(p+1)E(0)\\ &= (p-1)F(t) - 2(p+1)E(0)\end{aligned}$$

will eventually become positive, and will remain positive thereafter. Thus for $t$ large enough from (4.3) and $J(\phi) < d(\phi \in R)$, we would have

$$F''(t) \geq (p+3)\int |\phi_t|^2\,dx. \tag{4.4}$$

In view of (2.5), (2.6) and (4.4), using the Hölder's inequality, one has

$$\begin{aligned}F(t)F''(t) &\geq (p+3)\int |\phi_t|^2\,dx \cdot \int |\phi|^2\,dx\\ &= (p+3)\int |\phi_t|^2\,dx \cdot \int |\bar{\phi}|^2\,dx\\ &\geq (p+3)\left(\int |\phi_t\bar{\phi}|dx\right)^2\\ &\geq (p+3)\left(Re\int \phi_t\bar{\phi}\,dx\right)^2\\ &= (p+3)\left(\int \frac{1}{2}(\phi_t\bar{\phi} + \bar{\phi}_t\phi)\,dx\right)^2\\ &= (p+3)\left(\frac{1}{2}\frac{d}{dt}\int |\phi|^2\,dx\right)^2\\ &= \frac{p+3}{4}(F'(t))^2.\end{aligned} \tag{4.5}$$

Since

$$\left[F^{-\frac{p-1}{4}}(t)\right]'' = -\frac{p-1}{4}F^{-\frac{p+1}{4}}(t)\left[F(t)F''(t) - \frac{p+3}{4}(F'(t))^2\right],$$

we see that

$$\left[F^{-\frac{p-1}{4}}(t)\right]'' \leq 0.$$

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Therefore $F^{-\frac{p-1}{4}}(t)$ is concave for sufficiently large $t$, and there exists a finite time $T^*$ such that

$$\lim_{t\to T^*} F^{-\frac{p-1}{4}}(t) = 0.$$

In other words,

$$\lim_{t\to T^*} F(t) = \infty.$$

Thus one has $T < \infty$ and $\lim_{t\to T^-} \|\phi\|_H = \infty$.

The proof of Theorem 4.1 will be complete once we show that for some $t_1$, $F'(t)|_{t=t_1} > 0$. We prove it by contradiction. Suppose that for all $t$,

$$F'(t) \leq 0. \tag{4.6}$$

Then since $F(t) > 0$ and is convex, $F(t)$ must tend to a finite, non-negative limit $A$ as $t \to \infty$. By Proposition 3.2, we assert that $A > 0$. Therefore one has, as $t \to \infty$,

$$F(t) \to A > 0, \quad F'(t) \to 0, \quad F''(t) \to 0.$$

Thus from (4.2), we get

$$\lim_{t\to\infty} \int |\phi_t|^2\, dx = 0, \tag{4.7}$$

and

$$K(\phi) \to 0 \quad \textit{and} \quad I(\phi) \to 0 \quad \textit{as} \quad t \to \infty. \tag{4.8}$$

Now for any fixed $t > 0$, let $\phi_\lambda = \phi(x/\lambda)$. Because $K(\phi) < 0$ and $I(\phi) < 0$, we have

$$\begin{aligned} K(\phi_\lambda) = {} & \frac{1}{2}\lambda \int |\nabla\phi|^2\, dx + \frac{a}{2}\lambda \int |x|^{-2}|\phi|^2\, dx \\ & + \frac{3}{2}\lambda^3 \int |\phi|^2\, dx - \frac{3}{p+1}\lambda^3 \int |\phi|^{p+1}\, dx, \end{aligned} \tag{4.9}$$

$$\begin{aligned} I(\phi_\lambda) = {} & \frac{1}{2}\lambda \int |\nabla\phi|^2\, dx + \frac{a}{2}\lambda \int |x|^{-2}|\phi|^2\, dx \\ & - \frac{1}{2}\lambda^3 \int |\phi|^2\, dx - \frac{p-2}{p+1}\lambda^3 \int |\phi|^{p+1}\, dx. \end{aligned} \tag{4.10}$$

Since $I(\phi) < 0$, it yields that there exists $\lambda^* \in [0, 1)$ such that $I(\phi_{\lambda^*}) = 0$, and when $\lambda \in (\lambda^*, 1)$, $I(\phi_\lambda) < 0$. For $\lambda \in [\lambda^*, 1]$, since $K(\phi) < 0$, $K(\phi_\lambda)$ has the following two possibilities:

(i) $K(\phi_\lambda) < 0$ for $\lambda \in (\lambda^*, 1]$.

(ii) There exists $\mu \in [\lambda^*, 1)$ such that $K(\phi_\mu) = 0$.

For case (i), we have $I(\phi_{\lambda^*}) = 0$ and $K(\phi_{\lambda^*}) < 0$. From (3.8) and (3.19), it follows that

$$J(\phi_{\lambda^*}) \geq d_M \geq d.$$

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Moreover, from $K(\phi_{\lambda^*}) < 0$, we have

$$\begin{aligned}
J(\phi) - J(\phi_{\lambda^*}) &= \frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx \\
&\quad - \frac{1}{p+1}\int |\phi|^{p+1}\,dx - \frac{1}{2}\lambda^*\int |\nabla\phi|^2\,dx - \frac{a}{2}\lambda^*\int |x|^{-2}|\phi|^2\,dx \\
&\quad - \frac{1}{2}\lambda^{*3}\int |\phi|^2\,dx + \frac{1}{p+1}\lambda^{*3}\int |\phi|^{p+1}\,dx \\
&\geq \frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx \\
&\quad - \frac{1}{p+1}\int |\phi|^{p+1}\,dx - \frac{1}{2}\lambda^*\int |\nabla\phi|^2\,dx - \frac{a}{2}\lambda^*\int |x|^{-2}|\phi|^2\,dx \\
&= -\frac{1}{2}\lambda^{*3}\int |\phi|^2\,dx + \frac{1}{6}\lambda^*\int |\nabla\phi|^2\,dx + \frac{a}{6}\lambda^*\int |x|^{-2}|\phi|^2\,dx \\
&\quad + \frac{1}{2}\lambda^{*3}\int |\phi|^2\,dx \\
&= \frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx \\
&\quad - \frac{1}{p+1}\int |\phi|^{p+1}\,dx - \frac{1}{3}\lambda^*\int |\nabla\phi|^2\,dx - \frac{a}{3}\lambda^*\int |x|^{-2}|\phi|^2\,dx \\
&\geq \frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx \\
&\quad - \frac{1}{p+1}\int |\phi|^{p+1}\,dx - \frac{1}{3}\int |\nabla\phi|^2\,dx - \frac{a}{3}\int |x|^{-2}|\phi|^2\,dx \\
&= \frac{1}{6}\int |\nabla\phi|^2\,dx + \frac{a}{6}\int |x|^{-2}|\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx - \frac{1}{p+1}\int |\phi|^{p+1}\,dx \\
&= \frac{1}{3}K(\phi). \qquad (4.11)
\end{aligned}$$

For case (ii), we have $J(\phi_\mu) \geq d_B \geq d$. Referring to (4.11) and

$$\begin{aligned}
I(\phi) - I(\phi_{\lambda^*}) &= \frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi|^2\,dx - \frac{1}{2}\int |\phi|^2\,dx \\
&\quad - \frac{p-2}{p+1}\int |\phi|^{p+1}\,dx - \frac{1}{2}\lambda^*\int |\nabla\phi|^2\,dx - \frac{a}{2}\lambda^*\int |x|^{-2}|\phi|^2\,dx \\
&\quad + \frac{1}{2}\lambda^{*3}\int |\phi|^2\,dx + \frac{p-2}{p+1}\lambda^{*3}\int |\phi|^{p+1}\,dx, \qquad (4.12)
\end{aligned}$$

we have

$$J(\phi) - J(\phi_\mu) \geq \frac{1}{3}K(\phi). \qquad (4.13)$$

Since $J(\phi_{\lambda^*}) \geq d,\ J(\phi_\mu) \geq d$, from (4.11) and (4.13), we have

$$K(\phi) < J(\phi) < E(0) < d. \qquad (4.14)$$

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From (4.8), (4.11), (4.13), (3.7) and (3.19), we have

$$J(\phi) \geq J(\phi_\mu) \geq d \quad \text{as} \quad t \to \infty. \tag{4.15}$$

This is impossible from Proposition 3.2. So the supposition (4.6) is false. That is, there exists a $t_1$ such that $F'(t)|_{t=t_1} > 0$.

Thus we complete the proof of Theorem 4.1. $\square$

THEOREM 4.2. *Let* $3 < p < 5$ *and* $E(0) < d$. *If* $\phi_0 \in K_+ \bigcup R_+$, *then the solution* $\phi(t, x)$ *of the Cauchy problems* (1.1) *and* (1.2) *exists globally in* $t \in [0, \infty)$.

*Proof.* We prove this theorem through two steps.

*Step 1.* Let $\phi_0(x) \in K_+$. From Proposition 3.3, it follows that $\phi(t, x) \in K_+$ for $t \in [0, T)$. Thus for fixed $t \in [0, T)$, $J(\phi) < d$ and $K(\phi) > 0$, which imply that $\phi \neq 0$ and

$$\frac{1}{3}\int |\nabla\phi|^2\,dx + \frac{1}{3}\int |x|^{-2}|\phi|^2\,dx < J(\phi) < d. \tag{4.16}$$

Let $\phi_\lambda(x) = \lambda\phi(x)$, $K(\phi) > 0$ implies that there exists a $\lambda^* > 1$ such that

$$\begin{aligned} K(\phi_{\lambda^*}) &= \frac{1}{2}\int |\nabla\phi_{\lambda^*}|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi_{\lambda^*}|^2\,dx \\ &\quad + \frac{3}{2}\int |\phi_{\lambda^*}|^2\,dx - \frac{3}{p+1}\int |\phi_{\lambda^*}|^{p+1}\,dx \\ &= \frac{1}{2}\lambda^{*2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\lambda^{*2}\int |x|^{-2}|\phi|^2\,dx \\ &\quad + \frac{3}{2}\lambda^{*2}\int |\phi|^2\,dx - \frac{3}{p+1}\lambda^{*p+1}\int |\phi|^{p+1}\,dx \\ &= 0, \end{aligned} \tag{4.17}$$

and

$$J(\phi_{\lambda^*}) = \frac{1}{3}\lambda^{*2}\int |\nabla\phi|^2\,dx + \frac{a}{3}\lambda^{*2}\int |x|^{-2}|\phi|^2\,dx. \tag{4.18}$$

By (4.17) and (4.18), we get $\phi_{\lambda^*} \in N$ and

$$J(\phi_{\lambda^*}) \geq d_N \geq d > J(\phi). \tag{4.19}$$

Thus $J(\phi) - J(\phi_{\lambda^*}) < 0$, that is,

$$\begin{aligned} &\frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx - \frac{1}{p+1}\int |\phi|^{p+1}\,dx \\ &\quad -\frac{1}{3}\lambda^{*2}\int |\nabla\phi|^2\,dx - \frac{a}{3}\lambda^{*2}\int |x|^{-2}|\phi|^2\,dx < 0, \end{aligned} \tag{4.20}$$

so

$$\begin{aligned} &\frac{1}{2}\int |\nabla\phi|^2\,dx + \frac{a}{2}\int |x|^{-2}|\phi|^2\,dx + \frac{1}{2}\int |\phi|^2\,dx - \frac{1}{p+1}\int |\phi|^{p+1}\,dx \\ &\quad -\frac{1}{2}\lambda^{*2}\int |\nabla\phi|^2\,dx - \frac{a}{2}\lambda^{*2}\int |x|^{-2}|\phi|^2\,dx < 0. \end{aligned} \tag{4.21}$$

From (4.17), it follows that

$$\frac{1}{2}\int|\nabla\phi|^2\,dx+\frac{a}{2}\int|x|^{-2}|\phi|^2\,dx+\frac{1}{2}\int|\phi|^2\,dx \\ -\frac{1}{6\lambda^{*p-1}}\int|x|^{-2}|\phi|^2\,dx-\frac{1}{2\lambda^{*p-1}}\int|\phi|^2\,dx. \tag{4.22}$$

By (4.21) and (4.22), we get

$$\begin{aligned}-\frac{1}{p+1}\int|\phi|^{p+1}\,dx &= -\frac{1}{6\lambda^{*p-1}}\int|\nabla\phi|^2\,dx\\ &\quad -\frac{1}{6\lambda^{*p-1}}\int|\nabla\phi|^2\,dx-\frac{a}{6\lambda^{*p-1}}\int|x|^{-2}|\phi|^2\,dx\\ &\quad -\frac{1}{2\lambda^{*p-1}}\int|\phi|^2\,dx-\frac{1}{2}\lambda^{*2}\int|\nabla\phi|^2\,dx-\frac{a}{2}\lambda^{*2}\int|x|^{-2}|\phi|^2\,dx\\ &< 0,\end{aligned}$$

that is,

$$\left(\frac{1}{2}-\frac{1}{2\lambda^{*p-1}}\right)\int|\phi|^2\,dx \\ < \left(\frac{1}{6\lambda^{*p-1}}+\frac{1}{2}\lambda^{*2}-\frac{1}{2}\right)\left(\int|\nabla\phi|^2\,dx+a\int|x|^{-2}|\phi|^2\,dx\right). \tag{4.23}$$

Since $\lambda^*>1$ and $3<p<5$, (4.16) and (4.23) imply that

$$\int|\nabla\phi|^2\,dx+a\int|x|^{-2}|\phi|^2\,dx+\int|\phi|^2\,dx<C.$$

It implies that $\|\phi\|_H$ is bounded. So it must be $T=\infty$. Therefore, the solution $\phi(t,x)$ of the Cauchy problem (1.1), (1.2) globally exists in $t\in[0,\infty)$.

*Step 2.* Let $\phi_0\in R_+$. From Proposition 3.3, it follows that the solution $\phi(t,x)$ of the Cauchy problem (1.1), (1.2) satisfies $\phi(t,x)\in R_+$ for $t\in[0,T)$. Thus for fixed $t\in[0,T)$, we have $J(\phi)<d$, $I(\phi)>0$ and $K(\phi)<0$. From $J(\phi)<d$ and $I(\phi)>0$, it follows that

$$\left(\frac{1}{2}-\frac{1}{2(p-2)}\right)\int|\nabla\phi|^2\,dx+\left(\frac{1}{2}-\frac{1}{2(p-2)}\right)a\int|x|^{-2}|\phi|^2\,dx \\ +\left(\frac{1}{2}+\frac{1}{2(p-2)}\right)\int|\phi|^2\,dx<d,$$

which implies that $\phi(t,x)$ is bounded in $H$. So it must be $T=\infty$. Thus the solution $\phi(t,x)$ of the Cauchy problem (1.1), (1.2) exists globally in $t\in[0,\infty)$.

From Steps 1 and 2, we complete the proof of Theorem 4.2. □

By Proposition 3.4, Theorems 4.1 and 4.2, we get a necessary and sufficient condition of blowup for the solution to (1.1) and (1.2).

THEOREM 4.3. *Let* $3<p<5$ *and* $E(0)<d$. *Then the solution* $\phi(t,x)$ *of the Cauchy problem* (1.1), (1.2) *blows up in finite time if and only if* $\phi_0\in R$.

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By Proposition 3.2, we also get another sufficient condition of global existence for the solution to (1.1) and (1.2).

COROLLARY 4.1 (small data criterion). *If* $(\phi_0, \phi_1) \in H \times L^2(\mathbf{R}^3)$ *and satisfies*

$$\|\phi_0\|_H^2 + \|\phi_1\|_{L^2(\mathbf{R}^3)}^2 < d, \tag{4.24}$$

*then the solution* $\phi(t, x)$ *of the Cauchy problem* (1.1)*,* (1.2) *exists globally in* $t \in [0, \infty)$.

*Proof.* From (4.24) we have $J(\phi_0) < d$ and $E(0) < d$. Moreover we claim that $K(\phi_0) > 0$. Otherwise, there would be a $\lambda$ with $0 < \lambda \leq 1$ such that $K(\lambda\phi_0) = 0$. Thus $J(\lambda\phi_0) \geq d$. On the other hand, from (4.24),

$$\|\lambda\phi_0\|_H^2 + \|\phi_1\|_{L^2(\mathbf{R}^3)}^2 = \lambda^2\|\phi_0\|_H^2 + \|\phi_1\|_{L^2(\mathbf{R}^3)}^2 < d.$$

It follows that $J(\lambda\phi_0) < d$, which is contradictory with $J(\lambda\phi_0) \geq d$. Therefore, we have $\phi_0 \in K_+$. Thus Proposition 3.3 and Theorem 4.2 imply this corollary. □

ACKNOWLEDGEMENTS. This work is partially finished when the first author visited the IMS of the Chinese University of Hong Kong. She would like to express her thanks for the hospitality of the IMS. Especially, She would like to express her deep gratitude to Professor Zhouping Xin for his valuable suggestions and constant encouragements. Moreover, the authors thank the referee for his valuable suggestions.

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