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CLASSES OF OPERATOR-SMOOTH FUNCTIONS. III STABLE FUNCTIONS AND FUGLEDE IDEALS

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Abstract This paper continue to study the interrelation and hierarchy of the spaces of operator-Lipschitz functions and the spaces of functions closed to them: commutator bounded and operator stable. It also examines various properties of symmetrically normed ideals, introduces new classes of ideals: regular and Fuglede, and investigates them.

Keywords: operator; Lipschitz; functions; symmetrically normed ideals

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1. Introduction

This paper is a sequel to the article [**17**] on operator-Lipschitz functions. It continues to study operator-Lipschitz functions and functions close to them: commutator bounded and operator stable functions. It also examines various properties of symmetrically normed ideals. New classes of ideals (regular and Fuglede) are introduced and investigated in the paper. A summary of the main results for the most important class of Schatten ideals is given at the end of § 5. We refer the reader to [**17**] for a review of the history of the subject, the impetus behind our study, general notation and various technical results.

Recall that we denote by $B(H)$ the algebra of all bounded operators on a Hilbert space H, by $C(H)$ the ideal of all compact operators, and by $\mathcal{F}$ the ideal of all finite-rank operators in $B(H)$. A two-sided ideal J of $B(H)$ is *symmetrically normed* (see [**12**]) if it is a Banach space with respect to a norm $\|\cdot\|_J$ and

$$\|AXB\|_J \leqslant \|A\|\,\|X\|_J\|B\| \quad \text{for } A, B \in B(H) \text{ and } X \in J. \tag{1.1}$$

It is a $*$-ideal and, by the Calkin theorem, $\mathcal{F} \subset J \subseteq C(H)$. We denote by α compact subsets of $\mathbb{C}$ and set

$$J_{\mathrm{nor}}(\alpha) = \{A \in J : A \text{ is normal and } \mathrm{Sp}(A) \subseteq \alpha\}.$$

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Any continuous function g on α defines a map $T \to g(T)$ from $J_{\text{nor}}(\alpha)$ to $B(H)$. Various smoothness conditions when imposed on this map characterize important classes of operator-smooth functions. The condition that it is Lipschitzian, for example, defines the class of J-Lipschitz functions. Thus a function g is called J-*Lipschitzian* on α if there is $D > 0$ such that, for $A, B \in J_{\text{nor}}(\alpha)$,

$$g(A) - g(B) \in J \quad \text{and} \quad \|g(A) - g(B)\|_J \leqslant D\|A - B\|_J. \tag{1.2}$$

We denote by $J\text{-}\operatorname{Lip}(\alpha)$ the space of all J-Lipschitz functions on α.

A function g is called *commutator J-bounded* on α if there is $D > 0$ such that, for $A \in J_{\text{nor}}(\alpha)$ and $X \in B(H)$,

$$g(A)X - Xg(A) \in J \quad \text{and} \quad \|g(A)X - Xg(A)\|_J \leqslant D\|AX - XA\|_J. \tag{1.3}$$

We denote by $J\text{-}\operatorname{CB}(\alpha)$ the space of all commutator J-bounded functions on α.

In this paper we continue the study of intrinsic properties, interrelation and hierarchy of the spaces $J\text{-}\operatorname{Lip}(\alpha)$ and $J\text{-}\operatorname{CB}(\alpha)$, and consider also another class of functions close to them: operator J-stable functions.

It was shown in [**17**] that the space $J\text{-}\operatorname{Lip}(\alpha)$ always contains $J\text{-}\operatorname{CB}(\alpha)$ and that these spaces coincide if and only if α is a J-*Fuglede* set, that is, there is $C > 0$ such that

$$\|A^*X - XA^*\|_J \leqslant C\|AX - XA\|_J, \quad \text{for } A \in J_{\text{nor}}(\alpha) \text{ and } X \in B(H). \tag{1.4}$$

For example, all compact sets in $\mathbb{R}$ are J-Fuglede for all s.n. ideals J and for $J = B(H)$. The possibility to reduce the study of J-Lipschitz functions to the study of commutator J-bounded functions is very important, since it enables us to use the powerful techniques of the interpolation theory to compare the spaces $J\text{-}\operatorname{Lip}(\alpha)$ for different ideals. The conditions under which a compact subset of $\mathbb{C}$ is a J-Fuglede set were studied in [**17**] and will be further investigated here.

If all compact subsets of $\mathbb{C}$ are J-Fuglede, or, equivalently, if the unit disc is J-Fuglede, then J is called a *Fuglede ideal*. So Fuglede ideals are the ideals J for which $J\text{-}\operatorname{Lip}(\alpha) = J\text{-}\operatorname{CB}(\alpha)$ for all compacts α in $\mathbb{C}$.

In § 3 we obtain some sufficient condition for an ideal J to be Fuglede. We show that if the Boyd indices (see [**2**]) of J lie in $(1, \infty)$, then J is Fuglede. This extends the results of Weiss [**25**], Abdessemed and Davies [**1**] and Shulman [**24**] that all $\mathfrak{S}^p$, $p \in (1, \infty)$, are Fuglede ideals. It also shows that Lorentz ideals $\mathfrak{S}^{r,p}$ are Fuglede if $1 < p$. We establish that $\mathfrak{S}^1$ and $\mathfrak{S}^\infty$ are not Fuglede ideals (the fact that $\mathfrak{S}^b$ is not Fuglede follows from Corollary 3.3 of [**15**]).

We also use Hadamard multipliers and the interpolation theory to compare the spaces $J\text{-}\operatorname{CB}(\alpha)$ for various ideals J. We prove that if the Boyd indices of J lie between p and $p/(p-1)$, then $\mathfrak{S}^p\text{-}\operatorname{CB}(\alpha) \subseteq J\text{-}\operatorname{CB}(\alpha)$. This implies, in particular, that $\mathfrak{S}^p\text{-}\operatorname{CB}(\alpha) \subseteq \mathfrak{S}^q\text{-}\operatorname{CB}(\alpha)$ when

$$\min\left(p, \frac{p}{p-1}\right) \leqslant q \leqslant \max\left(p, \frac{p}{p-1}\right).$$

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The normal operators A, B in the definitions of J-Lipschitz and of commutator J-bounded functions in (1.2) and (1.3) belong to J. The question arises as to whether these inequalities hold for all normal $A, B \in B(H)$ with spectra in α. Using the theory of complex interpolation and the results of Bercovici and Voiculescu in [**3**] on quasi-diagonalization of operators modulo ideals, we obtain in § 4 that 'extended' inequalities (1.2) and (1.3) hold for all separable ideals and their duals. This generalizes the result of Kittaneh [**18**], who considered the case $J = \mathfrak{S}^2$. We use these results in § 5 to study J-stable functions.

A function g on α is called *J-stable* if the condition $A - B \in J$ implies $g(A) - g(B) \in J$ for all $A, B \in J_{\text{nor}}(\alpha)$. The property of J-stability of functions is important for various applications in mathematical physics. It was studied by Daletskii and Krein [**8**], Birman and Solomyak [**6**], Farforovskaya [**10**], Peller [**22**] and others. For separable ideals and their duals (see Corollary 4.6) this property of functions is, generally speaking, weaker than the property to be J-Lipschitzian. In § 5 we show that in many important cases ($J = \mathfrak{S}^p$, $1 < p < \infty$, for example) they are equivalent.

Johnson and Williams [**15**] constructed a normal operator A and a bounded X such that $[A, X] \in \mathfrak{S}^1$ and $[A^*, X] \notin \mathfrak{S}^1$. Weiss [**25**] asked whether for compact X, the condition $[A, X] \in \mathfrak{S}^1$ always implies $[A^*, X] \in \mathfrak{S}^1$. A negative answer to this question was given in [**23**]. It was shown in [**16**] that, for any $p > 1$, one can find $X \in \mathfrak{S}^p$ such that $[A, X] \in \mathfrak{S}^1$ and $[A^*, X] \notin \mathfrak{S}^1$. In § 5 we construct a normal *compact* operator A and a compact operator X such that $[A, X] \in \mathfrak{S}^1$, while $[A^*, X] \notin \mathfrak{S}^1$.

2. Preliminaries

Let c_0 be the space of all sequences of real numbers converging to 0, let $\hat{c}$ be the subspace of c_0 of sequences with a finite number of non-zero elements, and let Φ be the set of all symmetric norming (s.n.) functions on $\hat{c}$ (see [**12**, § III.3]). For $\xi = \{\xi_i\} \in c_0$, set $\xi^{(n)} = \{\xi_1, \dots, \xi_n, 0, \dots\}$. Then $\xi^{(n)} \in \hat{c}$. For $\phi \in \Phi$, the sequence $\phi(\xi^{(n)})$ does not decrease. Set $\phi(\xi) = \lim \phi(\xi^{(n)})$ and $c^\phi = \{\xi \in c_0 : \phi(\xi) < \infty\}$.

For $A \in C(H)$, let $s(A) = \{s_i(A)\}$ be the non-increasing sequence of all eigenvalues of $(A^*A)^{1/2}$ repeated according to multiplicity. For $\phi \in \Phi$, the set $J^\phi = \{A \in C(H) : s(A) \in c^\phi\}$ with norm $\|A\|_{J^\phi} = \phi(s(A))$ is an s.n. ideal. The closure J_0^ϕ of $\mathcal{F}$ in $\|\cdot\|_{J^\phi}$ is a separable ideal and $J_0^\phi \subseteq J^\phi$. An s.n. ideal is separable if and only if it coincides with some J_0^ϕ (see [**12**]). For some ϕ, $J^\phi = J_0^\phi$. An important class of such functions consists of

$$\phi_p(\xi) = \left(\sum_{i=1}^{\infty} |\xi_i|^p\right)^{1/p} \quad \text{for } 1 \leqslant p < \infty, \quad \text{and} \quad \phi_\infty(\xi) = \sup \|\xi_i\|.$$

The corresponding ideals $\mathfrak{S}^p$ are called Schatten ideals. We denote $C(H)$ by $\mathfrak{S}^\infty$ and $B(H)$ by $\mathfrak{S}^b$.

For $\phi \in \Phi$, there is the adjoint function ϕ^* such that the ideal J^{ϕ^*} is isomorphic to the dual space of J_0^ϕ: any bounded functional on J_0^ϕ has the form

$$F(X) = \mathrm{Tr}(XT) = \mathrm{Tr}(TX), \quad \text{where } T \in J^{\phi^*} \text{ and } \|F\| = \|T\|_{J^{\phi^*}}. \tag{2.1}$$

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Let

$$p' = \begin{cases} \dfrac{p}{p-1}, & \text{for } 1 < p < \infty, \\ 1, & \text{for } p = \infty, \\ b, & \text{for } p = 1. \end{cases}$$

Then $\phi_{p'} = (\phi_p)^*$, so $\mathfrak{S}^{p'}$ is isometrically isomorphic to the dual space of $\mathfrak{S}^p$.

The results below are known. We include their proof for the reader's convenience.

Proposition 2.1. *Let J and I be s.n. ideals.*

(i) *If $J \subseteq I$, then there exists $c > 0$ such that $\|X\|_I \leqslant c\|X\|_J$ for $X \in J$.*

(ii) *There is a unique, up to equivalence (see [**12**, III.3.4]), function $\phi \in \Phi$ such that $J_0^\phi \subseteq J \subseteq J^\phi$. The norms $\|\cdot\|_{J^\phi}, \|\cdot\|_J$ coincide on J_0^ϕ and $\|T\|_{J^\phi} \leqslant \|T\|_J$ for $T \in J$.*

(iii) *If J is reflexive, then there is $\phi \in \Phi$ such that $J_0^\phi = J = J^\phi$.*

Proof. Consider the norm $\|X\|^{\tilde{}} = \max(\|X\|_J, \|X\|_I)$ on J. Since $\|X\| \leqslant \|X\|_J$ and $\|X\| \leqslant \|X\|_I$, for $X \in J$, one can easily check that $(J, \|\cdot\|^{\tilde{}})$ is a Banach space. The identity operator from $(J, \|\cdot\|^{\tilde{}})$ to $(J, \|\cdot\|_J)$ is bounded. By Banach's theorem, the inverse is also bounded, so there is $c > 0$ such that $\|X\|_I \leqslant c\|X\|_J$. Part (i) is proved.

Consider $\|\cdot\|_J$ on $\mathcal{F}$. It follows from III.3.1 of [**12**] that there is $\phi \in \Phi$ such that $\|X\|_J = \|X\|_{J^\phi}$, for $X \in \mathcal{F}$. Hence $J_0^\phi \subseteq J$ and the norms $\|\cdot\|_J$, $\|\cdot\|_{J^\phi}$ coincide on J_0^ϕ. Let finite-dimensional projections P_n strongly converge to $\mathbf{1}$. For $T \in J$, the operators P_nT belong to $\mathcal{F}$, strongly converge to T and

$$\|P_nT\|_{J^\phi} = \|P_nT\|_J \leqslant \|P_n\|\,\|T\|_J = \|T\|_J.$$

By Theorem III.5.1 of [**12**], $T \in J^\phi$ and $\|T\|_{J^\phi} \leqslant \|T\|_J$. Thus $J_0^\phi \subseteq J \subseteq J^\phi$.

Suppose that also $J_0^\psi \subseteq J \subseteq J^\psi$. By (i), there are $c_1, c_2 > 0$ such that $\|X\|_{J^\psi} \leqslant c_1\|X\|_J$, for $X \in J$, and $\|X\|_J \leqslant c_2\|X\|_{J_0^\psi}$ for $X \in J_0^\psi$. Hence the norms $\|\cdot\|_J$ and $\|\cdot\|_{J^\psi}$ are equivalent on $\mathcal{F}$, so the norms $\|\cdot\|_{J^\phi}$ and $\|\cdot\|_{J^\psi}$ are equivalent on $\mathcal{F}$. Thus (see [**12**, III.4.2]) ϕ, ψ are equivalent and $J^\phi = J^\psi$. Part (ii) is proved.

Let J be reflexive. By (ii), $J_0^\phi \subseteq J \subseteq J^\phi$. Since J_0^ϕ is a closed subspace of J, J_0^ϕ is also reflexive (see [**13**, Proposition 67]). Since J^{ϕ^*} is the dual of J_0^ϕ, it is reflexive. Hence its closed subspace $J_0^{\phi^*}$ is also reflexive. Since J^ϕ is the dual of $J_0^{\phi^*}$, the ideal $J_0^{\phi^*}$ is the dual of J^ϕ. If the dual of a Banach space is separable, the space itself is separable. Thus, since $J_0^{\phi^*}$ is separable, J^ϕ is separable, so $J_0^\phi = J = J^\phi$. □

3. Boyd indices and Fuglede ideals

Let $J \subset I$ be s.n. ideals. A linear operator T on I is called *bounded on* (J, I) if it is bounded on I, preserves J and its restriction $T|_J$ to J is bounded in $\|\cdot\|_J$. We will denote by $\|T\|_I$ and $\|T\|_J$ the norms of the operator T on I and J. The set $L(J, I)$ of all bounded operators on (J, I) is a Banach space with norm $\max(\|T\|_I, \|T\|_J)$. We consider (in a simple form needed for our purposes) the notion of an interpolation space (see [**19**]).

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Definition 3.1. Let $J \subset I$ be s.n. ideals. A Banach space $(K, \|\cdot\|_K)$, $J \subset K \subset I$, is called an interpolation space for the pair (J, I) if any bounded operator T on (J, I) preserves K and the restriction $T|_K$ is a bounded operator on K.

Lemma 3.2. *Let $(K, \|\cdot\|_K)$ be an interpolation space for a pair (J, I) of s.n. ideals.*

(i) *There is a norm $\|\cdot\|'_K$ on K equivalent to $\|\cdot\|_K$ such that $(K, \|\cdot\|'_K)$ is an s.n. ideal.*

(ii) *If for each $T \in L(J, I)$, $\|T\|_K \leqslant \|T\|_I^t \|T\|_J^{1-t}$ for some $t \in [0, 1]$, then $(K, \|\cdot\|_K)$ is an s.n. ideal.*

Proof. The map $T \to T|_K$ from $L(J, I)$ into $B(K)$ is closed. By the closed graph theorem, there is $C > 0$ such that

$$\|T\|_K \leqslant C \max(\|T\|_I, \|T\|_J) \quad \text{for all } T \in L(J, I). \tag{3.1}$$

For $A \in B(H)$, the left and right multiplication operators L_A, R_A preserve I and J and $\|A\| = \|L_A\|_I = \|L_A\|_J = \|R_A\|_I = \|R_A\|_J$. Therefore, L_A and R_A are bounded on (J, I). Hence they preserve K, so that K is an ideal.

For all $A, B \in B(H)$ and $X \in K$, it follows from (3.1) that

$$\|AXB\|_K = \|L_A R_B X\|_K \leqslant C \max(\|L_A R_B\|_I, \|L_A R_B\|_J)\|X\|_K \leqslant C\|A\|\,\|B\|\,\|X\|_K.$$

Define a new norm on K by the formula

$$\|X\|'_K = \sup\{\|AXB\|_K : A, B \in B(H),\ \|A\| \leqslant 1,\ \|B\| \leqslant 1\}.$$

It is equivalent to $\|\cdot\|_K$, since $\|X\|_K = \|\mathbf{1}X\mathbf{1}\|_K \leqslant \|X\|'_K \leqslant C\|X\|_K$. For $F, G \in B(H)$,

$$\|FXG\|'_K = \|F\|\,\|G\| \left\| \frac{F}{\|F\|} X \frac{G}{\|G\|} \right\|'_K \leqslant \|F\|\,\|G\|\,\|X\|'_K.$$

Thus $(K, \|\cdot\|'_K)$ is an s.n. ideal. Part (i) is proved.

For $A, B \in B(H)$ and $X \in K$, we have

$$\begin{aligned} \|AXB\|_K &= \|L_A R_B X\|_K \leqslant \|L_A R_B\|_K \|X\|_K \\ &\leqslant \|L_A R_B\|_I^t \|L_A R_B\|_J^{1-t} \|X\|_K \leqslant \|L_A\|_I^t \|R_B\|_I^t \|L_A\|_J^{1-t} \|R_B\|_J^{1-t} \|X\|_K \\ &= \|A\|^t \|B\|^t \|A\|^{1-t} \|B\|^{1-t} \|X\|_K = \|A\|\,\|B\|\,\|X\|_K. \end{aligned}$$

Thus $(K, \|\cdot\|_K)$ is an s.n. ideal. □

Let $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n, \dots)$ be a diagonal operator with respect to a basis $\{e_n\}$ and let a compact α in $\mathbb{C}$ contain all λ_n. For a continuous function g on α, set

$$m_{ij} = \begin{cases} \dfrac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j}, & \text{if } \lambda_i \neq \lambda_j, \\ 0, & \text{if } \lambda_i = \lambda_j, \end{cases}$$

and consider the matrix $M(A, g) = (m_{ij})$. With each $X \in B(H)$ we associate the matrix (x_{ij}), with $x_{ij} = (Xe_j, e_i)$, and set $M(A, g) \circ X = (m_{ij} x_{ij})$. The matrix $M(A, g)$ is called a Hadamard J-multiplier (see [**19**]), if $M(A, g) \circ X$ belongs to J for each $X \in J$.

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Theorem 3.3. *Let ϕ be an s.n. function, let ϕ^* be its adjoint, and suppose that $J_0^\phi \subset J^{\phi^*}$. Let g be a commutator J^ϕ-bounded function on α. If an s.n. ideal K is an interpolation space for the pair (J_0^ϕ, J^{ϕ^*}), then there is $D > 0$ such that*

$$\|[g(A), X]\|_K \leqslant D\|[A, X]\|_K \quad \text{for } A \in K_{nor}(\alpha) \text{ and } X \in C(H).$$

If K is J^ψ or J_0^ψ, for some s.n. function ψ, then g is commutator K-bounded.

Proof. By Corollary 5.4 of [**17**], g is commutator J^{ϕ^*}-bounded on α. Since $K \subset J^{\phi^*}$, it follows from Corollary 5.3 of [**17**] that, for all $A \in K_{nor}(\alpha)$, the matrices $M(A, g)$ are Hadamard J^{ϕ^*}-multipliers and there is $\mathcal{D} > 0$ such that $\|M(A, g)\|_{J^{\phi^*}} \leqslant \mathcal{D}$. By Lemma 5.1 of [**17**], $M(A, g)$ are also Hadamard J_0^ϕ-multipliers and $\|M(A, g)\|_{J_0^\phi} \leqslant \mathcal{D}$. Hence all $M(A, g)$ are bounded on the pair (J_0^ϕ, J^{ϕ^*}). Since K is an interpolation space for this pair, $M(A, g)$ are Hadamard K-multipliers. By (3.1), $\|M(A, g)\|_K \leqslant C\mathcal{D}$. Hence, by Proposition 5.2 of [**17**], $\|[g(A), X]\|_K \leqslant D\|[A, X]\|_K$ for $A \in K_{nor}(\alpha)$, $X \in K$ and $D = C\mathcal{D}$.

Let $X \in C(H)$. Choose X_n in K such that $\|X - X_n\| \to 0$. Since g is commutator J^{ϕ^*}-bounded on α, it is J^{ϕ^*}-Lipschitzian. Hence it is Lipschitzian on α and it follows from Lemma 3.1 of [**17**] that $g(A) - g(0)1 \in K$. Hence

$$\begin{aligned} \|[g(A), X]\|_K &\leqslant \|[g(A), X_n]\|_K + \|[g(A) - g(0)\mathbf{1}, X - X_n]\|_K \\ &\leqslant D\|[A, X_n]\|_K + 2\|g(A) - g(0)\mathbf{1}\|_K \|X - X_n\|. \end{aligned}$$

Since $\|[A, X_n] - [A, X]\|_K \leqslant 2\|X - X_n\|\,\|A\|_K$, we have $\|[A, X_n]\|_K \to \|[A, X]\|_K$. Therefore, $\|[g(A), X]\|_K \leqslant D\|[A, X]\|_K$.

If I is J^ψ or J_0^ψ, then, by Proposition 3.4 of [**17**], g is commutator K-bounded on α. □

Corollary 3.4. *Let $p \in \{1, \infty, b\}$. Every commutator $\mathfrak{S}^p$-bounded function g on α is commutator J^ϕ-bounded on α for each norming function ϕ.*

Proof. By Corollary 5.4 of [**17**], g is commutator $\mathfrak{S}^1$- and $\mathfrak{S}^\infty$-bounded. Mityagin [**21**] obtained (see also Theorem 3.B in [**5**]) that any ideal J^ϕ is an interpolation space for the pair $(\mathfrak{S}^1, \mathfrak{S}^\infty)$. Applying Theorem 3.3, we complete the proof. □

For further applications of Theorem 3.3 we consider the Boyd indices (see [**2, 20**]) of s.n. ideals J. Set

$$\beta_J^*(n) = \inf_{X \in J} \frac{\|\overbrace{X \oplus \cdots \oplus X}^{n}\|_J}{\|X\|_J} \quad \text{and} \quad \gamma_J^*(n) = \sup_{X \in J} \frac{\|\overbrace{X \oplus \cdots \oplus X}^{n}\|_J}{\|X\|_J} \tag{3.2}$$

(see Remark 2.1 in [**17**]). The Boyd indices of J are defined by the formulae

$$p_J = \sup_n \left(\frac{\ln(n)}{\ln(\gamma_J^*(n))}\right) \quad \text{and} \quad q_J = \inf_n \left(\frac{\ln(n)}{\ln(\beta_J^*(n))}\right). \tag{3.3}$$

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For an s.n. function ϕ, the symmetric sequence space c_0^ϕ is the closure of the space $\hat{c}$ with respect to the norm $\|\xi\| = \phi(\xi)$, for $\xi \in \hat{c}$. Clearly, $c_0^\phi \subseteq c^\phi$ (see § 2). Consider the *dilation maps* D_n on c_0^ϕ defined by the formula

$$D_n(\xi) = \Bigl(\overbrace{\xi_1, \dots, \xi_1}^{n}, \overbrace{\xi_2, \dots, \xi_2}^{n}, \dots, \overbrace{\xi_k, \dots, \xi_k}^{n}, \dots\Bigr)$$

and their left inverses

$$D_{1/n}(\xi) = \frac{1}{n}\Biggl(\sum_{i=1}^{n} \xi_i, \sum_{i=n+1}^{2n} \xi_i, \dots, \sum_{i=(k-1)n+1}^{kn} \xi_i, \dots\Biggr),$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_k, \dots)$. The Boyd indices of c_0^ϕ are defined by the formulae

$$p_\phi = \sup_n \biggl(\frac{\ln(n)}{\ln \|D_n\|}\biggr) = \lim_{n\to\infty} \biggl(\frac{\ln(n)}{\ln \|D_n\|}\biggr),$$
$$q_\phi = \inf_n \biggl(\frac{\ln(1/n)}{\ln \|D_{1/n}\|}\biggr) = \lim_{n\to\infty} \biggl(\frac{\ln(1/n)}{\ln \|D_{1/n}\|}\biggr).$$

The norm $\|D_{1/n}\|$ can be expressed in terms of the operator D_n:

$$\|D_{1/n}\|^{-1} = \inf_{\xi \in c_0^\phi} \frac{\|D_n(\xi)\|}{\|\xi\|}. \tag{3.4}$$

Indeed, let π be the cyclic permutation *modulo* n of the set of positive integers: $\pi(k) = k+1$, if k is not divisible by n, and $\pi(k) = k - n + 1$, if k is divisible by n. For $\xi = (\xi_1, \dots, \xi_k, \dots)$, set $\Pi(\xi) = (\xi_{\pi(1)}, \dots, \xi_{\pi(k)}, \dots)$. Then

$$D_{1/n} D_n(\xi) = \xi \quad \text{and} \quad D_n D_{1/n}(\xi) = \frac{1}{n}(\xi + \Pi(\xi) + \Pi^2(\xi) + \dots + \Pi^{n-1}(\xi)).$$

Since $\phi(\xi) = \|\xi\| = \|\Pi^i(\xi)\|$, for all i, we have $\|D_n D_{1/n}(\xi)\| \leqslant \|\xi\|$. Denote by λ the right-hand side expression in (3.4). Then

$$\lambda \leqslant \inf_{\eta \in c_0^\phi} \frac{\|D_n D_{1/n}(\eta)\|}{\|D_{1/n}(\eta)\|} \leqslant \inf_{\eta \in c_0^\phi} \frac{\|\eta\|}{\|D_{1/n}(\eta)\|} = \biggl(\sup_{\eta \in c_0^\phi} \frac{\|D_{1/n}(\eta)\|}{\|\eta\|}\biggr)^{-1} = \|D_{1/n}\|^{-1}.$$

In order to finish the proof of (3.4), observe that, on the other hand,

$$\lambda = \inf_{\xi \in c_0^\phi} \biggl(\frac{\|D_{1/n} D_n(\xi)\|}{\|D_n(\xi)\|}\biggr)^{-1} = \biggl(\sup_{\xi \in c_0^\phi} \frac{\|D_{1/n} D_n(\xi)\|}{\|D_n(\xi)\|}\biggr)^{-1}$$
$$\geqslant \biggl(\sup_{\eta \in c_0^\phi} \frac{\|D_{1/n}(\eta)\|}{\|\eta\|}\biggr)^{-1} = \|D_{1/n}\|^{-1}.$$

Let $J = J_0^\phi$ be separable. For $X \in J$, let $s(X)$ be the non-increasing sequence of the eigenvalues of the operator $(X^*X)^{1/2}$. Then $s : X \to s(X)$ maps J onto c_0^ϕ, $\|X\|_J = \|s(X)\|$ and

$$s(\overbrace{X \oplus \dots \oplus X}^{n}) = D_n(s(X)).$$

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Using this and (3.4) we obtain

$$\|D_{1/n}\|^{-1} = \inf_{X\in J} \frac{\|D_n(s(X))\|}{\|s(X)\|} = \beta_J^*(n) \quad \text{and} \quad \|D_n\| = \sup_{X\in J} \frac{\|D_n(s(X))\|}{\|s(X)\|} = \gamma_J^*(n).$$

Therefore, $p_{J_0^\phi} = p_\phi$ and $q_{J_0^\phi} = q_\phi$. We also have

$$1 \leqslant \|D_{1/n}\|^{-1} \leqslant \|D_n\| \leqslant n \quad \text{and} \quad \|D_{n+1}\| \leqslant \|D_n\| + 1, \quad \text{so } p_\phi \leqslant q_\phi. \tag{3.5}$$

If $J \neq \mathfrak{S}^\infty$, then $\|D_n\| \to \infty$, as $n \to \infty$ (see [**12**, III.§ 3]).

For $\mathfrak{S}^r$, with $r \in [1,\infty)$, we have $\|D_{1/n}\| = \|D_n\| = n^{1/r}$, so $p_{\mathfrak{S}^r} = q_{\mathfrak{S}^r} = r$.

The Lorentz space $l_{r,q}$, for $1 \leqslant r, q < \infty$, consists of all sequences $\xi = (\xi_1, \xi_2, \dots) \in c_0$ such that

$$\|\xi\|_{r,q} = \phi(\xi) = \phi(\xi^*) = \left(\sum_{k=1}^{\infty} k^{(r/q)-1} (\xi_k^*)^r\right)^{1/r} < \infty,$$

where ξ^* is the non-increasing rearrangement of $(|\xi_1|, |\xi_2|, \dots)$. The Lorentz s.n. ideal $\mathfrak{S}^{r,q}$ (see [**4**]) consists of all compact operators X such that $\|X\|_{r,q} = \phi(s(X)) < \infty$.

Proposition 3.5. *For $J = \mathfrak{S}^{r,q}$, $p_J = q_J = r$.*

Proof. Set $\beta = (r/q) - 1$ and $\lambda_{k,n} = \sum_{i=1}^{n} [(k-1)n + i]^\beta$. We have

$$\|D_n\xi\|_{r,q}^r = \sum_{k=1}^{\infty} (\xi_k^*)^r \sum_{i=1}^{n} [(k-1)n + i]^\beta = \sum_{k=1}^{\infty} (\xi_k^*)^r \lambda_{k,n}. \tag{3.6}$$

(1) Let $r < q$, so that $-1 < \beta < 0$. Then

$$k^\beta n^{1+\beta} = n(kn)^\beta \leqslant \lambda_{k,n} \leqslant \sum_{i=1}^{n} (ki)^\beta = k^\beta \sum_{i=1}^{n} i^\beta \leqslant k^\beta \left(1 + \int_1^n x^\beta \,\mathrm{d}x\right) = \frac{n^{1+\beta}}{1+\beta} k^\beta.$$

Therefore,

$$n^{1+\beta} (\|\xi\|_{r,q})^r \leqslant (\|D_n\xi\|_{r,q})^r \leqslant \frac{n^{1+\beta}}{1+\beta} (\|\xi\|_{r,q})^r.$$

Thus

$$\|D_n\| \leqslant \left(\frac{n^{1+\beta}}{1+\beta}\right)^{1/r}$$

and, by (3.4), $n^{(1+\beta)/r} \leqslant \|D_{1/n}\|^{-1}$. Using (3.5), we have

$$q \leqslant \lim_{n\to\infty} \left(\frac{\ln(n)}{\ln \|D_n\|}\right) = p_\phi \leqslant q_\phi = \lim_{n\to\infty} \left(\frac{\ln(1/n)}{\ln \|D_{1/n}\|}\right) \leqslant q.$$

(2) If $r = q$, then $\mathfrak{S}^{q,q} = \mathfrak{S}^q$. By (3.6), $\|D_n\| = n^{1/q}$, so $p_\phi = q_\phi = q$.

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(3) Let $r > q$, so $0 < \beta$. Then

$$\frac{n^{1+\beta}}{1+\beta} k^{\beta} \leqslant \lambda_{k,n} \leqslant n^{1+\beta} k^{\beta},$$

since

$$\frac{n^{1+\beta}}{1+\beta}[k^{1+\beta} - (k-1)^{1+\beta}] = \int_0^n [(k-1)n + x]^{\beta} \, \mathrm{d}x \leqslant \lambda_{k,n} \leqslant n(kn)^{\beta} = n^{1+\beta} k^{\beta}.$$

Repeating the argument of part (1), we obtain that $p_\phi = q_\phi = q$. □

For $p \in (1, \infty)$, set

$$p_- = \min(p, p'), \quad p_+ = \max(p, p'), \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1. \tag{3.7}$$

Corollary 3.6. *Let J be a separable s.n. ideal. If $p_- < p_J$ and $q_J < p_+$, then each commutator $\mathfrak{S}^p$-bounded function g on $\alpha \subset \mathbb{C}$ is commutator J-bounded on α. In particular, g is commutator $\mathfrak{S}^q$- and $\mathfrak{S}^{r,q}$-bounded on α for $p_- < q < p_+$ and $1 \leqslant r < \infty$.*

Proof. Let $p < p'$. By Corollary 3.4 of [**2**], J is an interpolation space for $(\mathfrak{S}^p, \mathfrak{S}^{p'})$. Since $\phi_{p'} = (\phi_p)^*$ and $\mathfrak{S}^p \subset \mathfrak{S}^{p'}$, the result follows from Theorem 3.3. □

Recall that $\alpha \subset \mathbb{C}$ is J-Fuglede if the function $h(z) = \bar{z}$ is commutator J-bounded on α, that is, (1.4) holds.

Definition 3.7. An s.n. ideal J is called Fuglede if there is $C > 0$ such that, for all normal $A \in J$ and all $X \in B(H)$,

$$\|A^* X - X A^*\|_J \leqslant C \|AX - XA\|_J.$$

Evidently, an ideal J is Fuglede if and only if all compact subsets of $\mathbb{C}$ are J-Fuglede. This is equivalent to the condition that the unit disc of $\mathbb{C}$ is J-Fuglede. If J is Fuglede, it follows from Proposition 4.5 of [**17**] that the spaces of J-Lipschitz and of commutator J-bounded functions coincide on each compact in $\mathbb{C}$. It was proved in [**1, 24, 25**] that all Schatten ideals $\mathfrak{S}^p$, $1 < p < \infty$, are Fuglede. It follows from the results of [**15**] that $\mathfrak{S}^b = B(H)$ is not Fuglede. From this and from Corollary 5.4 of [**17**] and Corollary 2.5 we obtain the following result.

Corollary 3.8.

(i) *Let ϕ^* be the adjoint of ϕ. If one of the ideals J^ϕ, J^{ϕ^*}, J_0^ϕ, $J_0^{\phi^*}$ is Fuglede, then the others are also Fuglede ideals.*

(ii) *If $1 < p_J$ and $q_J < \infty$, for a separable ideal J, then J is a Fuglede ideal. In particular, all Lorentz ideals $\mathfrak{S}^{r,q}$ with $q > 1$ are Fuglede ideals.*

(iii) *The ideals $\mathfrak{S}^1$, $\mathfrak{S}^\infty$ and $\mathfrak{S}^b$ are not Fuglede ideals.*

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Problem 3.9. Are the Lorentz ideals $\mathfrak{S}^{r,1}$ Fuglede ideals if $1 < r$?

We now return to the subject of J-Lipschitz functions. Combining the results of Proposition 4.5 of [**17**], of Corollaries 3.6 and 5.4 of [**17**], and of Corollaries 2.3 and 2.5 yields the following corollary.

Corollary 3.10. *Let g be a $\mathfrak{S}^p$-Lipschitz function on $\alpha \subset \mathbb{C}$.*

(i) *Let $p \in (1, \infty)$. If $p_- < p_J$ and $q_J < p_+$, for a separable ideal $J = J_0^\phi$, then g is a J_0^ϕ-, $J_0^{\phi^*}$-, J^ϕ- and J^{ϕ^*}-Lipschitz function on α. In particular, g is $\mathfrak{S}^{r,q}$-Lipschitzian on α for $p_- < q < p_+$ (see Proposition 3.5).*

(ii) *Let $p \in \{1, \infty, b\}$ and let α be a $\mathfrak{S}^p$-Fuglede set. Then g is J_0^ϕ- and J^ϕ-Lipschitzian on α for any ϕ.*

Let J-$\mathrm{Lip}(a, b)$ be the space of all J-Lipschitz functions on $[a, b]$. By Corollary 3.6 of [**17**], functions in $\mathfrak{S}^b$-$\mathrm{Lip}(a, b)$ are differentiable. For $p \in (1, \infty)$, the spaces $\mathfrak{S}^p$-$\mathrm{Lip}(a, b)$ are larger than $\mathfrak{S}^b$-$\mathrm{Lip}(a, b)$ and contain Lipschitz, non-differentiable functions (see [**9**]). Peller [**22**] showed that $B^1_{\infty 1}(a, b) \subseteq \mathfrak{S}^b\text{-}\mathrm{Lip}(a, b) \subseteq B^1_{11}(a, b)$, where $B^1_{\infty 1}(a, b)$ and $B^1_{11}(a, b)$ are Besov classes of functions on $[a, b]$ (for another proof and some interesting related results, see [**7**]). Combining this with Proposition 5.5 (ii) of [**17**] and with Corollary 2.9 (ii) yields the following corollary.

Corollary 3.11. *For any s.n. function ϕ,*

$$\begin{aligned} B^1_{\infty 1}(a, b) \subseteq \mathfrak{S}^b\text{-}\mathrm{Lip}(a, b) = \mathfrak{S}^1\text{-}\mathrm{Lip}(a, b) \subseteq J^\phi\text{-}\mathrm{Lip}(a, b) = J_0^\phi\text{-}\mathrm{Lip}(a, b) \\ = J^{\phi^*}\text{-}\mathrm{Lip}(a, b) = J_0^{\phi^*}\text{-}\mathrm{Lip}(a, b) \subseteq \mathfrak{S}^2\text{-}\mathrm{Lip}(a, b) = \mathrm{Lip}(a, b). \end{aligned}$$

4. Extension of domains of inequalities (1.2) and (1.3)

The normal operators in the definitions of J-Lipschitz and of commutator J-bounded functions (see (1.2) and (1.3)) belong to the ideal J. The question arises as to whether inequalities (1.2) and (1.3) hold for all normal operators in $B(H)$. More precisely, we study for which s.n. ideals J the following properties hold.

(i) If g is J-Lipschitzian on α, then there is $\mathcal{D} > 0$ such that, for all normal operators A, B with spectrum in α, the condition $A - B \in J$ implies

$$g(A) - g(B) \in J \quad \text{and} \quad \|g(A) - g(B)\|_J \leqslant \mathcal{D}\|A - B\|_J.$$

(ii) If g is commutator J-bounded on α, then there is $\mathcal{D} > 0$ such that, for any normal A with $\mathrm{Sp}(A) \subseteq \alpha$ and any X in $B(H)$, the condition $[A, X] \in J$ implies

$$[g(A), X] \in J \quad \text{and} \quad \|[g(A), X]\|_J \leqslant \mathcal{D}\|[A, X]\|_J.$$

It will be shown that (i) and (ii) hold for all ideals J^ϕ and J_0^ϕ. This extends the result of Kittaneh [**18**] (see also Jocic [**14**]), who considered the case $J = \mathfrak{S}^2$.

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Let $J = J_0^\phi$. For $A \in B(H)$, the operator $\delta_A : X \to [A, X]$ is bounded on J. If $T \in J^{\phi^*}$, it follows from (2.1) that, for $X \in J$,

$$F_T(\delta_A(X)) = \mathrm{Tr}(AXT) - \mathrm{Tr}(XAT) = \mathrm{Tr}(XTA) - \mathrm{Tr}(XAT) = F_{-\delta_A(T)}(X). \qquad (4.1)$$

Let L be a $*$-subspace of J: $X \in L$ implies $X^* \in L$. Denote by $L^\perp$ its annihilator in J^{ϕ^*}:

$$L^\perp = \{T \in J^{\phi^*} : F_T(X) = \mathrm{Tr}(XT) = 0 \text{ for } X \in L\}.$$

If $T \in L^\perp$, then

$$F_{T^*}(X) = \mathrm{Tr}(T^*X) = \mathrm{Tr}((X^*T)^*) = \overline{\mathrm{Tr}(X^*T)} = \overline{F_T(X^*)} = 0,$$

for $X \in L$. Therefore, $T^* \in L^\perp$, so $L^\perp$ is a $*$-subspace of J^{ϕ^*}.

Let A be a normal operator and $\{A\}'$ be its commutant. Let $\{\mu_i\}_{i\in I}$ be the set of all eigenvalues of A and let Q_i be the projections on the corresponding eigenspaces. Then

$$B(H) \ni X \to \Psi_A(X) = \sum_{i\in I} Q_i X Q_i$$

is a map from $B(H)$ into $\{A\}'$ and $(\Psi_A)^2 = \Psi_A$. It follows from Theorem III.4.2 of [**12**] that $\Psi_A(X) \in J$, for $X \in J$, and $\|\Psi_A(X)\|_J \leqslant \|X\|_J$. Hence Ψ_A maps J into $\{A\}' \cap J$ and $\|\Psi_A\| \leqslant 1$. By $\overline{\delta_A(J)}$ we denote the closure of $\delta_A(J)$ in $\|\cdot\|_J$.

We will now generalize Theorem 2.2 of [**16**] to all separable s.n. ideals.

Theorem 4.1. *Let $J = J_0^\phi \neq \mathfrak{S}^1$ and let A be a normal operator in $B(H)$. Then*

(i) $\mathrm{Ker}(\delta_A) = \{A\}' \cap J = \{X \in J : \Psi_A(X) = X\}$ *so Ψ_A is a projection on* $\mathrm{Ker}(\delta_A)$*;*

(ii) $\mathrm{Ker}(\Psi_A) = \overline{\delta_A(J)}$ *and J is the direct sum of $\overline{\delta_A(J)}$ and* $\mathrm{Ker}(\delta_A)$.

Proof. Since Ψ_A maps J to $\{A\}' \cap J$, we have $\{X \in J : \Psi_A(X) = X\} \subseteq \{A\}' \cap J$.

Conversely, let $X = X^* \in \{A\}' \cap J$. Then $X = \bigoplus_{j=1}^\infty \lambda_j P_j$, where P_j are the mutually orthogonal projections on the finite-dimensional eigenspaces L_j of X. All $P_j \in \{A\}' \cap J$. Fix j. The subspace L_j is invariant for A and, therefore, decomposes in the orthogonal sum of eigenspaces of the operator $A|_{L_j}$. Hence $P_j = \Psi_A(P_j)$.

The finite sums $X_n = \bigoplus_{j=1}^n \lambda_j P_j$ converge to X in $\|\cdot\|_J$. Since $\|\Psi_A\| \leqslant 1$,

$$\begin{aligned}\|X - \Psi_A(X)\|_J &\leqslant \|X - X_n\|_J + \|\Psi_A(X) - X_n\|_J \\ &= \|X - X_n\|_J + \left\|\Psi_A\left(X - \bigoplus_{j=1}^n \lambda_j P_j\right)\right\|_J \leqslant 2\|X - X_n\|_J \to 0.\end{aligned}$$

Hence $\Psi_A(X) = X$. By Fuglede's theorem, $\{A\}' \cap J$ is a $*$-algebra. Thus $\{A\}' \cap J \subseteq \{X \in J : \Psi_A(X) = X\}$ and part (i) is proved.

Let us show that $L = \delta_A(J) + (\{A\}' \cap J)$ is dense in J. We have

$$L^\perp = \delta_A(J)^\perp \cap (\{A\}' \cap J)^\perp.$$

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Since $\{A\}' \cap J$ is a $*$-subspace, $(\{A\}' \cap J)^{\perp}$ is a $*$-subspace of J^{ϕ^*}. If $T \in \delta_A(J)^{\perp}$, it follows from (4.1) that $\delta_A(T) = 0$. Hence $T \in \{A\}' \cap J^{\phi^*}$, so $(\delta_A(J))^{\perp} = \{A\}' \cap J^{\phi^*}$ is a $*$-algebra. Thus $L^{\perp}$ is a $*$-subspace of $\{A\}' \cap J^{\phi^*}$.

Let $T = T^* \in L^{\perp} \subseteq J^{\phi^*}$. It is compact, so $T = \bigoplus_{i=1}^{\infty} \lambda_i P_i$, where the P_i are mutually orthogonal finite-dimensional projections. Since $T \in \{A\}'$, all P_i belong to $\{A\}' \cap J \subseteq L$. Hence $0 = F_T(P_i) = \mathrm{Tr}(P_i T) = \lambda_i \dim(P_i)$. Thus $T = 0$. Since $L^{\perp}$ is a $*$-space, $L^{\perp} = \{0\}$, so L is dense in J.

We now proceed with the proof of (ii). Since Ψ_A is a projection on $\{A\}' \cap J$,

$$J = \mathrm{Ker}(\Psi_A) \dotplus (\{A\}' \cap J). \tag{4.2}$$

Taking into account that $Q_i A = A Q_i = \mu_i Q_i$, for all Q_i, we obtain that

$$\Psi_A(\delta_A(X)) = \sum_{i \in I} (Q_i A X Q_i - Q_i X A Q_i) = \sum_{i \in I} (\mu_i Q_i X Q_i - \mu_i Q_i X Q_i) = 0$$

for any $X \in J$. Since $\|\Psi_A\| \leqslant 1$, we conclude that $\overline{\delta_A(J)} \subseteq \mathrm{Ker}(\Psi_A)$.

Let $X \in \mathrm{Ker}(\Psi_A)$. Since L is dense in J, there are $X_n \in \delta_A(J)$ and $Y_n \in \{A\}' \cap J$ such that $\|X - X_n - Y_n\|_J \to 0$. Since $\Psi_A(Y_n) = Y_n$ and $\Psi_A(X_n) = 0$, it follows that

$$\|Y_n\|_J = \|\Psi_A(Y_n)\|_J = \|\Psi_A(X - X_n - Y_n)\|_J \leqslant \|X - X_n - Y_n\|_J \to 0.$$

Hence $\|X - X_n\|_J \to 0$, so $\mathrm{Ker}(\Psi_A) = \overline{\delta_A(J)}$ and part (ii) follows from (4.2). □

Theorem 1 of **[24]** is a special case of the following result.

Proposition 4.2. *Let $\mathfrak{Y}$ be a linear manifold in a Banach space $\mathfrak{X}$ and a Banach space with norm $\|\cdot\|_{\mathfrak{Y}}$. Let S and T be commuting operators on $\mathfrak{X}$. Assume that*

(i) $\mathrm{Ker}(S) \subseteq \mathrm{Ker}(T)$;

(ii) $\mathrm{Ker}(T) \cap \overline{T\mathfrak{X}} = \{0\}$, *where $\overline{T\mathfrak{X}}$ is the norm closure of $T\mathfrak{X}$;*

(iii) *S and T preserve $\mathfrak{Y}$ and the operators $S|_{\mathfrak{Y}}$ and $T|_{\mathfrak{Y}}$ are bounded on $\mathfrak{Y}$;*

(iv) $\|y\|_{\mathfrak{X}} \leqslant \|y\|_{\mathfrak{Y}}$ *and* $\|Ty\|_{\mathfrak{Y}} \leqslant \|Sy\|_{\mathfrak{Y}}$, *for $y \in \mathfrak{Y}$;*

(v) *$\mathfrak{Y}$ is the direct sum of $\mathrm{Ker}(S|_{\mathfrak{Y}})$ and the closure $\overline{S\mathfrak{Y}}$ of $S\mathfrak{Y}$ in $\|\cdot\|_{\mathfrak{Y}}$.*

Let P be the projection in $\mathfrak{Y}$ on $\mathrm{Ker}(S|_{\mathfrak{Y}})$ along $\overline{S\mathfrak{Y}}$. Then

$$Sx \in \mathfrak{Y}, \quad \text{for } x \in \mathfrak{X}, \quad \text{implies } Tx \in \mathfrak{Y} \text{ and } \|Tx\|_{\mathfrak{Y}} \leqslant (1 + \|P\|)\|Sx\|_{\mathfrak{Y}}.$$

Proof. Define an operator U on $\mathrm{Ker}(S|_{\mathfrak{Y}}) \dotplus S\mathfrak{Y}$ by the formula

$$Uz = 0, \quad \text{for } z \in \mathrm{Ker}(S|_{\mathfrak{Y}}), \quad \text{and} \quad USy = Ty, \quad \text{for } y \in \mathfrak{Y}.$$

By (iv),

$$\|U(z + Sy)\|_{\mathfrak{Y}} = \|Ty\|_{\mathfrak{Y}} \leqslant \|Sy\|_{\mathfrak{Y}} \leqslant \|z + Sy\|_{\mathfrak{Y}} + \|P(z + Sy)\|_{\mathfrak{Y}} \leqslant (1 + \|P\|)\|z + Sy\|_{\mathfrak{Y}}.$$

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Hence U extends to a bounded operator on $\mathfrak{Y}$ and $\|U\| \leqslant 1+\|P\|$. Since S, T commute and $\mathrm{Ker}(S) \subseteq \mathrm{Ker}(T)$, we have $SU(z+Sy) = STy = TSy = T(z+Sy)$. Thus

$$T|_{\mathfrak{Y}} = US|_{\mathfrak{Y}} = SU|_{\mathfrak{Y}}.$$

Let $x \in \mathfrak{X}$ and $Sx \in \mathfrak{Y}$. Then $S(Tx - USx) = STx - SUSx = STx - TSx = 0$. Hence $Tx - USx \in \mathrm{Ker}(S)$. By (i), $Tx - USx \in \mathrm{Ker}(T)$. Since $Sx \in \mathfrak{Y}$, there are $z \in \mathrm{Ker}(S|_{\mathfrak{Y}})$ and $y_n \in \mathfrak{Y}$ such that $\|Sx - (z + Sy_n)\|_{\mathfrak{Y}} \to 0$. Therefore,

$$\|USx - Ty_n\|_{\mathfrak{X}} \leqslant \|USx - Ty_n\|_{\mathfrak{Y}} = \|USx - U(z+Sy_n)\|_{\mathfrak{Y}} \leqslant \|U\|\,\|Sx-(z+Sy_n)\|_{\mathfrak{Y}} \to 0.$$

Thus $USx \in \overline{T\mathfrak{X}}$, so $Tx - USx \in \mathrm{Ker}(T) \cap \overline{T\mathfrak{X}}$. It follows from (ii) that $Tx = USx \in \mathfrak{Y}$ and $\|Tx\|_{\mathfrak{Y}} \leqslant \|U\|\,\|Sx\|_{\mathfrak{Y}} \leqslant (1+\|P\|)\|Sx\|_{\mathfrak{Y}}$. □

A bounded operator T on a Banach space $\mathfrak{X}$ is called *hermitian*, if $\|\exp(\mathrm{i}tT)\|_{\mathfrak{X}} = 1$ for $t \in \mathbb{R}$. It is called *normal* if $T = A + \mathrm{i}B$, where A and B are commuting hermitian operators. Fong [**11**] showed that if T is normal, then

$$\mathrm{Ker}(T) \cap \overline{T\mathfrak{X}} = \{0\}. \tag{4.3}$$

Let K be a self-adjoint operator in $B(H)$. The operator $\delta_K(X) = KX - XK$ is hermitian on any s.n. ideal J, since

$$\|\exp(\mathrm{i}t\delta_K)X\|_J = \|\mathrm{e}^{\mathrm{i}tK}X\mathrm{e}^{-\mathrm{i}tK}\|_J = \|X\|_J \quad \text{for } X \in J.$$

If S is a normal operator on H, $S = K + \mathrm{i}L$, where the operators K and L are self-adjoint and commute. Then the operators δ_K and δ_L on J are hermitian and commute, so the operator $\delta_S = \delta_K + \mathrm{i}\delta_L$ is normal on J. We will show now that 'extended' inequalities (1.2) and (1.3) hold for all separable s.n. ideals.

Theorem 4.3. *Let J be a separable s.n. ideal and let g be a commutator J-bounded function on $\alpha \subset \mathbb{C}$. There is $\mathcal{D} > 0$ such that, for $X \in B(H)$ and normal operators A, B with spectra in α, $AX - XB \in J$ implies*

$$g(A)X - Xg(B) \in J \quad \textit{and} \quad \|g(A)X - Xg(B)\|_J \leqslant \mathcal{D}\|AX - XB\|_J.$$

Proof. Since g is commutator J-bounded, it follows from Proposition 3.4 (iii) of [**17**] that there is $D > 0$ such that, for all normal A with $\mathrm{Sp}(A) \subseteq \alpha$ and all $X \in J$,

$$\|\delta_{g(A)}(X)\|_J = \|[g(A), X]\|_J \leqslant D\|[A, X]\|_J = D\|\delta_A(X)\|_J. \tag{4.4}$$

Let $J \neq \mathfrak{S}^1$. The operator $g(A)$ is normal, so that the operators δ_A, $\delta_{g(A)}$ are normal on J and on $B(H)$, commute and $\mathrm{Ker}(\delta_A) \subseteq \mathrm{Ker}(\delta_{g(A)})$. It follows from Theorem 4.1 that $J = \mathrm{Ker}(\delta_A|_J) \dot{+} \overline{\delta_A(J)}$ and the projection on $\mathrm{Ker}(\delta_A|_J)$ has norm 1. By (4.3), $\mathrm{Ker}(\delta_{g(A)}) \cap \overline{\delta_{g(A)}(B(H))} = \{0\}$. Replacing T in Proposition 4.2 by $\delta_{g(A)}$, S by δ_A, $\mathfrak{X}$ by $B(H)$ and $\mathfrak{Y}$ by J, we obtain that, for any $X \in B(H)$,

$$[A, X] \in J \quad \text{implies } [g(A), X] \in J \text{ and } \|[g(A), X]\|_J \leqslant 2D\|[A, X]\|_J.$$

Repeating now the argument of Proposition 4.1 of [**17**], we complete the proof of the case $J \neq \mathfrak{S}^1$. The case when $J = \mathfrak{S}^1$ will be considered in Theorem 4.5. □

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To prove an analogue of Theorem 4.3 for all ideals J^ϕ, we need the following result.

Lemma 4.4. *Let $J \subset I \subseteq C(H)$ and let $J = J^\phi$. Then there are s.n. ideals $\{J(t)\}_{t\in[0,1]}$ such that*

(i) *$J = J(1) \subseteq J(t) \subseteq J(s) \subseteq J(0) = I$, for $s \leqslant t$ in $(0,1)$;*

(ii) *all functions $\varphi_X(t) = \|X\|_{J(t)}$ are continuous on $[0,1]$ for $X \in J$;*

(iii) *all $J(t)$ are interpolation spaces for the pair (J,I) and if T is a bounded operator on (J,I), then $\|T\|_{J(t)} \leqslant \|T\|_I^t \|T\|_J^{1-t}$ for $t \in [0,1]$;*

(iv) *if I is reflexive, then all ideals $J(t)$, $t \in (0,1)$, are reflexive.*

Proof. It follows from Theorem III.5.1 of [**12**] that the unit ball $\mathbf{J}_1$ of J is closed in $(C(H), \|\cdot\|)$. By Proposition 2.1, there is $c > 0$ such that $\|X\| \leqslant c\|X\|_I$ for $X \in I$. This implies that $\mathbf{J}_1$ is closed in I in $\|\cdot\|_I$.

By Proposition 2.1 (i), there is $C > 0$ such that $\|X\|_I \leqslant C\|X\|_J$, for $X \in J$. Consider an equivalent norm $\|X\|_J' = C\|X\|_J$ on J. Then $\|X\|_I \leqslant \|X\|_J'$. It follows from Theorems IV.1.2 and IV.1.8 of [**19**] that there are Banach spaces $(J(t), \|\cdot\|_{J(t)})$, for $t \in [0,1]$, which satisfy (i), (ii) and (iii) and such that $\|X\|_{J(s)} \leqslant \|X\|_{J(t)}$, for $s \leqslant t$ and $X \in J(t)$. By Lemma 3.2, all $(J(t), \|\cdot\|_{J(t)})$ are s.n. ideals.

If I is reflexive, it follows from Theorem IV.1.4 of [**19**] that all s.n. ideals $J(t)$, $t \in (0,1)$, are also reflexive. □

All compact operators B such that $\sum_{n=1}^\infty n^{-1/2} s_n(B) < \infty$, where $s_n(B)$ are eigenvalues of $(B^*B)^{1/2}$, form a separable ideal $\mathfrak{S}^2_-$ (see [**12**, § 15]) contained in $\mathfrak{S}^2$. Let J be an s.n. ideal not contained in $\mathfrak{S}^2_-$. Bercovici and Voiculescu proved in [**3**] that, for any normal A in $B(H)$, there are finite-rank positive operators R_n such that

$$\text{the } R_n \text{ strongly converge to } \mathbf{1}, \quad \|R_n\| \leqslant 1 \quad \text{and} \quad \lim \|[A, R_n]\|_J = 0. \tag{4.5}$$

Theorem 4.5. *Let $J = J^\phi$. If g is commutator J-bounded on $\alpha \subset \mathbb{C}$, then there is $\mathcal{D} > 0$ such that, for all normal operators A, B with spectra in α and all $X \in B(H)$,*

$$AX - XB \in J \quad \text{implies } g(A)X - Xg(B) \in J \text{ and } \|g(A)X - Xg(B)\|_J \leqslant \mathcal{D}\|AX - XB\|_J.$$

Proof. We will prove the theorem for $A = B$. Then the general case for different A and B will follow as in the proof of Proposition 4.1 of [**17**].

(1) Let $J \nsubseteq \mathfrak{S}^2_-$. By Proposition 3.4 of [**17**], there is $D > 0$ such that $\|[g(A), Y]\|_J \leqslant D\|[A, Y]\|_J$ for all normal A with $\mathrm{Sp}(A) \subseteq \alpha$ and all $Y \in J$. Fix A. Let finite-rank operators R_n satisfy (4.5). Then, for all $X \in B(H)$,

$$\|[g(A), R_n]\|_J \leqslant D\|[A, R_n]\|_J \to 0 \quad \text{and} \quad \|[g(A), R_n X]\|_J \leqslant D\|[A, R_n X]\|_J.$$

Taking this into account, we have from Lemma 3.3 of [**17**]

$$\begin{aligned} \|[g(A), X]\|_J &= \lim \|R_n[g(A), X]\|_J = \lim \|[g(A), R_n X] - [g(A), R_n]X\|_J \\ &\leqslant D \lim \|[A, R_n X]\|_J \leqslant D \lim \|[A, R_n]X\|_J + D \lim \|R_n[A, X]\|_J \\ &= D\|[A, X]\|_J. \end{aligned}$$

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(2) Let $J \subseteq \mathfrak{S}^2_-$. By (1.3), there is $D > 0$ such that $\|[g(A), X]\|_J \leqslant D\|[A, X]\|_J$ for all $A \in J_{\mathrm{nor}}(\alpha)$ and $X \in B(H)$. Since A are diagonal, by Proposition 5.2 of [**17**], the matrices $M(A, g)$ (see § 3) are Hadamard J-multipliers and $\|M(A, g)\|_J \leqslant 2D$. Since g is a Lipschitz function, we have from (5.1) and (5.2) of [**17**] that $M(A, g)$ are also Hadamard $\mathfrak{S}^2$-multipliers and $\|M(A, g)\|_{\mathfrak{S}^2} \leqslant \|M(A, g)\|_J$.

Since $J \subset \mathfrak{S}^2$ and $\mathfrak{S}^2$ is reflexive, there are s.n. ideals $J(t)$, $t \in [0, 1]$, which satisfy conditions of Lemma 4.4: $J = J(1)$, $\mathfrak{S}^2 = J(0)$ and $J(t)$ are reflexive for $t \in (0, 1)$. By Lemma 4.4 (iii), the $M(A, g)$ are also Hadamard $J(t)$-multipliers, for $t \in (0, 1)$, and $\|M(A, g)\|_{J(t)} \leqslant 2D$. It follows from Proposition 5.2 of [**17**] that

$$\|[g(A), X]\|_{J(t)} \leqslant 2D\|[A, X]\|_{J(t)} \quad \text{for } A \in J_{\mathrm{nor}}(\alpha) \text{ and } X \in J(t). \tag{4.6}$$

By Proposition 2.1 (iii), all s.n. ideals $J(t)$, $t \in (0, 1)$, are separable. It follows from Proposition 3.4 of [**17**] that g is commutator $J(t)$-bounded and (4.6) holds for all normal operators A with $\mathrm{Sp}(A) \subseteq \alpha$ and all $X \in J(t)$. Since all $J(t)$ are reflexive, $J(t) \neq \mathfrak{S}^1$. It was proved in Theorem 4.3 that $[A, X] \in J(t)$ implies

$$[g(A), X] \in J(t) \quad \text{and} \quad \|[g(A), X]\|_{J(t)} \leqslant 4D\|[A, X]\|_{J(t)}, \tag{4.7}$$

for $t \in (0, 1)$ and for all normal A with $\mathrm{Sp}(A) \subseteq \alpha$ and all $X \in B(H)$.

If $[A, X] \in J$, then $[A, X] \in J(t)$, for $t \in (0, 1)$, so (4.7) holds. By (1.1) and (4.7), $\|[g(A), X]P\|_{J(t)} \leqslant 4D\|[A, X]\|_{J(t)}$ for any finite-dimensional projection P. It follows from Lemma 4.4 (ii) that

$$\|[g(A), X]P\|_J = \lim_{t \to 1} \|[g(A), X]P\|_{J(t)} \leqslant 4D \lim_{t \to 1} \|[A, X]\|_{J(t)} = 4D\|[A, X]\|_J.$$

Hence, by Theorem III.5.1 of [**12**], $[g(A), X] \in J$ and $\|[g(A), X]\|_J \leqslant 4D\|[A, X]\|_J$.

This completes the proof of the theorem in the case when $A = B$. □

To prove the results of Theorems 4.3 and 4.5 for J-Lipschitz functions we, as usual, have to restrict our consideration either to Fuglede ideals or to J-Fuglede sets.

Corollary 4.6. *Let J be J^{ϕ} or J^{ϕ}_0 and let g be a J-Lipschitz function on a J-Fuglede set α (for example, $\alpha \subset \mathbb{R}$). Then there is $\mathcal{D} > 0$ such that, for all normal operators A, B with spectra in α and all $X \in B(H)$, $AX - XB \in J$ implies*

$$g(A)X - Xg(B) \in J \quad \text{and} \quad \|g(A)X - Xg(B)\|_J \leqslant \mathcal{D}\|AX - XB\|_J.$$

In particular, $A - B \in J$ implies $g(A) - g(B) \in J$ and $\|g(A) - g(B)\|_J \leqslant \mathcal{D}\|A - B\|_J$.

5. *J*-stable and commutator *J*-stable functions

In this section we study J-stable functions.

Definition 5.1. Let g be a function on $\alpha \subset \mathbb{C}$ and let J be an s.n. ideal.

(i) g is called J-stable if, for all normal operators A, B with spectra in α, the condition $A - B \in J$ implies $g(A) - g(B) \in J$.

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(ii) g is called commutator J-stable if, for all normal operators A with $\mathrm{Sp}(A) \subseteq \alpha$ and all $X \in B(H)$, the condition $[A, X] \in J$ implies $[g(A), X] \in J$.

J-stable functions on $\mathbb{R}$ may be considered to be acting on $B(H)/J$. We show that in many important cases ($J = \mathfrak{S}^p$, $p \in (1, \infty)$, for example) a function is J-stable if and only if it is J-Lipschitzian.

It follows from Theorems 4.3 and 4.5 that if J is J^ϕ or J_0^ϕ, then commutator J-bounded functions are J-stable and commutator J-stable. If J is Fuglede, then, by Corollary 4.6, J-Lipschitz functions are J-stable and commutator J-stable. We will study the converse inclusion.

The following result establishes an important relation between the classes of J-stable and commutator J-stable functions. In particular, it shows that all commutator J-stable functions are J-stable.

Proposition 5.2. *Let J be an s.n. ideal and let g be a continuous function on $\alpha \subset \mathbb{C}$. The following conditions are equivalent:*

(i) *g is J-stable on α;*

(ii) *for any normal operator A with $\mathrm{Sp}(A) \subseteq \alpha$ and any $X = X^* \in B(H)$, the condition $[A, X] \in J$ implies $[g(A), X] \in J$.*

Proof. (i) $\Longrightarrow$ (ii). Let A be normal, $\mathrm{Sp}(A) \subseteq \alpha$ and let U be a unitary operator such that $[A, U] \in J$. Then $A - UAU^* = [A, U]U^* \in J$ and $\mathrm{Sp}(A) = \mathrm{Sp}(UAU^*)$. If g is J-stable, $g(A) - g(UAU^*) \in J$. Since $g(UAU^*) = Ug(A)U^*$, we have $[g(A), U] \in J$.

Let $X = X^*$, $\|X\| < 1$ and $[A, X] \in J$. The operator $U = X + \mathrm{i}(\mathbf{1} - X)^{1/2}$ is unitary. Since $f(t) = t + \mathrm{i}(1 - t)^{1/2}$ is an analytic function in a neighbourhood of $\mathrm{Sp}(X)$, it follows from Example 4.2 of [**17**] that $[A, U] = [A, f(X)] \in J$. By the above argument, $[g(A), U] \in J$. Similarly, $[g(A), U^*] \in J$, so that

$$[g(A), X] = \tfrac{1}{2}([g(A), U] + [g(A), U^*]) \in J.$$

From this it follows that, for each $X = X^*$, $[A, X] \in J$ implies $[g(A), X] \in J$.

(ii) $\Longrightarrow$ (i). Let A, B be normal operators with spectra in α and let $A - B \in J$. The operator

$$R = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is normal and $\mathrm{Sp}(R) \subseteq \alpha$. Set

$$X = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

Then $X = X^*$ and

$$[R, X] = \begin{pmatrix} 0 & A - B \\ B - A & 0 \end{pmatrix} \in J, \quad \text{so } [g(R), X] = \begin{pmatrix} 0 & g(A) - g(B) \\ g(B) - g(A) & 0 \end{pmatrix} \in J.$$

Hence $g(A) - g(B) \in J$. □

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Definition 5.3. Let $h(z) = \bar{z}$.

(i) A compact α in $\mathbb{C}$ is called weakly J-Fuglede if the function h is commutator J-stable on α, that is, for all normal A with $\mathrm{Sp}(A) \subseteq \alpha$ and all X in $B(H)$, $[A, X] \in J$ implies $[A^*, X] \in J$.

(ii) An s.n. ideal J is called weakly Fuglede if any compact subset of $\mathbb{C}$ is weakly J-Fuglede, that is, for all normal A and all $X \in B(H)$, $[A, X] \in J$ implies $[A^*, X] \in J$.

Clearly, any compact α in $\mathbb{R}$ is weakly J-Fuglede for any ideal J.

If J is J^ϕ or J_0^ϕ and $\alpha \subset \mathbb{C}$ is a J-Fuglede set, then the function h is commutator J-bounded on α. It follows from Theorems 4.3 and 4.5 that h is commutator J-stable, so α is weakly J-Fuglede.

Similarly, if J is a Fuglede ideal, it is also weakly Fuglede. The converse statement, however, is not true, which justifies our terminology.

Proposition 5.4. *The ideal $\mathfrak{S}^\infty$ is a weakly Fuglede but not Fuglede ideal.*

Proof. By Corollary 3.8, $\mathfrak{S}^\infty$ is not Fuglede. Let A be normal and $[A, X] \in \mathfrak{S}^\infty$. Let $\hat{A}$ and $\hat{X}$ be their images in the Calkin algebra $B(H)/\mathfrak{S}^\infty$. Then $[\hat{A}, \hat{X}] = 0$. By the Fuglede theorem for C^*-algebras, $[\hat{A}^*, \hat{X}] = 0$, so $[A^*, X] \in \mathfrak{S}^\infty$. □

We conclude from Proposition 5.4 that every compact in $\mathbb{C}$ is weakly $\mathfrak{S}^\infty$-Fuglede, but not every compact is $\mathfrak{S}^\infty$-Fuglede.

Proposition 5.5. *Let g be a continuous function on α. If α is weakly J-Fuglede, then the following conditions are equivalent:*

(i) *g is J-stable on α;*

(ii) *g is commutator J-stable on α.*

Proof. (ii) $\Longrightarrow$ (i). This follows from Proposition 5.2.

(i) $\Longrightarrow$ (ii). Let $X = Y + \mathrm{i}Z$, $Y = Y^*$, $Z = Z^*$, and let $[A, X] \in J$ for a normal A with $\mathrm{Sp}(A) \subseteq \alpha$. Since α is weakly J-Fuglede, $[A, X^*] \in J$. Hence $[A, Y], [A, Z] \in J$ and, by Proposition 5.2, $[g(A), Y] \in J$ and $[g(A), Z] \in J$. Therefore, $[g(A), X] \in J$. □

Let J be an s.n. ideal. For any n, set

$$\beta_J(n) = \inf_{X \in \mathcal{F}} \frac{\Big\| \overbrace{X \oplus \cdots \oplus X}^{n} \Big\|_J}{\|X\|_J} \quad \text{and} \quad \gamma_J(n) = \sup_{X \in \mathcal{F}} \frac{\Big\| \overbrace{X \oplus \cdots \oplus X}^{n} \Big\|_J}{\|X\|_J}.$$

Then (see (3.2)) $\beta_J^*(n) \leqslant \beta_J(n) \leqslant \gamma_J(n) \leqslant \gamma_J^*(n)$. If J is separable, then $\beta_J^*(n) = \beta_J(n)$ and $\gamma_J(n) = \gamma_J^*(n)$. We say that J is *regular*, if

$$J \subsetneq \mathfrak{S}^\infty \quad \text{and} \quad \varliminf_{n \to \infty} \frac{\beta_J(n)}{\gamma_J(n)} > 0. \tag{5.1}$$

All ideals $\mathfrak{S}^p$, $p \in [1, \infty)$, are regular with $\beta_{\mathfrak{S}^p}(n) = \gamma_{\mathfrak{S}^p}(n) = n^{1/p}$. All Lorentz ideals $J = \mathfrak{S}^{p,q}$, $1 \leqslant p, q < \infty$, are regular with $Cn^{1/q} \leqslant \beta_J(n) \leqslant \gamma_J(n) \leqslant Dn^{1/q}$, for some $0 < C \leqslant D$.

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Theorem 5.6. *Let J be J_0^ϕ or J^ϕ and let g be a continuous function on α. If J is regular, then the following conditions are equivalent:*

(i) *g is commutator J-bounded on α;*

(ii) *g is commutator J-stable on α.*

Proof. (i) $\Longrightarrow$ (ii). This follows from Theorems 4.3 and 4.5.

(ii) $\Longrightarrow$ (i). Let g be commutator J-stable, but not commutator J-bounded. By Proposition 3.4 of [**17**], there are some $A_i \in \mathcal{F}_{\mathrm{nor}}(\alpha)$ and $X_i \in \mathcal{F}$ such that $\|[g(A_i), X_i]\|_J \geqslant i^3\|[A_i, X_i]\|_J$. Set $\lambda_i = i^2\max(\|X_i\|, \|[A_i, X_i]\|_J)$ and $Z_i = X_i/\lambda_i$. Then

$$\|Z_i\| \leqslant i^{-2}, \quad \|[A_i, Z_i]\|_J \leqslant i^{-2} \quad \text{and} \quad \|[g(A_i), Z_i]\|_J \geqslant i^3\|[A_i, Z_i]\|_J. \tag{5.2}$$

Set

$$A = \overbrace{A_1 \oplus \cdots \oplus A_1}^{m_1} \oplus \cdots \oplus \overbrace{A_i \oplus \cdots \oplus A_i}^{m_i} \oplus \cdots$$

and

$$Z = \overbrace{Z_1 \oplus \cdots \oplus Z_1}^{m_1} \oplus \cdots \oplus \overbrace{Z_i \oplus \cdots \oplus Z_i}^{m_i} \oplus \cdots$$

(we will choose m_i later). Then $Z \in C(H)$. Since $A_i \in \mathcal{F}_{\mathrm{nor}}(\alpha)$, we have that $\|A_i\| \leqslant \sup_{\lambda\in\alpha}|\lambda|$, so that A is a normal bounded operator with $\mathrm{Sp}(A) \subseteq \alpha$. Set

$$[A, Z]^{(i)} = \overbrace{[A_1, Z_1] \oplus \cdots \oplus [A_1, Z_1]}^{m_1} \oplus \cdots \oplus \overbrace{[A_i, Z_i] \oplus \cdots \oplus [A_i, Z_i]}^{m_i}.$$

Then $[A, Z]^{(i)} \in J_0^\phi$ and, by (5.1),

$$\begin{aligned}\|[A, Z]^{(i+p)} - [A, Z]^{(i)}\|_J &\leqslant \sum_{k=i+1}^{i+p} \|\overbrace{[A_k, Z_k] \oplus \cdots \oplus [A_k, Z_k]}^{m_k}\|_J \\ &\leqslant \sum_{k=i+1}^{i+p} \gamma(m_k)\|[A_k, Z_k]\|_J.\end{aligned}$$

It follows from (5.2) that $1 \leqslant k^{-2}\|[A_k, Z_k]\|_J^{-1}$. Since $\gamma(i) \to \infty$ and $\gamma(i+1) \leqslant \gamma(i) + 1$ (see [**12**, III.§ 3]), we may choose m_k in such a way that

$$1 \leqslant k^{-2}\|[A_k, Z_k]\|_J^{-1} \leqslant \gamma(m_k) \leqslant k^{-2}\|[A_k, Z_k]\|_J^{-1}k^{1/2}.$$

Therefore,

$$k^{-2} \leqslant \gamma(m_k)\|[A_k, Z_k]\|_J \leqslant k^{-3/2}, \tag{5.3}$$

so that

$$\|[A, Z]^{(i+p)} - [A, Z]^{(i)}\|_J \leqslant \sum_{k=i+1}^{i+p} k^{-3/2} \to 0,$$

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as $i \to \infty$. Hence $[A, Z]^{(i)}$ converge to some $B \in J_0^\phi$. From (5.2) we have

$$\|[A, Z] - [A, Z]^{(i)}\| = \sup_{i+1 \leqslant k} \|[A_k, Z_k]\| \leqslant \sup_{i+1 \leqslant k} \|[A_k, Z_k]\|_J \to 0,$$

as $i \to \infty$. Thus $B = [A, Z] \in J_0^\phi \subseteq J$. Since g is commutator J-stable, $[g(A), Z] \in J$.

On the other hand,

$$\beta(m_k)\|[g(A_k), Z_k]\|_J \leqslant \|\overbrace{[g(A_k), Z_k] \oplus \cdots \oplus [g(A_k), Z_k]}^{m_k}\|_J \leqslant \|[g(A), Z]\|_J.$$

Hence it follows from (5.2) that

$$\beta(m_k)k^3\|[A_k, Z_k]\|_J \leqslant \|[g(A), Z]\|_J.$$

Therefore,

$$\frac{\beta_J(m_k)}{\gamma_J(m_k)} k(k^2\gamma(m_k)\|[A_k, Z_k]\|_J) \leqslant \|[g(A), Z]\|_J,$$

so that, by (5.3),

$$\frac{\beta_J(m_k)}{\gamma_J(m_k)} k \leqslant \|[g(A), Z]\|_J.$$

Since J is regular,

$$\frac{\beta_J(m_k)}{\gamma_J(m_k)} k \to \infty, \quad \text{as } k \to \infty.$$

Thus $[g(A), Z] \notin J$. This contradiction completes the proof. □

Corollary 5.7. *Let J be J_0^ϕ or J^ϕ and let it be regular. Then*

(i) *J is a Fuglede ideal if and only if it is weakly Fuglede;*

(ii) *a compact α in $\mathbb{C}$ is J-Fuglede if and only if it is weakly J-Fuglede.*

Since $\mathfrak{S}^1$ is a regular but not Fuglede ideal, we obtain the following result.

Corollary 5.8. *The ideal $\mathfrak{S}^1$ is not weakly Fuglede.*

The result in Corollary 5.8 means that $[A, X] \in \mathfrak{S}^1$, for a normal A and bounded X, does not always imply $[A^*, X] \in \mathfrak{S}^1$. Weiss [**25**] asked whether for compact X this implication always holds. A negative answer to this question was obtained in [**23**]. It was established later in [**16**] that there exists a normal operator A such that, for any $p > 1$, one can find an operator X in $\mathfrak{S}^p$ such that $[A, X] \in \mathfrak{S}^1$ and $[A^*, X] \notin \mathfrak{S}^1$.

Corollary 5.9. *There exist a compact normal operator A and a compact operator X such that $[A, X] \in \mathfrak{S}^1$ and $[A^*, X] \notin \mathfrak{S}^1$.*

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Proof. It follows from the discussion after Proposition 4.10 of [**17**] and from Corollary 5.4 of [**17**] that the sets

$$\alpha_k = \left\{ \pm \frac{1}{l} + \frac{\mathrm{i}}{l} : k \leqslant l < \infty \right\}, \quad k = 1, 2, \dots,$$

are not $\mathfrak{S}^1$-Fuglede, so the function $h(z) = \bar{z}$ is not commutator $\mathfrak{S}^1$-bounded on them. Set $\alpha_{k(1)} = \alpha_1$. By Proposition 3.4 of [**17**], there are $X_1 \in \mathcal{F}$ and $A_1 \in \mathcal{F}_{\mathrm{nor}}(\alpha_{k(1)})$ such that $\|[h(A_1), X_1]\|_{\mathfrak{S}^1} \geqslant \|[A_1, X_1]\|_{\mathfrak{S}^1}$. Let $k(2) \in \mathbb{N}$ be such that $\alpha_{k(2)} \cap \mathrm{Sp}(A_1) = 0$. By Proposition 3.4 of [**17**], there are $A_2 \in \mathcal{F}_{\mathrm{nor}}(\alpha_{k(2)})$ and $X_2 \in \mathcal{F}$ such that $\|[h(A_2), X_2]\|_{\mathfrak{S}^1} \geqslant 2^3 \|[A_2, X_2]\|_{\mathfrak{S}^1}$. Continuing this process we get a sequence of sets $\alpha_{k(1)} \supset \alpha_{k(2)} \supset \cdots \supset \alpha_{k(l)}$ and sequences of normal operators $A_l \in \mathcal{F}_{\mathrm{nor}}(\alpha_{k(l)})$ and of operators $X_l \in \mathcal{F}$ such that $\alpha_{k(l)} \cap \mathrm{Sp}(A_j) = 0$, for $j < l$, and

$$\|[A_l^*, X_l]\|_{\mathfrak{S}^1} = \|[h(A_l), X_l]\|_{\mathfrak{S}^1} \geqslant l^3 \|[A_l, X_l]\|_{\mathfrak{S}^1}.$$

Now, repeating the proof of Theorem 5.6, we construct operators

$$A = \overbrace{A_1 \oplus \cdots \oplus A_1}^{m_1} \oplus \cdots \oplus \overbrace{A_n \oplus \cdots \oplus A_n}^{m_n} \oplus \cdots$$

and

$$Z = \overbrace{Z_1 \oplus \cdots \oplus Z_1}^{m_1} \oplus \cdots \oplus \overbrace{Z_n \oplus \cdots \oplus Z_n}^{m_n} \oplus \cdots,$$

such that $Z \in C(H)$, $[A, Z] \in \mathfrak{S}^1$ and $[h(A), Z] = [A^*, Z] \notin \mathfrak{S}^1$. We also have that A is a normal operator, $\mathrm{Sp}(A) \subseteq \alpha_1$ and that all eigenspaces of A, corresponding to non-zero eigenvectors, are finite dimensional. Hence A is compact. $\square$

For regular ideals the J-stability of a function is equivalent to it being J-Lipschitzian.

Corollary 5.10. *Let a regular s.n. ideal J be J^ϕ or J_0^ϕ. Let g be a function on $\alpha \subset \mathbb{C}$. If α is weakly J-Fuglede, then the following are equivalent:*

(i) *g is a J-Lipschitz function on α;*

(ii) *g is J-stable on α.*

Proof. (i) $\Longrightarrow$ (ii). This follows from Corollaries 4.6 and 5.7. Conversely, by Proposition 5.5, if g is J-stable on α, it is commutator J-stable on α. By Theorem 5.6, g is commutator J-bounded on α. Hence it is J-Lipschitzian on α. $\square$

We summarize below our results for the important class of ideals $\mathfrak{S}^p$.

Corollary 5.11.

(i) *The ideals $\mathfrak{S}^p$, for $p \in (1, \infty)$, are regular, Fuglede and weakly Fuglede. The ideal $\mathfrak{S}^1$ is regular, but neither Fuglede, nor weakly Fuglede. The ideals $\mathfrak{S}^\infty$ and $\mathfrak{S}^b$ are weakly Fuglede, but neither regular, nor Fuglede ideals.*

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(ii) *If a set is $\mathfrak{S}^p$-Fuglede for some $p \in \{1, \infty, b\}$, it is Fuglede for all $p \in \{1, \infty, b\}$.*

(iii) *If $p \in (1, \infty)$, then the following are equivalent:*

(1) *g is $\mathfrak{S}^p$-Lipschitzian on α;*

(2) *g is commutator $\mathfrak{S}^p$-bounded on α;*

(3) *g is $\mathfrak{S}^p$-stable on α;*

(4) *g is commutator $\mathfrak{S}^p$-stable on α;*

(5) *g is $\mathfrak{S}^q$-Lipschitzian on α for $p_- \leqslant q \leqslant p_+$ (see (3.7));*

(6) *there is $D = D(\alpha, g, p) > 0$ such that, for $X \in B(H)$ and normal A, B with spectra in α, the condition $AX - XB \in \mathfrak{S}^p$ implies*

$$g(A)X - Xg(B) \in \mathfrak{S}^p \quad \text{and} \quad \|g(A) - g(B)\|_{\mathfrak{S}^p} \leqslant D\|AX - XB\|_{\mathfrak{S}^p}.$$

(iv) *If a function g is commutator $\mathfrak{S}^p$-bounded on α for some p in $\{1, \infty, b\}$, it is commutator $\mathfrak{S}^q$-bounded on α for all q in $[1, \infty] \cup b$.*

(v) *Let α be $\mathfrak{S}^p$-Fuglede, for $p \in \{1, \infty, b\}$. The following are equivalent:*

(1) *g is $\mathfrak{S}^p$-Lipschitzian on α;*

(2) *g is commutator $\mathfrak{S}^p$-bounded on α;*

(3) *g is commutator $\mathfrak{S}^1$-stable on α;*

(4) *there is $D > 0$ such that, for $X \in B(H)$ and normal A, B with spectra in α, the condition $AX - XB \in \mathfrak{S}^1$ implies*

$$g(A)X - Xg(B) \in \mathfrak{S}^1 \quad \text{and} \quad \|g(A)X - Xg(B)\|_{\mathfrak{S}^1} \leqslant D\|AX - XB\|_{\mathfrak{S}^1}.$$

If $\alpha \subset \mathbb{R}$, then the above conditions are equivalent to the following condition:

(5) *g is $\mathfrak{S}^1$-stable on α.*

(vi) *Let $\mathfrak{S}^p$-$\mathrm{Lip}(\alpha)$ be the space of all $\mathfrak{S}^p$-Lipschitz functions and let $\mathrm{Lip}(\alpha)$ be the space of all Lipschitz (in the usual sense) functions on α. Then*

$$\mathfrak{S}^b\text{-}\mathrm{Lip}(\alpha) \subseteq \mathfrak{S}^p\text{-}\mathrm{Lip}(\alpha) \subseteq \mathfrak{S}^2\text{-}\mathrm{Lip}(\alpha) = \mathrm{Lip}(\alpha) \quad \text{for } p \in [1, \infty] \cup b;$$
$$\mathfrak{S}^p\text{-}\mathrm{Lip}(\alpha) = \mathfrak{S}^{p'}\text{-}\mathrm{Lip}(\alpha) \subseteq \mathfrak{S}^q\text{-}\mathrm{Lip}(\alpha) \quad \text{for } p_- \leqslant q \leqslant p_+.$$

Problem 5.12.

(i) Do the spaces $\mathfrak{S}^p$-$\mathrm{Lip}(\alpha)$ differ for different $p \in (1, 2)$? A related question: are all functions in $\mathrm{Lip}(\alpha)$ $\mathfrak{S}^p$-Lipschitzian, for all $p \in (1, \infty)$?

(ii) The space $C^{(1)}(a, b)$ of all continuously differentiable functions on $[a, b]$ is not contained in $\mathfrak{S}^b$-$\mathrm{Lip}(a, b)$ (see [**10**]). On the other hand, $C^{(1)}(a, b) \subset \mathfrak{S}^2$-$\mathrm{Lip}(a, b)$. Do all spaces $\mathfrak{S}^p$-$\mathrm{Lip}(a, b)$, for $p \in (1, \infty)$, contain $C^{(1)}(a, b)$?

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Problem 5.13. Let g be a continuous function on $[-1,1]$ and $p \in [1,\infty]$. Let there exist $D > 0$ such that $\|g(A) - g(B)\|_{\mathfrak{S}^p} \leqslant D\|A - B\|_{\mathfrak{S}^p}$, for all self-adjoint A, B in the unit ball of $\mathfrak{S}^p$. Is g $\mathfrak{S}^p$-Lipschitzian on $[-1,1]$?

It is easy to verify that the answer is positive for $p = 2$ and $p = \infty$.

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