BULL. AUSTRAL. MATH. SOC. 16A66, 16A21
VOL. 35 (1987) 379-386.

ON ESSENTIAL EXTENSIONS OF RINGS

E.R. PUCZYŁOWSKI

This paper concerns the problem of description of the set of rings containing a given ring as an essential ideal. The results obtained are applied to some problems of ring theory and radicals.

All rings in this paper are associative. To denote that I is an ideal of a ring A we write $I \triangleleft A$. Recall that an ideal I of A is said to be essential or that the extension $I \triangleleft A$ is essential, if for every non-zero ideal J of A, $J \cap I \neq 0$. Given a ring I we write $EI = \{B \mid B \cong A/I$ for some essential extension $I \triangleleft A\}$. Clearly if $I \cong I'$ then $EI = EI'$.

It appears that many ring theoretic problems concern in fact a description of EI. Among them are those on the possession by I of a unity and on describing atoms in the lattice of radicals. In this paper we study EI, extending, in particular, results in both those areas. For some important results we give simpler proofs. In the second part of the paper special attention is paid to applications to radicals, which actually inspired the studies.

1.

A key role in this paper is played by the following elementary

---

Received 19 May 1986.

---

lemma.

LEMMA 1. *Suppose that* I *is a subring of a ring* P *and simultaneously* $I \lhd R$. *Given* $p \in P$, $r \in R$ *and a regular element* $i \in I$ *of* P *the following conditions are equivalent*

$i)$ $pi = ri$;

$ii)$ $ip = ir$;

$iii)$ $pi' = ri'$ *for each* $i' \in I$;

$iv)$ $i'p = i'r$ *for every* $i' \in I$.

Proof. Since I is an ideal of R , we have for every $i' \in I$ both $ri' \in I$ and $i'r \in I$. Now if $pi = ri$ then $i'pi = i'ri$. Hence, since i is regular in P , $i'p = i'r$. Thus i) implies iv). Obviously ii) is a special case of iv). Symmetric arguments give the other implications. □

It is clear that the element p of Lemma 1 belongs to the idealiser $Id_P I = \{x \in P \mid xI \subseteq I , Ix \subseteq I\}$ of I in P . The idealiser $Id_P I$ is the largest subring of P containing I as an ideal.

Taking in Lemma 1 $P = I$ and $p = 0$ one obtains that for every $r \in R$, $ri = 0$ if and only if $ir = 0$. This implies that $R(i) = \{r \in R \mid ir = 0\}$ is an ideal of R . Since i is a regular element of I, $I \cap R(i) = 0$. Thus we have

COROLLARY 2. *If* $I \lhd R$ *and* I *contains a regular element* i *then* I *is an essential ideal of* R *if and only if* $R(i) = 0$.

COROLLARY 3. $EI = \{0\}$ *if and only if* I *is a ring with unity.*

Proof. If e is a unity of I and $I \lhd R$ then for every $r \in R \setminus I$, $r - er \neq 0$. Obviously $r - er \in R(e)$ and e is a regular element of I . Thus I is not an essential ideal of R .

Suppose now that $EI = \{0\}$. Let I^1 be I with a unity adjoined and let M be a maximal ideal of I^1 satisfying $I \cap M = 0$. Clearly I can be treated as an essential ideal of I^1/M . Thus, since $EI = \{0\}$, we have $I = I^1/M$; in particular I is a ring with unity. □

For a given right R-module M , let $L(M_R)$ denote the lattice of R-submodules of M . We use $\approx$ to denote a lattice isomorphism.

https://doi.org/10.1017/S0004972700013368 Published online by Cambridge University Press

COROLLARY 4. *If i is a regular element of I and I is an essential ideal of R then*

a) the mapping $f: R \to I$ given by $f(r) = ir$ is an embedding of right R-modules;

b) $L((R/I)_Z) \approx L((iR/iI)_I)$, where Z is the ring of integers.

Proof. Obviously $\mathrm{Ker} f = R(i)$, so by Corollary 2, $\mathrm{Ker} f = 0$. This proves a). Now $L((R/I)_Z) \approx L((iR/iI)_Z)$. Since I is an ideal of R, every additive subgroup of iR/iI is a right I-submodule. Thus $L((iR/iI)_Z) \approx L((iR/iI)_I)$. □

Corollary 4 gives immediately relations between some properties of I and $B \in EI$ such as $\mathrm{card} B \le \mathrm{card} I$ or, if I is a right chain ring then the additive group of B is a chain.

The Krull dimension of a right R-module M is denoted by $\mathrm{Kdim} M_R$.

PROPOSITION 5. *If I contains a regular element i, $B \in EI$ and $\mathrm{Kdim}\, I_I$ is defined, then $\mathrm{Kdim}\, B_Z$ is also defined and $\mathrm{Kdim}\, B_Z \wedge \mathrm{Kdim}\, I_I$.*

Proof. Let I be an essential ideal of R with $R/I \cong B$. By Corollary 4 b), $\mathrm{Kdim}(i^n R/i^n I)_I = \mathrm{Kdim} B_Z$ for $n = 1,2,\dots$. Since I is an ideal of R and $i \in I$, $iR \supseteq iI \supseteq i^2 R \supseteq i^2 I \supseteq \dots$ is a chain of right I-submodules of I. Hence $\mathrm{Kdim} B_Z < \mathrm{Kdim} I_I$. □

COROLLARY 6. *Let I be a k-algebra over a field k containing a regular element i. Then*

a) if I is a right chain ring then the field k is finite and prime and if $0 \ne A \in EI$ then $A \cong k$ or $A \cong k^0$, where k^0 is the zero ring on the additive group of k;

b) if $\mathrm{Kdim}\, I_I$ is defined then every ring of EI is finite and $EI = \{0\}$ provided k is infinite.

Proof. Let $I \lhd R$ be an essential extension. Observe that $iR/iI \subseteq U = \{x \in I/iI \mid xI = 0\}$. Obviously U is a k-space and every additive subgroup of I/iI is its I-submodule. Hence if I is a right chain ring then $\dim_k U = 1$. In addition k must be a chain Z-module, which implies $\mathrm{card} k < \infty$. Thus $\mathrm{card} U = \mathrm{card} k < \infty$. Now by Corollary 4

https://doi.org/10.1017/S0004972700013368 Published online by Cambridge University Press

a), $\mathrm{card}R/I \le \mathrm{card}k < \infty$. This proves a). No infinite field and no infinite dimensional space over a field have Krull dimension as Z-modules. Hence if $\mathrm{Kdim}I_I$ is defined and $U \neq 0$ then k is a finite field and $\dim_k U < \infty$. Now Corollary 4 a) implies

$\mathrm{card}\ R/I \le \mathrm{card}\ k\ \dim_k U < \infty$. □

Corollaries 3 and 6 imply that (see [11, Section 3]) if a k-algebra I over an infinite field k contains a regular element and $\mathrm{Kdim}I_I$ is defined then I has a unity

The following result was proved in [6]. Here we give a direct proof not using multiplier algebras.

PROPOSITION 7 ([6]). *Let L be a left ideal of a ring P and suppose that L contains an element l regular in P. If $L \triangleleft R$ and $Rl \subseteq Pl$ then there exists a ring homomorphism $f : R \longrightarrow Id_P L$ such that $f|_L = id$.*

Proof. The assumption $Rl \subseteq Pl$ says that for every $r \in R$ there exists $p \in P$ with $rl = pl$. Regularity of l implies that p is uniquely determined; denote this element by $f(r)$. Thus $rl = f(r)l$ and by Lemma 1, $lr = lf(r)$ and $f(r) \in Id_P L$. If $r_1, r_2 \in R$ then $f(r_1+r_2)l = (r_1+r_2)l = r_1l + r_2l = f(r_1)l + f(r_2)l$, so $f(r_1+r_2) = f(r_1) + f(r_2)$. Now $lf(r_1r_2)l = lr_1r_2l = lf(r_1)r_2l = lf(r_1)f(r_2)l$, so $f(r_1r_2) = f(r_1)f(r_2)$. Hence f is a homomorphism of R into $Id_P L$. Obviously $f|_L = id$. □

The above result can obviously be applied to principal left ideals generated by regular elements of P and it was done in [6]. The following lemma shows that it can also be applied to some other left ideals of P .

LEMMA 8. *If L is a left ideal of a ring P with unity and $LP = P$ then*

a) L is a finitely generated right L-module;

b) if $L \triangleleft R$ then for every $l \in L$, $Pl \supseteq Rl$.

Proof. Since $LP = P$ and P is a ring with unity, there exist $p_1, \dots, p_n \in P$ and $l_1, \dots, l_n \in L$ such that $1 = l_1p_1 + \dots + l_np_n$. Hence

$L \subseteq l_1p_1L + \ldots + l_np_nL \subseteq l_nL + \ldots + l_nL \subseteq L$, so $l_1L + \ldots + l_nL = L$ and we obtain a). Now $Rl = R(1l) = R(l_1p_1 + \ldots + l_np_n)l$. Since for all i, $p_il \in L$, we have $R(l_1p_1l + \ldots + l_np_nl) \subseteq Rl_1(p_1l) + \ldots + Rl_n(p_nl) \subseteq L(p_1l) + \ldots + L(p_nl) = (Lp_1)l + \ldots + (Lp_n)l \subseteq Pl$. □

COROLLARY 9. *If* L *is a left ideal of a ring with unity* P , $LP = P$ *and* L *contains a regular element of* P *then* $B \in EL$ *if and only if for some subring* A *of* Id_PL , $L \subseteq A$ *and* $A/L \cong B$.

Proof. Let $L \lhd R$ be an essential extension with $R/L \cong B$. By Proposition 7 and Lemma 8 there exists a homomorphism $f : R \longrightarrow Id_PL$ such that $f|_L = id$. Obviously $\mathrm{Ker} f \cap L = 0$, so, since L is an essential ideal of R , $\mathrm{Ker} f = 0$. This proves that B is isomorphic to $f(R)$ which is a subring of Id_PL .

Conversely, if A is a subring of Id_PL containing L then $L \lhd A$. Now if $I \lhd A$ and $I \cap L = 0$ then $IL = 0$. Since L contains a regular element of P , we have $I = 0$. This proves that the extension $L \lhd A$ is essential. □

2.

Lemma 8 a) gives a very simple proof of the following result of Beidar.

PROPOSITION 10 ([2]). *If a left ideal* L *of a simple ring with unity is ring-isomorphic to a right ideal* I *of a ring* A *then the ideal* I *is finitely generated.*

Proof. By Lemma 8 a) L is a finitely generated right L-module. Hence I is also a finitely generated right I-module and, even more, I is a finitely generated right A-module. □

Using Proposition 10 Beidar answered a question of Sands [12] constructing a left stable and hereditary radical which is not right strong. Now we give another example of such a radical.

Recall that a radical S is <u>left strong</u> (<u>stable</u>) ([5,7]) if for every left ideal L of a ring R if $L \in S$ then $L \subseteq S(R)$ $(S(L) \subseteq S(R))$. Right strong (stable) radicals are defined dually. A class M of rings is <u>hereditary</u> if $I \triangleleft R$ and $R \in M$ imply $I \in M$.

EXAMPLE 1. Let D be a simple domain with unity containing a right ideal I which is not finitely generated. The ring I is simple ([5]). For, if $0 \neq J \triangleleft I$ then $DJI = D$. Hence $I = ID = IDJI \subseteq J$ and $I = J$. This implies in particular that the class $M = \{I\} \cup N$, where N is the class of nilpotent rings, is hereditary. Now the lower left stable radical S determined by M is hereditary (see [7]). However the radical S is not right strong. For I is a non-zero right S-ideal of D, so if S is right strong, $D \in S$. This implies that D contains subrings $0 \neq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n = D$ with $A_1 \in M$ and such that A_i is a left ideal of A_{i+1} for $1 \leq i \leq n-1$. Since D is a domain, $A_1 \cong I$. Now $A_1 = A_1^n \subseteq A_n A_{n-1} \ldots A_1$, so $A_1 = A_n A_{n-1} \ldots A_1$ is a left ideal of D. Hence by Proposition 10, the ideal I is finitely generated, a contradiction.

Remark. Using Proposition 10 and the idea of Example 2 of [8] one proves easily that the lower radical determined by the class $M = \{J \mid (\exists$ subrings $J = J_1 \subseteq \ldots \subseteq J_n = D)$ $(J_i$ is a left ideal of J_{i+1}, $1 \leq i \leq n-1)\}$, where D is the ring of Example 1, is left but not right hereditary. The first example of such a radical was constructed by Beidar in [2].

In [1] Andrunakievich and Ryabukhin asked if there is a simple ring without unity whose lower radical is an atom in the lattice of all radicals. The first example of such a ring was pointed out by Gardner in [4]. He also proved that the lower radical determined by a simple ring A is an atom in the lattice of radicals when A satisfies the following condition: if $A \triangleleft R$ and $R/A \cong A$ then there exists $I \triangleleft R$ such that $R = A \oplus I$. Obviously the condition is satisfied if and only if $A \notin EA$. A large part of this paper was inspired by this result. The foregoing and Gardner's result give

COROLLARY 11. *The lower radical determined by a simple idempotent ring A is an atom in the lattice of radicals provided one of the following conditions is satisfied*

a) ([4]) A has a unity;

b) $K\dim A_A$ *is defined;*

c) ([6]) A *is a left ideal of a simple ring* P *with unity, it contains a regular element of* P *and* $Id_P A/A$ *is a commutative ring;*

d) ([3]) A *is a right chain ring;*

e) A *is a maximal left ideal of a simple ring* P *with unity containing a regular element of* P *but* A *is not a domain.*

Proof. Conditions a), b) and c) are immediate consequences of Gardner's result and Corollaries 3, 6 and 9 respectively. To prove d) assume that A is a right chain ring and $A \in EA$. By Corollary 4 a) the additive group of A is chain. Since A, as a simple ring, is an algebra over a field F, we have $\dim_F A = 1$. Thus $A \cong F$ and the result is a consequence of a). Suppose now that A satisfies condition a). It is well known ([10]) that $Id_P A/A = \mathrm{End}_P(P/A)$. Since P/A is a simple P-module, $\mathrm{End}_P(P/A)$ is a division ring. Thus Corollary 9 implies that every ring $B \in EA$ is a domain. In particular $A \notin EA$ and Gardner's result completes the proof. □

Examples of rings without unity elements satisfying the assumptions of Corollary 11 b), c) and d) were constructed respectively by Robson ([9]), Leavitt and VanLeeuwen ([6]) and Dubrovin ([3]). To construct an example for Corollary 11 e) let us take a simple domain D with unity. Then the ring D_2 of 2×2-matrices over D is simple. Let L be a maximal left ideal of D. It is clear that $A = \begin{pmatrix} L & D \\ L & D \end{pmatrix}$ is a maximal left ideal of D_2 containing a regular element of D_2 and that A is not a domain.

References

[1] V. A. Andrunakievich and J. M. Ryabukhin, *Radicals of algebras and structural theory*, Nauka, Moscow, 1979 (Russian).

[2] K. I. Beidar, "Examples of rings and radicals", *Radical Theory, Colloq. Math. Soc. Janos Bolyai* 38 (1982), 19-46.

[3] N. I. Dubrovin, "Chain domains", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, 1980, no. 2, 51-54 (Russian).

[4] B. J. Gardner, "Simple rings whose lower radicals are atoms", *Acta Math. Hungar.* 43 (1984), 131-135.

[5] N. Divinsky, J. Krempa and A. Sulinski, "Strong radical properties of alternative and associative rings", *J. Algebra* 17 (1971), 369-388.

[6] W. G. Leavitt and L. C. A. VanLeeuwen, "Multiplier algebras and minimal embeddability", *Publ. Math. Debrecen* 29 (1982), 95-99.

[7] E. R. Puczyłowski, "Remarks on stable radicals", *Bull. Acad. Polon. Sci. Sér. Sci. Math.* 28 (1980), 11-16.

[8] E. R. Puczyłowski, "On Sands' questions concerning strong and hereditary radicals", *Glasgow Math. J.* 28 (1986), 1-3.

[9] J. C. Robson, "Simple Noetherian rings need not have unity elements", *Bull. London Math. Soc.* 7 (1975), 269-270.

[10] J. C. Robson, "Idealizers and hereditary Noetherian prime rings", *J. Algebra* 22 (1972), 45-81

[11] J. C. Robson, "A unified approach to unity", *Comm. Algebra* 7 (1979), 1245-1255.

[12] A. D. Sands, "On relations among radical properties", *Glasgow Math. J.* 18 (1977), 17-23.

Institute of Mathematics
University of Warsaw
PKiN
00-901 Warsaw, Poland