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Homological Dimensions of Local (Co)homology Over Commutative DG-rings

Liran Shaul

Abstract. Let A be a commutative noetherian ring, let $\mathfrak{a} \subseteq A$ be an ideal, and let I be an injective A-module. A basic result in the structure theory of injective modules states that the A-module $\Gamma_{\mathfrak{a}}(I)$ consisting of $\mathfrak{a}$-torsion elements is also an injective A-module. Recently, de Jong proved a dual result: If F is a flat A-module, then the $\mathfrak{a}$-adic completion of F is also a flat A-module. In this paper we generalize these facts to commutative noetherian DG-rings: let A be a commutative non-positive DG-ring such that $\mathrm{H}^0(A)$ is a noetherian ring and for each $i < 0$, the $\mathrm{H}^0(A)$-module $\mathrm{H}^i(A)$ is finitely generated. Given an ideal $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$, we show that the local cohomology functor $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}$ associated with $\bar{\mathfrak{a}}$ does not increase injective dimension. Dually, the derived $\bar{\mathfrak{a}}$-adic completion functor $\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}$ does not increase flat dimension.

1 Introduction

1.1 Torsion of Injective Modules

Let A be a commutative noetherian ring, and let $\mathfrak{a} \subseteq A$ be an ideal. The $\mathfrak{a}$-torsion functor is the functor $\Gamma_{\mathfrak{a}}(M) := \varinjlim \operatorname{Hom}_A(A/\mathfrak{a}^n, M)$, which maps an A-module M to its submodule consisting $\mathfrak{a}$-torsion elements. An important consequence of Matlis' structure theory of injective modules is the following result.

Theorem A *Let A be a commutative noetherian ring, let $\mathfrak{a} \subseteq A$ be an ideal, and let J be an injective A-module. Then $\Gamma_{\mathfrak{a}}(J)$ is also an injective A-module.*

This basic result depends on A being noetherian. It is false in general if A is not noetherian, even if $\mathfrak{a}$ is finitely generated (see Remark 3.6).

We denote by $\mathsf{D}(A)$ the unbounded derived category of A-modules, and by $\mathsf{D}^{\mathrm{b}}(A)$ its full triangulated subcategory of complexes with bounded cohomology. For $M \in \mathsf{D}(A)$, its injective dimension, denoted by $\operatorname{inj\,dim}_A(M)$ was defined in [1, Definition 2.1.I]. The functor $\Gamma_{\mathfrak{a}}$ has a right derived functor

$$\mathrm{R}\Gamma_{\mathfrak{a}} \colon \mathsf{D}(A) \to \mathsf{D}(A).$$

It is calculated using K-injective resolutions. The following result is an immediate corollary of Theorem A.

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Corollary 1.1 *Let A be a commutative noetherian ring, and let $\mathfrak{a} \subseteq A$ be an ideal. Then for any $M \in \mathsf{D}^{\mathrm{b}}(A)$, we have the following inequality:*

$$\operatorname{inj\,dim}_A(\mathrm{R}\Gamma_{\mathfrak{a}}(M)) \leq \operatorname{inj\,dim}_A(M).$$

We now switch our attention to commutative non-positive noetherian DG-rings. The definition of a commutative DG-ring is recalled in Section 2.1. Just as commutative rings represent affine schemes, commutative DG-rings represent affine derived schemes. Given a commutative DG-ring A, the fact that $A^i = 0$ for $i > 0$ implies that $\mathrm{H}^0(A)$ is a commutative ring. Following [11], we set $\overline{A} := \mathrm{H}^0(A)$. A commutative DG-ring A is called *noetherian* if the commutative ring $\overline{A}$ is noetherian, and $\mathrm{H}^i(A)$ is a finitely generated $\overline{A}$-module for every $i < 0$. We will denote by $\mathsf{D}(A)$ the unbounded derived category of DG-modules over A, and by $\mathsf{D}^{\mathrm{b}}(A)$ its full triangulated subcategory of DG-modules with bounded cohomology.

Commutative noetherian DG-rings arise naturally in algebraic geometry. If $\mathbb{K}$ is a commutative noetherian ring and A, B are finite type $\mathbb{K}$-algebras, then $A \otimes^{\mathrm{L}}_{\mathbb{K}} B$ is a commutative noetherian DG-ring, and if $\mathbb{K}$ is not a field, it often cannot be represented using ordinary commutative rings.

Let A be a commutative DG-ring, and let $\bar{\mathfrak{a}} \subseteq \overline{A}$ be a finitely generated ideal. In our recent paper [9] we introduced a triangulated functor

$$\mathrm{R}\Gamma_{\bar{\mathfrak{a}}} \colon \mathsf{D}(A) \to \mathsf{D}(A)$$

called the right derived $\bar{\mathfrak{a}}$-torsion or local cohomology at $\bar{\mathfrak{a}}$ functor. When A is a commutative noetherian ring, this functor coincides with the usual local cohomology functor. The definition of $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}$, as well as an explicit construction of it using the telescope complex, is recalled in Section 2.2. With any $M \in \mathsf{D}(A)$ we can associate its injective dimension, denoted by $\operatorname{inj\,dim}_A(M)$. The definition is recalled in Section 2.3.

The first main result of this paper generalizes Corollary 1.1 to commutative noetherian DG-rings.

Theorem 1.2 *Let A be a commutative noetherian DG-ring, and let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. Then for any $M \in \mathsf{D}^{\mathrm{b}}(A)$, there is an inequality*

$$\operatorname{inj\,dim}_A(\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}(M)) \leq \operatorname{inj\,dim}_A(M).$$

We will prove this result in Theorem 3.5.

1.2 Completion of Flat Modules

Given a commutative ring A and a finitely generated ideal $\mathfrak{a} \subseteq A$, the $\mathfrak{a}$-adic completion functor is the functor $\Lambda_{\mathfrak{a}}(M) := \varprojlim A/\mathfrak{a}^n \otimes_A M$. This construction is dual to the $\mathfrak{a}$-torsion functor. This is best demonstrated by the Greenlees–May duality [5], which states that if A is noetherian, then the derived functors of $\mathfrak{a}$-torsion and $\mathfrak{a}$-adic completion are adjoint to each other. This intimate connection between $\mathfrak{a}$-torsion and $\mathfrak{a}$-adic completion raises the question: is there a dual result to Theorem A for the $\mathfrak{a}$-adic completion functor? We have the following result.

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Theorem B *Let A be a commutative noetherian ring, let $\mathfrak{a} \subseteq A$ be an ideal, and let F be a flat A-module. Then $\Lambda_{\mathfrak{a}}(F)$ is also a flat A-module.*

Theorem A is a classical result that has been known for many decades. In contrast, Theorem B was first proved by de Jong in 2013 (see [10, Tag 0AGW]). If F is a finitely generated A-module, then this result is trivial. Prior to de Jong's proof, the best known result was by Enochs, who showed in [3] that this holds under the additional assumption that A has finite Krull dimension. After de Jong's proof, other proofs were given by Gabber and Ramero, and by Yekutieli ([12, Theorem 0.1]).

For a complex of A-modules M, its flat dimension, denoted by $\operatorname{fl\,dim}_A(M)$ was defined in [1, Definition 2.1.F]. The functor $\Lambda_{\mathfrak{a}}$ has a left derived functor

$$\mathrm{L}\Lambda_{\mathfrak{a}}: \mathsf{D}(A) \to \mathsf{D}(A).$$

It is calculated using K-flat resolutions. The following is an immediate corollary of Theorem B.

Corollary 1.3 *Let A be a commutative noetherian ring, and let $\mathfrak{a} \subseteq A$ be an ideal. Then for any $M \in \mathsf{D}^{\mathrm{b}}(A)$,*

$$\operatorname{fl\,dim}_A\left(\mathrm{L}\Lambda_{\mathfrak{a}}(M)\right) \le \operatorname{fl\,dim}_A(M).$$

Now let A be a commutative DG-ring. Given a finitely generated ideal $\bar{\mathfrak{a}} \subseteq \bar{A}$, there is a triangulated functor

$$\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}: \mathsf{D}(A) \to \mathsf{D}(A)$$

called the *derived $\bar{\mathfrak{a}}$-adic completion functor*. Its definition is recalled in Section 2.2 below. If A is a commutative noetherian ring, it coincides with the usual left derived functor of adic completion. With any $M \in \mathsf{D}(A)$ we can associate its flat dimension, denoted by $\operatorname{fl\,dim}_A(M)$. The definition is recalled in Section 2.3.

The second main result of this paper generalizes Corollary 1.3 to commutative noetherian DG-rings.

Theorem 1.4 *Let A be a commutative noetherian DG-ring, and let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. Then for any $M \in \mathsf{D}^{\mathrm{b}}(A)$,*

$$\operatorname{fl\,dim}_A\left(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(M)\right) \le \operatorname{fl\,dim}_A(M).$$

This will be proved in Theorem 4.3.

2 Preliminaries

In this section we recall some basic facts concerning commutative DG-rings.

2.1 Commutative DG-rings

A DG-ring A is a $\mathbb{Z}$-graded ring $A = \bigoplus_{n=-\infty}^{\infty} A^n$ together with an additive differential $d: A \to A$ of degree +1, such that $d \circ d = 0$ and such that the Leibniz rule holds: $d(a \cdot b) = d(a) \cdot b + (-1)^i \cdot a \cdot d(b)$ for all $a \in A^i$ and $b \in A^j$. A DG-ring A is called *non-positive* if $A^i = 0$ for all $i > 0$. We say that A is *commutative* if $b \cdot a = (-1)^{i \cdot j} \cdot a \cdot b$ for all $a \in A^i$ and $b \in A^j$, and moreover $a \cdot a = 0$ if i is odd.

In this paper, all DG-rings are assumed to be commutative and non-positive. Given a commutative DG-ring A, a DG-module over it is graded A-module M with a differential $d\colon M \to M$ of degree $+1$ satisfying a graded Leibniz rule. The category of all DG-modules is denoted by $\mathrm{DGMod}(A)$. Inverting quasi-isomorphisms in it, we obtain the derived category of DG-modules over A, denoted by $\mathsf{D}(A)$.

If A is a commutative DG-ring, recall from the introduction that we denote by $\overline{A}$ the commutative ring $\mathrm{H}^0(A)$. The DG-ring A is called *noetherian* if $\overline{A}$ is noetherian and $\mathrm{H}^i(A)$ is a finitely generated $\overline{A}$-module for all $i < 0$.

2.2 Local (Co)homology Over Commutative DG-rings and the Telescope Complex

Let A be a commutative DG-ring, and let $\bar{\mathfrak{a}} \subseteq \overline{A}$ be a finitely generated ideal. The category of derived $\bar{\mathfrak{a}}$-torsion DG-modules, denoted by $\mathsf{D}_{\bar{\mathfrak{a}}\text{-tor}}(A)$, is the full triangulated subcategory of $\mathsf{D}(A)$, consisting of DG-modules M, such that for all $n \in \mathbb{Z}$, the $\overline{A}$-module $\mathrm{H}^n(M)$ is $\bar{\mathfrak{a}}$-torsion.

According to [9, Theorem 2.13(1)], the inclusion functor $i_{\bar{\mathfrak{a}}} : \mathsf{D}_{\bar{\mathfrak{a}}\text{-tor}}(A) \hookrightarrow \mathsf{D}(A)$ has a right adjoint $F_{\bar{\mathfrak{a}}}\colon \mathsf{D}(A) \to \mathsf{D}_{\bar{\mathfrak{a}}\text{-tor}}(A)$. The composition $i_{\bar{\mathfrak{a}}} \circ F_{\bar{\mathfrak{a}}}\colon \mathsf{D}(A) \to \mathsf{D}(A)$ is denoted by $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}\colon \mathsf{D}(A) \to \mathsf{D}(A)$ and called the *local cohomology functor of A with respect to* $\bar{\mathfrak{a}}$.

By [9, Theorem 2.13(2)], the functor $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}$ has a left adjoint $\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}\colon \mathsf{D}(A) \to \mathsf{D}(A)$ called the *derived completion* (or *local homology*) *functor of A with respect to* $\bar{\mathfrak{a}}$.

Below, we will give explicit formulas for the functors $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}$ and $\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}$.

Remark 2.1 If A is an ordinary commutative noetherian ring, by [6, Theorem 7.12], these constructions of local cohomology and derived completion with respect to $\bar{\mathfrak{a}}$ coincide with the usual right derived functor of the functor $\Gamma_{\bar{\mathfrak{a}}}$ and left derived functor of $\Lambda_{\bar{\mathfrak{a}}}$.

To give an explicit formula for the local cohomology $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}$ and derived completion $\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}$, we recall the construction of the telescope complex from [6, Section 5]. Given a commutative ring A and some $a \in A$, the telescope complex $\mathrm{Tel}(A; a)$ is the cochain complex

$$0 \longrightarrow \bigoplus_{i=0}^{\infty} A \xrightarrow{d} \bigoplus_{i=0}^{\infty} A \longrightarrow 0$$

with non-zero components in degrees $0, 1$. Letting $\{\delta_i \mid i \geq 0\}$ be the basis of the countably generated free A-module $\bigoplus_{i=0}^{\infty} A$, the differential d is defined by

$$d(\delta_i) = \begin{cases} \delta_0 & \text{if } i = 0, \\ \delta_{i-1} - a \cdot \delta_i & \text{if } i \geq 1. \end{cases}$$

Given a finite sequence $\mathbf{a} = (a_1, \dots, a_n)$ of elements of A, the telescope complex associated with $\mathbf{a}$ is the complex

$$\mathrm{Tel}(A; \mathbf{a}) := \mathrm{Tel}(A; a_1) \otimes_A \mathrm{Tel}(A; a_2) \otimes_A \cdots \otimes_A \mathrm{Tel}(A; a_n).$$

This is a bounded complex of free A-modules. The telescope complex has the following base change property: if $f\colon A \to B$ is a homomorphism between commutative rings, and $\mathbf{b} = (f(a_1), f(a_2), \dots, f(a_n))$, there is an isomorphism of complexes of B-modules $\operatorname{Tel}(A;\mathbf{a}) \otimes_A B \cong \operatorname{Tel}(B;\mathbf{b})$.

Let A be a commutative DG-ring, and let $\bar{\mathfrak{a}} \subseteq \bar{A}$ be a finitely generated ideal. Let $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_n)$ be a finite sequence of elements in $\bar{A}$ that generates $\bar{\mathfrak{a}}$, and using the surjection $A^0 \to \bar{A}$, choose some lifts $\mathbf{a} = (a_1, \dots, a_n)$ of $\bar{\mathbf{a}}$ to the commutative ring A^0. By [9, Proposition 2.4] and the base change property of the telescope complex, there are isomorphisms

$$\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}(-) \cong \operatorname{Tel}(A^0;\mathbf{a}) \otimes_{A^0} -, \tag{2.1}$$

$$\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(-) \cong \operatorname{Hom}_{A^0}\big(\operatorname{Tel}(A^0;\mathbf{a}), -\big) \tag{2.2}$$

of functors $\mathsf{D}(A) \to \mathsf{D}(A)$.

2.3 Homological Dimensions Over Commutative DG-rings

Let A be a commutative DG-ring. Given a DG-module M over A, we set

$$\inf(M) := \inf\{i \in \mathbb{Z} \mid \mathrm{H}^i(M) \neq 0\}, \quad \sup(M) := \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i(M) \neq 0\},$$

and $\operatorname{amp}(M) := \sup(M) - \inf(M)$.

Let M be a DG-module over A. Following [1, Section 2.I], we let the injective dimension of M, denoted by $\operatorname{inj\,dim}_A(M)$ to be the number

$$\inf\{n \in \mathbb{Z} \mid \operatorname{Ext}^i_A(N, M) = 0 \text{ for any bounded DG-module } N \text{ and any } i > n - \inf N\},$$

where $\operatorname{Ext}^i_A(N, M) := \mathrm{H}^i(\mathrm{R}\operatorname{Hom}_A(N, M))$. By [1, Theorem 2.4.I][1] this coincides with the usual definition in case A is a commutative ring. Similarly, we let the flat dimension of M, denoted by $\operatorname{fl\,dim}_A(M)$ be the number

$$\inf\{n \in \mathbb{Z} \mid \operatorname{Tor}^i_A(N, M) = 0 \text{ for any bounded DG-module } N \text{ and any } i > n - \inf N\}$$

where $\operatorname{Tor}^i_A(N, M) := \mathrm{H}^{-i}(N \otimes^{\mathrm{L}}_A M)$. The projective dimension of M, denoted by $\operatorname{proj\,dim}_A(M)$ is defined similarly.

2.4 Subcategories of $\mathsf{D}(A)$

Given a commutative DG-ring A, we denote by $\mathsf{D}^-(A)$ (resp. $\mathsf{D}^+(A)$) the full triangulated subcategory of A consisting of DG-modules with bounded above (resp. bounded below) cohomology. The full triangulated subcategory of DG-modules with bounded cohomology is $\mathsf{D}^{\mathrm{b}}(A) = \mathsf{D}^-(A) \cap \mathsf{D}^+(A)$. Assume further that A is noetherian. In particular, $\bar{A}$ is a noetherian ring. We say that $M \in \mathsf{D}(A)$ has finitely generated cohomology if for all $i \in \mathbb{Z}$, $\mathrm{H}^i(M)$ is a finitely generated $\bar{A}$-module. The full triangulated subcategory of $\mathsf{D}(A)$ consisting of DG-modules with finitely generated cohomologies is denoted by $\mathsf{D}_{\mathrm{f}}(A)$. We let $\mathsf{D}^+_{\mathrm{f}}(A) = \mathsf{D}^+(A) \cap \mathsf{D}_{\mathrm{f}}(A)$, $\mathsf{D}^-_{\mathrm{f}}(A) = \mathsf{D}^-(A) \cap \mathsf{D}_{\mathrm{f}}(A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A) = \mathsf{D}^b(A) \cap \mathsf{D}_{\mathrm{f}}(A)$.

[1]Unlike [1], in this paper we use a cohomological notation, hence the difference between the formulas.

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2.5 The Tensor-evaluation Morphism

Let A be a commutative DG-ring. Given $M, N, K \in \mathrm{D}(A)$, there is a natural morphism

$$\eta_{M,N,K} \colon \mathrm{R}\operatorname{Hom}_A(M,N) \otimes^{\mathrm{L}}_A K \to \mathrm{R}\operatorname{Hom}_A(M, N \otimes^{\mathrm{L}}_A K)$$

in $\mathrm{D}(A)$, defined as follows. Let $P \cong M$ be a K-projective resolution, and let $F \cong K$ be a K-flat resolution. Then $\eta_{M,N,K}$ is the composition

$$\mathrm{R}\operatorname{Hom}_A(M,N) \otimes^{\mathrm{L}}_A K \cong \operatorname{Hom}_A(P,N) \otimes_A F \xrightarrow{\phi} \operatorname{Hom}_A(P, N \otimes_A F) \cong \mathrm{R}\operatorname{Hom}_A(M, N \otimes^{\mathrm{L}}_A K),$$

where the map ϕ is the usual tensor-evaluation morphism (see [11, Equation (5.6)] for its formula in the DG-case). The next result generalizes [8, Proposition 6.7].

Proposition 2.2 *Let A be a commutative noetherian DG-ring, and let $M, N, K \in \mathrm{D}(A)$. Assume one of the following holds:*

(i) $M \in \mathrm{D}^-_{\mathrm{f}}(A)$, $N \in \mathrm{D}^{\mathrm{b}}(A)$ *and* $\operatorname{fl\,dim}_A(K) < \infty$,

(ii) $\operatorname{proj\,dim}_A(M) < \infty$, $N \in \mathrm{D}^{\mathrm{b}}(A)$ *and* $K \in \mathrm{D}^-_{\mathrm{f}}(A)$.

Then the morphism

$$\eta_{M,N,K} \colon \mathrm{R}\operatorname{Hom}_A(M,N) \otimes^{\mathrm{L}}_A K \to \mathrm{R}\operatorname{Hom}_A(M, N \otimes^{\mathrm{L}}_A K)$$

is an isomorphism in $\mathrm{D}(A)$.

Proof (i) Fixing such N, K, we have a natural morphism

$$\zeta_M \colon \mathrm{R}\operatorname{Hom}_A(M,N) \otimes^{\mathrm{L}}_A K \to \mathrm{R}\operatorname{Hom}_A(M, N \otimes^{\mathrm{L}}_A K).$$

These assumptions on N and K ensure that the functors $\mathrm{R}\operatorname{Hom}_A(-,N) \otimes^{\mathrm{L}}_A K$ and $\mathrm{R}\operatorname{Hom}_A(-, N \otimes^{\mathrm{L}}_A K)$ are both contravariant way-out right functors. Clearly, ζ_A is an isomorphism. Hence, by a DG-version of the lemma on way-out functors (for instance, [11, Theorem 2.11]), we deduce that ζ_M is an isomorphism for any $M \in \mathrm{D}^-_{\mathrm{f}}(A)$.

(ii) Fixing such M, N, we have a natural morphism

$$\iota_K \colon \mathrm{R}\operatorname{Hom}_A(M,N) \otimes^{\mathrm{L}}_A K \to \mathrm{R}\operatorname{Hom}_A(M, N \otimes^{\mathrm{L}}_A K).$$

These assumptions on M and N ensure that the functors $\mathrm{R}\operatorname{Hom}_A(M,N) \otimes^{\mathrm{L}}_A -$ and $\mathrm{R}\operatorname{Hom}_A(M, N \otimes^{\mathrm{L}}_A -)$ are both covariant way-out left functors, and it is clear that ι_A is an isomorphism. Hence, by the lemma on way-out functors, ι_K is an isomorphism for any $K \in \mathrm{D}^-_{\mathrm{f}}(A)$. ■

3 Injective Dimension of Local Cohomology

In this section we will prove Theorem 1.2. We begin by proving some basic results about injective dimension over commutative DG-rings.

Proposition 3.1 *Let $A \to B$ be a homomorphism between commutative DG-rings, and let M be a DG-module over A. Then*

$$\operatorname{inj\,dim}_B\big(\mathrm{R}\operatorname{Hom}_A(B,M)\big) \le \operatorname{inj\,dim}_A(M).$$

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Proof This follows from the adjunction

$$\mathrm{R}\operatorname{Hom}_B\big(-, \mathrm{R}\operatorname{Hom}_A(B, M)\big) \simeq \mathrm{R}\operatorname{Hom}_A(-, M). \qquad \blacksquare$$

Over a commutative ring A, it is well known that one can detect the injective dimension of a complex M by checking the vanishing of $\operatorname{Ext}^i_A(N, M)$ for all A-modules N (that is, complexes with zero amplitude). The proof of the next result is based on the same idea over a DG-ring A, together with the observation that a DG-module whose amplitude is zero is isomorphic in the derived category to the shift of an $\overline{A}$-module.

Theorem 3.2 *Let A be a commutative DG-ring, and let M be a DG-module over A. Then there is an equality*

$$\operatorname{inj}\dim_A(M) = \operatorname{inj}\dim_{\mathrm{H}^0(A)}\big(\mathrm{R}\operatorname{Hom}_A(\mathrm{H}^0(A), M)\big).$$

Proof Applying Proposition 3.1 to the map $A \to \overline{A}$, we have that

$$\operatorname{inj}\dim_A(M) \geq \operatorname{inj}\dim_{\overline{A}}\big(\mathrm{R}\operatorname{Hom}_A(\overline{A}, M)\big).$$

To prove the converse, assume $\operatorname{inj}\dim_{\overline{A}}(\mathrm{R}\operatorname{Hom}_A(\overline{A}, M)) = n < \infty$. Let N be a bounded DG-module. We must show that $\operatorname{Ext}^i_A(N, M) = 0$ for all $i > n - \inf N$. We will prove this by induction on $\operatorname{amp} N$. If $\operatorname{amp} N = 0$, there is some $m \in \mathbb{Z}$ such that $N \simeq \mathrm{H}^m(N)[-m]$. It follows by adjunction that

$$\begin{aligned}\mathrm{R}\operatorname{Hom}_A(N, M) &\simeq \mathrm{R}\operatorname{Hom}_A\big(\mathrm{H}^m(N)[-m], M\big)\\ &\simeq \mathrm{R}\operatorname{Hom}_{\overline{A}}\big(\mathrm{H}^m(N)[-m], \mathrm{R}\operatorname{Hom}_A(\overline{A}, M)\big)\end{aligned}$$

so the fact that $\operatorname{inj}\dim_{\overline{A}}(\mathrm{R}\operatorname{Hom}_A(\overline{A}, M)) = n$ implies that $\operatorname{Ext}^i_A(N, M) = 0$ for all $i > n - \inf N$.

Given $l > 0$, assume now that for any bounded DG-module N with $\operatorname{amp} N < l$ we have that $\operatorname{Ext}^i_A(N, M) = 0$ for all $i > n - \inf N$, and let N be a bounded DG-module with $\operatorname{amp} N = l$.

According to [4, p. 299], using truncations, the DG-module N fits into a distinguished triangle

$$N' \longrightarrow N \longrightarrow N'' \longrightarrow N[1] \tag{3.1}$$

in $\mathsf{D}(A)$, such that the $\operatorname{amp} N' < l$, $\operatorname{amp} N'' < l$, and moreover $\inf N' \geq \inf N$ and $\inf N'' \geq \inf N$. Applying the contravariant triangulated functor $\mathrm{R}\operatorname{Hom}_A(-, M)$ to the triangle (3.1), we obtain a distinguished triangle

$$\begin{aligned}&\mathrm{R}\operatorname{Hom}_A(N'', M) \longrightarrow \mathrm{R}\operatorname{Hom}_A(N, M)\\ &\qquad\longrightarrow \mathrm{R}\operatorname{Hom}_A(N', M) \longrightarrow \mathrm{R}\operatorname{Hom}_A(N'', M)[1].\end{aligned} \tag{3.2}$$

The result now follows from the induction hypothesis and the long exact sequence in cohomology associated with the distinguished triangle (3.2). $\blacksquare$

Before stating the next lemma, we shall need the following terminology.

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Remark 3.3 If A is a commutative DG-ring, and $\bar{\mathfrak{a}} \subseteq \overline{A}$ is a finitely generated ideal, we can form the local cohomology functor of A with respect to $\bar{\mathfrak{a}}$ and the local cohomology functor of $\overline{A}$ with respect to $\bar{\mathfrak{a}}$. The former is a functor $\mathsf{D}(A) \to \mathsf{D}(A)$, while the latter is a functor $\mathsf{D}(\overline{A}) \to \mathsf{D}(\overline{A})$.

According to our notation, both should be denoted by $\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}$. In cases where there will be such ambiguity, we solve it by using the notation

$$\mathrm{R}\Gamma^{\overline{A}}_{\bar{\mathfrak{a}}}: \mathsf{D}(\overline{A}) \to \mathsf{D}(\overline{A})$$

for the local cohomology functor of $\overline{A}$ with respect to $\bar{\mathfrak{a}}$, and the notation

$$\mathrm{R}\Gamma^{A}_{\bar{\mathfrak{a}}}: \mathsf{D}(A) \to \mathsf{D}(A)$$

for the local cohomology functor of A with respect to $\bar{\mathfrak{a}}$. Similarly, we will write

$$\mathrm{L}\Lambda^{\overline{A}}_{\bar{\mathfrak{a}}}: \mathsf{D}(\overline{A}) \to \mathsf{D}(\overline{A})$$

for the derived completion functor of $\overline{A}$ with respect to $\bar{\mathfrak{a}}$, and

$$\mathrm{L}\Lambda^{A}_{\bar{\mathfrak{a}}}: \mathsf{D}(A) \to \mathsf{D}(A)$$

for the derived completion functor of A with respect to $\bar{\mathfrak{a}}$.

Lemma 3.4 *Let A be a commutative noetherian DG-ring, and let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. Then for any DG-module M over A with bounded cohomology, there is a natural isomorphism*

$$\mathrm{R}\Gamma^{\overline{A}}_{\bar{\mathfrak{a}}}\big(\mathrm{R}\operatorname{Hom}_A(\overline{A}, M)\big) \cong \mathrm{R}\operatorname{Hom}_A\big(\overline{A}, \mathrm{R}\Gamma^{A}_{\bar{\mathfrak{a}}}(M)\big)$$

in $\mathsf{D}(\overline{A})$.

Proof Let $\mathbf{a}$ be a finite sequence of elements of the ring A^0 whose image in $\overline{A}$ generates the ideal $\bar{\mathfrak{a}}$, and let $\bar{\mathbf{a}}$ be its image in $\overline{A}$. The latter is a finite sequence of elements of the ring $\overline{A}$. By (2.1), there is a natural isomorphism

$$\mathrm{R}\Gamma^{\overline{A}}_{\bar{\mathfrak{a}}}\big(\mathrm{R}\operatorname{Hom}_A(\overline{A}, M)\big) \cong \operatorname{Tel}(\overline{A}; \bar{\mathbf{a}}) \otimes_{\overline{A}} \mathrm{R}\operatorname{Hom}_A(\overline{A}, M).$$

By the base change property of the telescope complex, there is an isomorphism of complexes of $\overline{A}$-modules

$$\operatorname{Tel}(A^0; \mathbf{a}) \otimes_{A^0} \overline{A} \cong \operatorname{Tel}(\overline{A}; \bar{\mathbf{a}}).$$

This implies that

$$\operatorname{Tel}(\overline{A}; \bar{\mathbf{a}}) \otimes_{\overline{A}} \mathrm{R}\operatorname{Hom}_A(\overline{A}, M) \cong \operatorname{Tel}(A^0; \mathbf{a}) \otimes_{A^0} \mathrm{R}\operatorname{Hom}_A(\overline{A}, M).$$

Set $T := A \otimes_{A^0} \operatorname{Tel}(A^0; \mathbf{a})$. Since $\operatorname{Tel}(A^0; \mathbf{a})$ is a K-flat complex of finite flat dimension over A^0, it follows that the DG-module T is K-flat over A and that $\operatorname{fl dim}_A(T) < \infty$. Hence, it holds that

$$\operatorname{Tel}(A^0; \mathbf{a}) \otimes_{A^0} \mathrm{R}\operatorname{Hom}_A(\overline{A}, M) \cong T \otimes^{\mathrm{L}}_A \mathrm{R}\operatorname{Hom}_A(\overline{A}, M).$$

Let $M \cong I$ be a K-injective resolution over A, let $I \otimes_A T \cong J$ be a K-injective resolution over A, and let $A \to B \xrightarrow{\simeq} \overline{A}$ be a semi-free DG-algebra resolution of $\overline{A}$ over A. In

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particular, B is K-projective over A. These resolutions and the naturality of the tensor evaluation morphism induce a commutative diagram in $\mathrm{DGMod}(A)$:

$$\begin{array}{ccccc}
\mathrm{Hom}_A(\overline{A}, I) \otimes_A T & \xrightarrow{\beta} & \mathrm{Hom}_A(\overline{A}, I \otimes_A T) & \xrightarrow{\varphi} & \mathrm{Hom}_A(\overline{A}, J) \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\psi} \\
\mathrm{Hom}_A(B, I) \otimes_A T & \xrightarrow[\gamma]{} & \mathrm{Hom}_A(B, I \otimes_A T) & \xrightarrow[\chi]{} & \mathrm{Hom}_A(B, J).
\end{array}$$

Since I is K-injective over A and T is K-projective over A, it follows that α is a quasi-isomorphism. Similarly, since B is K-projective over A, it follows that χ is a quasi-isomorphism. K-injectivity of J over A implies that ψ is a quasi-isomorphism. Finally, since A is noetherian and T has finite flat dimension over A, it follows by Proposition 2.2(i) that γ is a quasi-isomorphism. Hence, by the 2-out-of-3 property, the $\overline{A}$-linear map $\varphi \circ \beta$ is also a quasi-isomorphism. Hence, there are natural isomorphisms

$$\begin{aligned}
\mathrm{R}\mathrm{Hom}_A(\overline{A}, M) \otimes_A T &\cong \mathrm{Hom}_A(\overline{A}, I) \otimes_A T \\
&\cong \mathrm{Hom}_A(\overline{A}, J) \cong \mathrm{R}\mathrm{Hom}_A(\overline{A}, M \otimes_A T)
\end{aligned}$$

in $\mathsf{D}(\overline{A})$. Finally, note that

$$M \otimes_A T = M \otimes_A (A \otimes_{A^0} \mathrm{Tel}(A^0; \mathbf{a})) \cong M \otimes_{A^0} \mathrm{Tel}(A^0; \mathbf{a}) \cong \mathrm{R}\Gamma^{A}_{\bar{\mathfrak{a}}}(M).$$

Combining all the above natural isomorphisms gives the required result. ∎

We now prove the first main result of this paper.

Theorem 3.5 *Let A be a commutative noetherian DG-ring, and let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. Then for any $M \in \mathsf{D}^{\mathrm{b}}(A)$, there is an inequality*

$$\mathrm{inj\,dim}_A\big(\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}(M)\big) \le \mathrm{inj\,dim}_A(M).$$

Proof According to Theorem 3.2, we have that

$$\begin{aligned}
\mathrm{inj\,dim}_A\big(\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}(M)\big) &= \mathrm{inj\,dim}_A\big(\mathrm{R}\Gamma^{A}_{\bar{\mathfrak{a}}}(M)\big) \\
&= \mathrm{inj\,dim}_{\overline{A}}\big(\mathrm{R}\mathrm{Hom}_A(\overline{A}, \mathrm{R}\Gamma^{A}_{\bar{\mathfrak{a}}}(M))\big).
\end{aligned}$$

Using Lemma 3.4, we have that

$$\mathrm{inj\,dim}_{\overline{A}}\big(\mathrm{R}\mathrm{Hom}_A(\overline{A}, \mathrm{R}\Gamma^{A}_{\bar{\mathfrak{a}}}(M))\big) = \mathrm{inj\,dim}_{\overline{A}}\big(\mathrm{R}\Gamma^{\overline{A}}_{\bar{\mathfrak{a}}}(\mathrm{R}\mathrm{Hom}_A(\overline{A}, M))\big),$$

and by Corollary 1.1 we obtain that

$$\mathrm{inj\,dim}_{\overline{A}}\big(\mathrm{R}\Gamma^{\overline{A}}_{\bar{\mathfrak{a}}}(\mathrm{R}\mathrm{Hom}_A(\overline{A}, M))\big) \le \mathrm{inj\,dim}_{\overline{A}}\big(\mathrm{R}\mathrm{Hom}_A(\overline{A}, M)\big) = \mathrm{inj\,dim}_A(M),$$

where the last equality follows from Theorem 3.2. Combining all of the above, we obtain that

$$\mathrm{inj\,dim}_A\big(\mathrm{R}\Gamma_{\bar{\mathfrak{a}}}(M)\big) \le \mathrm{inj\,dim}_A(M),$$

as claimed. ∎

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Remark 3.6 Given a commutative ring A, and a finitely generated ideal $\mathfrak{a} \subseteq A$, unlike Remark 2.1, if A is non-noetherian, in general the right derived functor of the $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}$ might be different from the local cohomology functor with respect to $\mathfrak{a}$ from Section 2.2. However, if $\mathfrak{a}$ satisfies a technical condition called weak proregularity (see [6, Definition 4.21]), these functors coincide. In [7, Proposition 3.1], there is an example of a commutative ring A, a finitely generated (in fact, principal) weakly proregular ideal $\mathfrak{a} \subseteq A$, and an injective A-module I such that $\Gamma_{\mathfrak{a}}(I)$ is not an injective A-module.

4 Flat Dimension of Derived Completion

The aim of this section is to prove Theorem 1.4. The next result is dual to Theorem 3.2.

Theorem 4.1 *Let A be a commutative DG-ring, and let M be a DG-module over A. Then there is an equality*

$$\operatorname{fl\,dim}_A(M) = \operatorname{fl\,dim}_{\mathrm{H}^0(A)}\left(\mathrm{H}^0(A) \otimes^{\mathrm{L}}_A M\right).$$

Proof For any $N \in \mathsf{D}(\overline{A})$, the isomorphism

$$N \otimes^{\mathrm{L}}_A M \cong N \otimes^{\mathrm{L}}_{\overline{A}} (\overline{A} \otimes^{\mathrm{L}}_A M)$$

shows that

$$\operatorname{fl\,dim}_A(M) \geq \operatorname{fl\,dim}_{\overline{A}}\left(\overline{A} \otimes^{\mathrm{L}}_A M\right).$$

To prove the converse, assume that $\operatorname{fl\,dim}_{\overline{A}}(\overline{A} \otimes^{\mathrm{L}}_A M) = n < \infty$, and let N be a bounded DG-module. If $\operatorname{amp}(N) = 0$, there is an isomorphism $N \cong \mathrm{H}^m(N)[-m]$ for some $m \in \mathbb{Z}$, which implies that

$$M \otimes^{\mathrm{L}}_A N \cong (M \otimes^{\mathrm{L}}_A \overline{A}) \otimes^{\mathrm{L}}_{\overline{A}} \mathrm{H}^m(N)[-m].$$

Hence, the fact that

$$\operatorname{fl\,dim}_{\overline{A}}(\overline{A} \otimes^{\mathrm{L}}_A M) = n$$

implies that in this case $\operatorname{Tor}^i_A(N, M) = 0$ for all $i > n - \inf N$. Now, proceeding by induction on $\operatorname{amp}(N)$ exactly as in the proof of Theorem 3.2, we obtain the general case for an arbitrary bounded DG-module N. ∎

We will now use the terminology introduced in Remark 3.3.

Lemma 4.2 *Let A be a commutative noetherian DG-ring, and let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. Then for any DG-module M over A with bounded cohomology, there is a natural isomorphism*

$$\mathrm{L}\Lambda^{\overline{A}}_{\bar{\mathfrak{a}}}(\overline{A} \otimes^{\mathrm{L}}_A M) \cong \overline{A} \otimes^{\mathrm{L}}_A \left(\mathrm{L}\Lambda^{A}_{\bar{\mathfrak{a}}}(M)\right)$$

in $\mathsf{D}(\overline{A})$.

Proof Let $\mathbf{a}$ be a finite sequence of elements of the ring A^0 whose image in $\overline{A}$ generates the ideal $\bar{\mathfrak{a}}$, and let $\bar{\mathbf{a}}$ be its image in $\overline{A}$. The latter is a finite sequence of elements of the ring $\overline{A}$. By (2.2), there is a natural isomorphism

$$\mathrm{L}\Lambda^{\overline{A}}_{\bar{\mathfrak{a}}}(\overline{A} \otimes^{\mathrm{L}}_A M) \cong \operatorname{Hom}_{\overline{A}}\left(\operatorname{Tel}(\overline{A}; \bar{\mathbf{a}}), \overline{A} \otimes^{\mathrm{L}}_A M\right).$$

As in the proof of Lemma 3.4, setting $T := \operatorname{Tel}(A^0;\mathbf{a}) \otimes_{A^0} A$, observing that T is K-projective over A, and using adjunctions, this is naturally isomorphic to

$$\operatorname{Hom}_A(T, \overline{A} \otimes^{\mathrm{L}}_A M).$$

Let $P \cong M$ be a K-flat resolution over A, let $F \cong \operatorname{Hom}_A(T,P)$ be a K-flat resolution over A, and let $A \to B \cong \overline{A}$ be a semi-free DG-algebra resolution of $\overline{A}$ over A. We obtain a commutative diagram in $\operatorname{DGMod}(A)$:

$$\begin{array}{ccccc}
F \otimes_A \overline{A} & \xrightarrow{\beta} & \operatorname{Hom}_A(T,P) \otimes_A \overline{A} & \xrightarrow{\varphi} & \operatorname{Hom}_A(T, P \otimes_A \overline{A}) \\
\uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\psi} \\
F \otimes_A B & \xrightarrow[\gamma]{} & \operatorname{Hom}_A(T,P) \otimes_A B & \xrightarrow[\chi]{} & \operatorname{Hom}_A(T, P \otimes_A B).
\end{array}$$

The fact that F is K-flat over A implies that α is a quasi-isomorphism, while K-flatness of B over A implies that γ is a quasi-isomorphism. The fact that P is K-flat over A and T is K-projective over A implies that ψ is a quasi-isomorphism. Finally, since A is noetherian and $\operatorname{proj\,dim}_A(T) < \infty$, by Proposition 2.2(ii), the map χ is a quasi-isomorphism. It follows by the 2-out-of-3 property that the $\overline{A}$-linear map $\varphi \circ \beta$ is a quasi-isomorphism. Hence, there are natural isomorphisms

$$\begin{aligned}
\mathrm{R}\operatorname{Hom}_A(T,M) \otimes^{\mathrm{L}}_A \overline{A} &\cong F \otimes_A \overline{A} \cong \operatorname{Hom}_A(T, P \otimes_A \overline{A}) \\
&\cong \mathrm{R}\operatorname{Hom}_A(T, M \otimes^{\mathrm{L}}_A \overline{A})
\end{aligned}$$

in $\mathsf{D}(\overline{A})$. The result now follows by combining all these isomorphisms and using the fact that there is a natural isomorphism $\mathrm{L}\Lambda^{A}_{\bar{\mathfrak{a}}}(M) \cong \mathrm{R}\operatorname{Hom}_A(T,M)$ in $\mathsf{D}(A)$. ∎

Here is the second main result of this paper.

Theorem 4.3 *Let A be a commutative noetherian DG-ring, and let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. Then for any $M \in \mathsf{D}^{\mathrm{b}}(A)$, there is an inequality*

$$\operatorname{fl\,dim}_A\big(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(M)\big) \le \operatorname{fl\,dim}_A(M).$$

Proof By Theorem 4.1,

$$\begin{aligned}
\operatorname{fl\,dim}_A\big(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(M)\big) &= \operatorname{fl\,dim}_A\big(\mathrm{L}\Lambda^{A}_{\bar{\mathfrak{a}}}(M)\big) \\
&= \operatorname{fl\,dim}_{\overline{A}}\big(\overline{A} \otimes^{\mathrm{L}}_A \mathrm{L}\Lambda^{A}_{\bar{\mathfrak{a}}}(M)\big).
\end{aligned}$$

Using Lemma 4.2, we have that

$$\operatorname{fl\,dim}_{\overline{A}}\big(\overline{A} \otimes^{\mathrm{L}}_A \mathrm{L}\Lambda^{A}_{\bar{\mathfrak{a}}}(M)\big) = \operatorname{fl\,dim}_{\overline{A}}\big(\mathrm{L}\Lambda^{\overline{A}}_{\bar{\mathfrak{a}}}(\overline{A} \otimes^{\mathrm{L}}_A M)\big),$$

and by Corollary 1.3 we obtain that

$$\operatorname{fl\,dim}_{\overline{A}}\big(\mathrm{L}\Lambda^{\overline{A}}_{\bar{\mathfrak{a}}}(\overline{A} \otimes^{\mathrm{L}}_A M)\big) \le \operatorname{fl\,dim}_{\overline{A}}(\overline{A} \otimes^{\mathrm{L}}_A M) = \operatorname{fl\,dim}_A(M),$$

where the last equality follows from Theorem 4.1. Combining all of the above, we obtain that

$$\operatorname{fl\,dim}_A\big(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(M)\big) \le \operatorname{fl\,dim}_A(M),$$

as claimed. ∎

Remark 4.4 As in Remark 3.6, this result is false in general if the (DG-)ring is not assumed to be noetherian (even if the ideal is finitely generated by a regular sequence). See [12, Theorem 6.2] for an example.

Remark 4.5 Let A be a commutative noetherian ring such that the Krull dimension of A is ≥ 1, and let $B = A[x]$. Then B is a projective B-module, and it follows from [2, Theorem 1] that the B-module $\Lambda_{(x)}(B)$ is not projective.

We finish the paper with an important application of Theorem 4.3. A basic result in commutative algebra states that if A is a commutative noetherian ring and $\mathfrak{a} \subseteq A$ is an ideal, then the canonical map $A \to \widehat{A}$ from A to its $\mathfrak{a}$-adic completion is flat. Here is the analogue of this result in derived commutative algebra.

Let A be a commutative noetherian DG-ring, and let $\bar{\mathfrak{a}} \subseteq \overline{A}$ be an ideal. The derived $\bar{\mathfrak{a}}$-adic completion of A is a commutative DG-ring, denoted by $\mathrm{L}\Lambda(A, \bar{\mathfrak{a}})$. It was defined in [9, Theorem 0.2]. It is a derived analogue of the $\mathfrak{a}$-adic completion $\widehat{A}$. There is a natural map $A \to \mathrm{L}\Lambda(A, \bar{\mathfrak{a}})$, but this is not a map of DG-rings. Instead, it only exists in a suitable homotopy category (see [9] for details). Concretely, one can realize it as follows: there is a commutative DG-ring P, a quasi-isomorphism $P \to A$, and a natural map of DG-rings $P \to \mathrm{L}\Lambda(A, \bar{\mathfrak{a}})$. Since P and A are quasi-isomorphic, the triangulated categories $\mathrm{D}(A)$ and $\mathrm{D}(P)$ are isomorphic. Using this isomorphism and the map $P \to \mathrm{L}\Lambda(A, \bar{\mathfrak{a}})$, we can view $\mathrm{L}\Lambda(A, \bar{\mathfrak{a}})$ as an object of $\mathrm{D}(A)$. According to [9, Proposition 3.58], this object is isomorphic to $\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A)$.

The above paragraph explains that the analogue of the number $\operatorname{fl\,dim}_A(\widehat{A})$ in derived commutative algebra is the number $\operatorname{fl\,dim}_A(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A))$. This explains the importance of our final result.

Corollary 4.6 *Let A be a commutative noetherian DG-ring, and assume that A has bounded cohomology. Let $\bar{\mathfrak{a}} \subseteq \mathrm{H}^0(A)$ be an ideal. Then*

$$\operatorname{fl\,dim}_A\left(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A)\right) = 0.$$

Proof Since $\operatorname{fl\,dim}_A(A) = 0$, it follows from Theorem 4.3 that

$$\operatorname{fl\,dim}_A\left(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A)\right) \leq 0.$$

On the other hand, by [9, Proposition 6.1], we have that

$$\mathrm{H}^0\big(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A)\big) \cong \Lambda_{\bar{\mathfrak{a}}}(\mathrm{H}^0(A)) \ncong 0.$$

Hence, $\sup(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A)) = 0$, and we deduce that

$$\mathrm{H}^0(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A) \otimes^{\mathrm{L}}_A \mathrm{H}^0(A)) \neq 0,$$

which implies that

$$\operatorname{fl\,dim}_A\left(\mathrm{L}\Lambda_{\bar{\mathfrak{a}}}(A)\right) \geq 0.$$ ∎

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Faculty of Mathematics, Bielefeld University, 33501 Bielefeld, Germany
e-mail: LShaul@math.uni-bielefeld.de

(Current Address) *Department of Mathematics, Ben Gurion University of the Negev, Beer Sheva 84105, Israel*
e-mail: shlir@post.bgu.ac.il

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