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THE TYPE I PART OF THE REGULAR REPRESENTATION

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Let G be a discrete group and let $H = L^2(G)$, with norm $|\ |$. Let $B(H)$ be the ring of bounded operators on H with the norm

$$||T|| = \sup\{|T(x)| : x \in H, |x| = 1\}.$$

The right regular representation of G on H induces an injection $\rho : C[G] \rightarrow B(H)$, and $W(G)$ is the closure of the image of ρ in the weak operator topology on $B(H)$ (C = complex numbers). Using ρ, we identify $C[G]$ with its image in $W(G)$. The techniques of this paper are taken from [**3**], so familiarity with it would be helpful. [**4**] is a general reference for W^*-algebras.

$W(G)$ is a W^*-algebra of finite type. Hence there are mutually orthogonal central projections $e, e_1, e_2, \ldots$ in $W(G)$ whose least upper bound is the identity, and such that $e\,W(G)$ is of type II and $e_nW(G)$ is of type I_n; more precisely, $e_nW(G)$ is isomorphic to the ring of $n \times n$ matrices over its center. Kaniuth has characterized those groups for which $W(G)$ is purely of type I or type II as follows.

THEOREM 1 (Kaniuth [**2**]; Smith [**3**]). *Let Δ be the subgroup of G consisting of those elements with only finitely many conjugates. Then*

(A) *$W(G)$ is of type* II *if and only if either*

(i) *$[G : \Delta] = \infty$, or*

(ii) *$[G : \Delta] < \infty$ and Δ' is infinite.*

(B) *$W(G)$ is of type* I *if and only if G has an abelian subgroup of finite index.*

Martha Smith's later proof is more direct than Kaniuth's original proof and reveals two interesting facts: (1) The support of all the central projections $e, e_1, e_2, \ldots$ lies in a finite subgroup of G; (2) There are only finitely many n for which e_n is nonzero. We will use these facts and the methods of [**3**] to prove the following result, which identifies the type I part of $W(G)$ when it is nonzero.

THEOREM 2. *Suppose G is a group with $[G : \Delta] < \infty$ and Δ' finite. Let*

$$K = \bigcap \{L' : [G : L] < \infty\}, \quad e_K = \frac{1}{|K|} \sum \{g : g \in K\}.$$

Then $e_KW(G)$ is the type I *part of $W(G)$.*

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The rest of the paper is devoted to proving Theorem 2. Note that since $[G : \Delta] < \infty$ and Δ' is finite, K is a finite subgroup of Δ. Thus $[G : C_G(K)] < \infty$; moreover

$$K = \bigcap \{ (\Delta \cap L)' : [G : L] < \infty \}$$

and since K and each such $(\Delta \cap L)'$ is finite there must be a finite set $L_1, \ldots, L_k$ of such L such that $K = \bigcap (\Delta \cap L_i)'$. Then

$$N = \Delta \cap L_1 \cap \ldots \cap L_k \cap C_G(K)$$

is a subgroup of finite index in $C_G(K)$ such that $N' = K$. Further,

(1) N is of finite index in G;
(2) N is nilpotent of class $\leqq 2$;
(3) $N' = K$, a finite group in the center of N;
(4) If L is a subgroup of finite index in N, $L' = K$.

LEMMA 3. *Suppose L is a subgroup of finite index in G. Then every $h \in K$ is a commutator $h = x^{-1}y^{-1}xy = \langle x, y \rangle$ of two elements of L.*

Proof. We can assume $L \subseteq N$, and we choose a finite generating set for K recursively as follows: let $x_1, y_1 \in L$ be such that $\langle x_1, y_1 \rangle = h_1, h_1 \neq 1$. If $K = \text{gp}(h_1)$, we are done; if not, note that $[G : C_L(x_1, y_1)] < \infty$, so $C_L(x_1, y_1)' = K$ and we can choose $x_2, y_2 \in C_L(x_1, y_1)$ with $\langle x_2, y_2 \rangle = h_2, h_2 \notin \text{gp}(h_1)$. If $K \neq \text{gp}(h_1, h_2)$, continue by choosing $x_3, y_3 \in C_L(x_1, y_1, x_2, y_2)$ with $\langle x_3, y_3 \rangle = h_3, h_3 \notin \text{gp}(h_1, h_2)$, etc. Since K is finite we eventually get $\text{gp}(h_1, \ldots, h_n) = K$ and the elements $x_1, y_1, \ldots, x_n, y_n$ we have chosen satisfy

$$\langle x_i, y_i \rangle = h_i,$$
$$\langle x_i, y_j \rangle = \langle x_i, x_j \rangle = \langle y_i, y_j \rangle = 1 \text{ for } i \neq j.$$

Finally, suppose $h \in K$, and let $h = h_1^{i_1} \ldots h_n^{i_n}$. It is easy to see that the commutativity conditions above imply that $h = \langle x_1^{i_1} \ldots x_n^{i_n}, y_1 \ldots y_n \rangle$.

Let S_{2n} denote the standard identity

$$S_{2n}(x_1, \ldots, x_{2n}) = \sum \pm x_{\pi(1)} \ldots x_{\pi(2n)},$$

where the sum is over all permutations π of $\{1, \ldots, 2n\}$ and the sign is positive for even permutations and negative for odd permutations, and let $A_{2n}(G)$ denote the ideal of $W(G)$ generated by all $S_{2n}(x_1, \ldots, x_{2n}), x_i \in W(G)$.

LEMMA 4. *For all n, $1 - e_K \in A_{2n}(G)$.*

Proof. Let $B = (1 - e_K)\, C[K]$. B is the augmentation ideal of $C[K]$, the ideal generated by all $\{1 - h, h \in K\}$. We are going to show that

$$A_{2n}(G) \supseteq B^n\, C[G] = (1 - e_K)^n C[G] = (1 - e_K) C[G].$$

First note that B^n is generated by all n-fold products $(h_1 - 1) \ldots (h_n - 1)$ with $h_1, \ldots, h_n \in K$ and choose $x_1, y_1, \ldots, x_n, y_n \in G$ as follows, using Lemma 3: let $h_1 = \langle x_1, y_1 \rangle$, $h_2 = \langle x_2, y_2 \rangle$ where $x_2, y_2 \in C_G(x_1, y_1)$, $h_3 =$

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$\langle x_3, y_3 \rangle$, where $x_3, y_3 \in C_G(x_1, y_1, x_2, y_2)$ etc. Letting $[x, y] = xy - yx = yx(\langle x, y \rangle - 1)$, these elements satisfy

$$(*) \quad [x_i, y_i] = y_i x_i (h_i - 1)$$
$$\langle x_i, y_j \rangle = \langle x_i, x_j \rangle = \langle y_i, y_j \rangle = 1 \text{ for } i \neq j.$$

Next consider the standard identity

$$S_{2n}(g_1, \ldots, g_{2n}) = \sum \pm g_{\pi(1)} \ldots g_{\pi(2n)}.$$

It can be re-expressed as

$$S_{2n}(g_1, \ldots, g_{2n}) = \sum \pm [g_{i\pi(1)}, g_{j\pi(1)}] \ldots [g_{i\pi(n)}, g_{j\pi(n)}]$$

where the sum ranges over all partitions of the set $\{g_1, \ldots, g_{2n}\}$ into n disjoint two-element sets $\{g_{i1}, g_{j1}\} \ldots \{g_{in}, g_{jn}\}$ (labelled arbitrarily) and all permutations π of $\{1, \ldots, n\}$. The sign in front of each product depends only on the partition and the choice of labelling for $\{g_{it}, g_{jt}\}$. It is independent of π since $g_{is}g_{js}g_{it}g_{jt}$ and $g_{it}g_{jt}g_{is}g_{js}$ (corresponding to a transposition of pairs) differ by an even permutation when considered as permutations of all four letters.

When we use the above expression to evaluate $S_{2n}(x_1, y_1, \ldots, x_n, y_n)$ and invoke the commutativity relations $(*)$, we see that the only surviving terms of the right hand side are permutations of $[x_1, y_1], \ldots, [x_n, y_n]$ and these all have the sign $+1$. Moreover, for any permutation π of $\{1, \ldots, n\}$ invoking $(*)$ yields

$$\begin{aligned}[x_{\pi(1)}, y_{\pi(1)}] \ldots [x_{\pi(n)}, y_{\pi(n)}] \\ &= y_{\pi(1)}x_{\pi(1)}(h_{\pi(1)} - 1) \ldots y_{\pi(n)}x_{\pi(n)}(h_{\pi(n)} - 1) \\ &= (h_1 - 1) \ldots (h_n - 1) y_1 x_1 \ldots y_n x_n.\end{aligned}$$

Therefore,

$$\begin{aligned}S_{2n}(x_1, y_1, \ldots, x_n, y_n) &= \sum [x_{\pi(1)}, y_{\pi(1)}] \ldots [x_{\pi(n)}, y_{\pi(n)}] \\ &= n!(h_i - 1) \ldots (h_n - 1) y_1 x_1 \ldots y_n x_n.\end{aligned}$$

Recalling that B^n is generated by all n-fold products $(h_1 - 1) \ldots (h_n - 1)$ with $h_1, \ldots, h_n \in K$, this shows that

$$A_{2n}(G) \supseteq B^n C[G] = (1 - e_K)^n C[G] = (1 - e_K) C[G].$$

Proof of Theorem 2. Since K is finite, there is a natural homomorphism

$$W(G) \to W(G/K).$$

G/K contains an abelian subgroup of finite index and hence $W(G/K)$ is of type I, by Theorem 1(B). The kernel of the above homomorphism is $(1 - e_K) W(G)$ and its restriction to $e_K W(G)$ induces an isomorphism between $e_K W(G)$ and $W(G/K)$. Thus $e_K W(G)$ is a type I summand of $W(G)$.

Now Theorem 2 will be proved if we show that $(1 - e_K) W(G)$ contains no type I summand. Suppose conversely that it contains a nonzero type I_n part

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$f_n W(G)$ for some n. Then $f_n W(G)$ is isomorphic to the ring of $n \times n$ matrices over its center and hence satisfies the standard identity S_{2n} [**1**]. This says that $f_n A_{2n}(G) = 0$ and contradicts Lemma 4, which says that $1 - e_K \in A_{2n}(G)$.

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