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MATHIEU GROUP M_{24} AND MODULAR FORMS

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§ 0. Introduction

In [6], Mason reported some connections between sporadic simple group M_{24} and certain cusp forms which appear in the 'denominator' of Thompson series assigned to Fisher-Griess's group F_1. In this paper, we discuss the 'numerator' of these Thompson series.

We state our result precisely. Since M_{24} is a subgroup of the symmetric group S_{24} of degree 24, we can write for each $m \in M_{24}$,

$$m = (n_1)(n_2) \cdots (n_s), \qquad n_1 \geqq \cdots \geqq n_s \geqq 1,$$

to mean that m is a product of cycle of length n_i, $1 \leqq i \leqq s$. To each $m = (n_1) \cdots (n_s)$, we associate modular forms $\eta_m(z)$ and $\vartheta_m(z)$ as follows; let

$$\eta_m(z) = \eta(n_1 z) \cdots \eta(n_s z),$$

where $\eta(z)$ is the Dedekind η-function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = \exp(2\pi\sqrt{-1}z)$ and $z \in \boldsymbol{H} = \{z \in \boldsymbol{C} \mid \operatorname{Im} z > 0\}$. Then, in [6], Mason showed that $\eta_m(z)$ is a cusp form of weight $s/2$ on $\Gamma_0(n_1 n_s)$ with some character ε_m and is also a common eigenfunction of all Hecke operators, (see also [3]).

On the other hand, it is well-known that M_{24} acts on the Leech lattice L as isometries. To each $m \in M_{24}$, put

$$L^m = \{x \in L \mid m \cdot x = x\}.$$

Then L^m is an even integral, positive definite quadratic lattice of rank s. Let $\vartheta_m(z)$ denote the theta function of L^m:

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$$\vartheta_m(z) = \sum_{x \in L^m} \exp(\pi\sqrt{-1}z\langle x, x\rangle)$$

where $\langle\ ,\ \rangle$ is the bilinear form on L. Then $\vartheta_m(z)$ is a modular form of weight $s/2$, however, it is not so easy to determine $\vartheta_m(z)$ explicitly.

Here comes Conway-Norton's remarkable discoveries 'Monstrous Moonshine'. Especially the following conjecture is very important:

CONJECTURE 0.1. For each $m \in M_{24}$, put

$$j_m(z) = \frac{\vartheta_m(z)}{\eta_m(z)} .$$

Then there exists an element g in F_1 such that the Thompson series $T_g(z)$ assigned to g in [1] coincides with $j_m(z)$ up to a constant term.

In this paper, we describe $\vartheta_m(z)$ explicitly as a linear sum of Eisenstein series and $\eta_m(z)$ assuming the above conjecture. The main result is as follows:

THEOREM 0.1. *For* $m \in M_{24}$, $m \neq 12^2, 4^6, 2^{12}, 10^2\cdot 2^2, 12\cdot 6\cdot 4\cdot 2, 4^4\cdot 2^4$, *there exists a unique modular form* $\theta_m(z) = 1 + \sum_{n\geqq 1} a_n(m)q^n$, $a_n(m) \in Z$ *satisfying the following conditions.*

(0.1) *There exists* $g \in F_1$ *such that*

$$\frac{\theta_m(z)}{\eta_m(z)} = T_g(z) + c, \text{ for some constant } c .$$

(0.2) $a_1(m) = 0$.

(0.3) $a_n(m)$ *are even integers for all* n .

(0.4) $a_n(m) \geqq 0$ *for all* n .

(0.5) *If* $m^r = m'$ *for some* $r \in Z$, *then* $a_n(m) \leqq a_n(m')$ *for all* n .

For the remaining 6 cases, if we add one more condition that

(0.6) $a_2(m) =$ the number of elements in $\{x \in L^m \mid \langle x, x\rangle = 4\}$,

we can prove that there exists a unique $\theta_m(z)$ satisfying (0.1) $\sim$ (0.6).

We already applied these result to construct moonshines for $PSL_2(F_7)$ ([4]).

In the subsequent paper, we shall apply the same argument to all the elements of the automorphism group of the Leech lattice. In this case, we need to modify the above conjecture slightly (see [5]).

§1.

Let G be a finite group and Λ be an even integral, positive definite quadratic lattice on which G acts as isometries of Λ. For any $g \in G$, Λ^g is the set of fixed points of g on Λ. Then Λ^g is also an even integral, positive definite quadratic lattice. Let $(\Lambda^g)^\#$ be the dual lattice of Λ^g; $(\Lambda^g)^\# = \{x \in \boldsymbol{Q}\Lambda^g \mid \langle x, y\rangle \in \boldsymbol{Z}$ for all $y \in \Lambda^g\}$. Let ℓ_g denote the exponent of $(\Lambda^g)^\#/\Lambda^g$. The theta function $\theta(z; \Lambda^g)$ of Λ^g is defined by

$$\theta(z; \Lambda^g) = \sum_{x \in \Lambda^g} \exp(\pi\sqrt{-1}z\langle x, x\rangle) .$$

We assume that the rank of Λ^g is always even which is denoted by $2k_g$. Let $\{u_i\}$ be a basis of Λ^g and put $A_g = (\langle u_i, u_j\rangle)$. Then A_g is an even integral, positive definite symmetric matrix. Let N_g be the smallest positive integer such that $N_g \cdot A_g^{-1}$ is even integral. Then $\theta(z; \Lambda^g)$ is a modular form on $\Gamma_0(N_g)$ of weight k_g with some character.

LEMMA 1.1. *Suppose that g is of order d. Then N_g divides $2dN_e$ where e is the identity element of G.*

Proof. It is easily seen that ℓ_g divides N_g and N_g divides $2\ell_g$. Combining this with Lemma 2 in [7], we get the proof.

COROLLARY 1.1. *Let m be any element in M_{24} of order d. Then $\vartheta_m(z)$ is a modular form on $\Gamma_0(2d)$ with some character.*

Proof. The pair M_{24} and the Leech lattice L satisfy the above situation in Lemma 1.1. Since L is unimodular, $N_e = 1$. Hence we get the proof.

LEMMA 1.2. *Let $\theta(z; \Lambda^g) = 1 + \sum_{n \geqq 1} b_n(g)q^n$ be the Fourier expansion. Then we have*

(1.1) *$b_n(g)$ are even integers for all n .*

(1.2) *$b_n(g) \geqq 0$.*

(1.3) *If $g^r = g'$, then $b_n(g) \leqq b_n(g')$ for all n .*

Proof. These are obvious.

COROLLARY 1.2. *Let $\vartheta_m(z) = 1 + \sum_{n \geqq 1} a_n(m)q^n$ be as in the Introduction. Then $a_n(g)$ satisfy the conditions* (0.2) (0.3) (0.4) (0.5) *in Theorem* 0.1.

Proof. It is well known that L has no vectors of length 2, so $a_1(m) = 0$ for all m. Other statements follow from Lemma 1.2.

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LEMMA 1.3. *The notation being as above, suppose that* $\vartheta_m(z)/\eta_m(z) = T_g(z) - c$. *Then c is equal to the Fourier coefficient of q^2 in $\eta_m(z)$.*

Proof. This follows from the fact that $a_1(m) = 0$.

The proof of Theorem 0.1 is done by computation with the help of the above lemmas. We explain the argument of the proof only by taking a few examples. For $q^n(a_0, a_1, a_2, \cdots)$, we mean the Fourier expansion $a_0q^n + a_1q^{n+1} + a_2q^{n+2} + \cdots$. For mA, mB, $\cdots$, we mean the Atlas name of elements of F_1 in [1]. Take $m = 3^6 \cdot 1^6$. Then $\eta_{36.16}(z) = q(1, -6, 9, 4, 6, \cdots)$. Elements g of F_1 satisfying (0.1) (0.2) and (0.3) are 3A, 6A, 6C, 12A, 12C, 12E, 24A, 24B, 24D and 48A. Among these, only 3A satisfies the condition (0.4). Take $m = 2^8 \cdot 1^8$. Then $\eta_{28.18}(z) = q(1, -8, 12, 64, -210, \cdots)$. Elements g of F_1 satisfying (0.1) (0.2) (0.3) and (0.4) are 1A and 2A. $\eta_{28.18}(z) \times (T_{1A}(z) + 8) = q^0(1, 0, 196832, \cdots)$, but $\vartheta_{124}(z) = q^0(1, 0, 196560, \cdots)$ and $196560 < 196832$; this contradicts the condition (0.5). Similar argument can be applied to all $m \in M_{24}$, except 12^2, 6^4, 4^6, 2^{12}, $10^2 \cdot 2^2$, $12 \cdot 6 \cdot 4 \cdot 2$, to determine uniquely the solution $\theta_m(z)$ which satisfies (0.1) $\sim$ (0.5).

For the remaining cases, the solution $\theta_m(z)$ which satisfies (0.1) $\sim$ (0.5) is not uniquely determined. To choose the unique solution, we need one more condition (0.6). To state all the argument and computations is too tedious, so we state only the results in Table I in Appendix.

§ 2.

We give several remarks

Remark 2.1. In [2], Mckay, Dummit and Kisilevsky considered the products of η-functions which have multiplicative Fourier coefficients. There are 30 such functions which are called multiplicative products of η-functions. Among them, 2 cases are modular forms of half integral weight, and the remaining 28 cases are characterized by the property that they are primitive cusp forms, (see [3]). On the other hand, there are close connections between these and Frame shape associated to rational representations of finite groups: for example, for all $m \in M_{24}$, $\eta_m(z)$ have multiplicative Fourier coefficients.

Therefore, we consider whether all the multiplicative products of η-functions have the similar property to Theorem 0.1. The result is as follows:

PROPOSITION 2.1. *Let $f(z)$ be a multiplicative product of η-functions*

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which does not coincides with $\eta_m(z)$ for $m \in M_{24}$. Then there exists a theta function $\vartheta(z)$ such that $\vartheta(z)/f(z)$ is a generator of the modular function field of Γ which is of genus 0 and contains $\Gamma_0(N)$ for some N.

Proof. The proof is done by giving such $\vartheta(z)$ explicitly as follows;

m	$\vartheta(z)$	symbol
$18\cdot 6$	$\theta(18z)\theta(6z)$	$36\|3+$
$9^2\cdot 3^2$	$\theta\left(z;\begin{bmatrix}2&1\\1&2\end{bmatrix}\right)^2$	$9\|3+$
$6^3\cdot 2^3$	$\theta(6z)^3\theta(2z)^3$	$12+$
$16\cdot 8$	$\theta(16z)\theta(8z)$	$32\|4+$
$8^2\cdot 4^2$	$\theta(8z)^2\theta(4z)^2$	$16\|2+$
$20\cdot 4$	$\theta(20z)\theta(4z)$	$40\|2+$
$22\cdot 2$	$\theta(22z)\theta(2z)$	$44+$
8^3	$\theta(8z)^3$	$16\|4+$
24	$\theta(24z)$	$48\|12+$

Here the symbol means the same as in [1] and

$$\theta(z) = \sum_{n\in Z} \exp\,(\pi\sqrt{-1}\,zn^2)\,.$$

Remark 2.2. To prove Conjecture 0.1, we need only to compute Fourier coefficients of q^n of $\vartheta_m(z)$ for a few small n and to check that these coincide with $a_n(m)$ of $\theta_m(z)$. This computation may be possible to use the explicit description of L and M_{24} given in [8], but we do not yet run this computation.

So, for the time being, it is not yet proved that $\theta_m(z)$ in Theorem 0.1 are theta functions of some even integral, positive definite quadratic lattices, for some $m \in M_{24}$, for example $m = 2^8\cdot 1^8,\ 5^4\cdot 1^4,\ 7^3\cdot 1^3$, etc. However, there is a following fact.

PROPOSITION 2.2. *Let m be $2^8\cdot 1^8$, $3^6\cdot 1^6$, $5^4\cdot 1^4$, $7^3\cdot 1^3$, $11^2\cdot 1^2$, and $23\cdot 1$. Then there exists a theta function $\theta(z; T_m)$ such that*

$$\theta(z; T_m) = \theta_m(z) + c_m\eta_m(z)$$

where c_m is a non-zero constant.

Proof. The proof is done by giving T_m explicitly as follows:

m	T_m	c_m
$2^8\cdot 1^8$	$\begin{bmatrix}2&1&1&1\\1&2&0&0\\1&0&2&0\\1&0&0&2\end{bmatrix}^4$	96

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m	T_m	c_m
$3^6\cdot 1^6$	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^6$	36
$5^4\cdot 1^4$	$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix}^2$	12
$7^3\cdot 1^3$	$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}^3$	6
$11^2\cdot 1^2$	$\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}^2$	4
$23\cdot 1$	$\begin{bmatrix} 2 & 1 \\ 1 & 12 \end{bmatrix}$	2

Here T_m are given by the corresponding even integral, positive definite symmetric matrix, and A^n means n-times direct sum of A. The reason why we can find such theta functions is the following: let A_m be the same as in Remark 2.3. If we assume that the conjecture 0.1 is true, we can compute the determinant of A_m in Proposition 2.4. We choose T_m whose determinant, level and rank are the same as those of A_m. Then, since the level of the associated theta functions is a prime number, it is proved that Proposition 2.2 is true. The detail will appear in the subsequent paper.

The similar phenomena can be found when the level is not a prime. The existence of such theta functions is closely related to the existence of various moonshines of $PSL_2(F_q)$.

PROPOSITION 2.3. *Let m be $6^2\cdot 3^2\cdot 2^2\cdot 1^2$. $15\cdot 5\cdot 3\cdot 1$, $14\cdot 7\cdot 2\cdot 1$ and $10^2\cdot 2^2$. Then there exists a theta function $\theta(z;T_m)$ such that*

$$\theta(z;T_m) = \theta_m(z) + c_m\eta_m(z)\,,$$

with some constant c_m.

Proof. The proof is done by giving T_m explicitly as follows:

m	T_m	c_m
$6^2\cdot 3^2\cdot 2^2\cdot 1^2$	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^2 \oplus \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^2$	12
$15\cdot 5\cdot 3\cdot 1$	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \oplus \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$	6
$14\cdot 7\cdot 2\cdot 1$	$\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \oplus \begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}$	2
$10^2\cdot 2^2$	$\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix}^2$	4 .

Remark 2.3. We consider the index of L^m in $(L^m)^\sharp$. Let A_m denote the corresponding matrix to L^m. Then the determinant of A_m is equal to the index of L^m in $(L^m)^\sharp$.

PROPOSITION 2.4. *For any $m = (n_1) \cdots (n_s)$ in M_{24}, assume that $\vartheta_m(z) = \theta_m(z)$ in Theorem* 0.1. *Then it holds that the index of L^m in $(L^m)^\sharp = n_1 n_2 \cdots n_s$.*

Proof. It is well-known that

$$\theta(z; A_m)(-iz)^{k_m} = (\det A_m)^{-\frac{1}{2}}\theta\left(-\frac{1}{N_m z},\ N_m^{-1}\cdot A_m\right)$$

where N_m, $2k_m$ denote the level and the rank of A_m respectively. Hence, by calculating $\theta_m(-1/z)$, we can know the determinant of A_m. Since we know the explicit description of $\theta_m(z)$ by the linear sum of Eisenstein series and $\eta_m(z)$, it is easy to calculate $\theta_m(-1/z)$.

Appendix; Table I, II, III

Table I; For any $m \in M_{24}$, we describe $\theta_m(z)$ in Theorem 0.1 as the linear sum of Eisenstein series and $\eta_m(z)$ and also give the corresponding element g in F_1 which satisfies the condition (0.1). For Eisenstein series, we use the following notation: For even $k \geqq 4$,

$$E_k(z) = 1 - \frac{2_k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\,,$$

is the Eisenstein series of weight k on $SL_2(Z)$ where B_k is the k-th Bernoulli number and $\sigma_r(n) = \sum_{\substack{d|n\\ d>0}} d^r$.

Let

$$G_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma_1(n)q^n\,.$$

For any characters χ and ψ defined modulo N and M, and for any odd integer k,

$$E_{\chi,\psi}^{(k)}(z) = c_{k,\chi,\psi} + \sum_{\substack{m>0\\ n>0}}\chi(m)\psi(n)n^{k-1}q^{mn}$$

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is the Eisenstein series on $\Gamma_0(NM)$ of weight k with character $\chi\psi$ where $c_{k,\chi,\psi}$ is a constant related to generalized Bernoulli numbers. In table I, we use the following notation; χ, ρ, ψ and φ are real primitive characters defined modulo 4, 8, 7 and 23 respectively.

For $g \in F_1$, and symbol, we mean the same as in [1].

Table II; We give a few Fourier coefficients $a_n(m)$ $0 \leqq n \leqq 9$ for $\theta_m(z) = \sum_{n=0}^{\infty} a_n(m) q^n$.

Table III; We give a few Fourier coefficients $c_n(m)$ $1 \leqq n \leqq 10$ for $\eta_m(z) = \sum_{n=1}^{\infty} c_n(m) q^n$.

Table I

m	g	symbol	$\theta_m(z)$
1^{24}	$1A$		$E_{12}(z) - \frac{65520}{691}\eta_{1^{24}}(z)$
$2^8\cdot 1^8$	$2A$	$2+$	$\frac{1}{17}\{E_8(z) + 16E_8(2z) - 480\eta_{2^8\cdot 1^8}(z)\}$
$3^6\cdot 1^6$	$3A$	$3+$	$-\frac{1}{26}\{E_6(z) - 27E_6(3z) + 504\eta_{3^6\cdot 1^6}(z)\}$
$5^4\cdot 1^4$	$5A$	$5+$	$\frac{1}{26}\{E_4(z) + 25E_4(5z) - 240\eta_{5^4\cdot 1^4}(z)\}$
$4^4\cdot 2^2\cdot 1^4$	$4A$	$4+$	$\frac{1}{5}\{4E_{1,\chi}^{(5)}(z) + 64E_{\chi,1}^{(5)}(z) - 68\eta_{4^4\cdot 2^2\cdot 1^4}(z)\}$
$7^3\cdot 1^3$	$7A$	$7+$	$-\frac{1}{8}\{7E_{1,\psi}^{(3)}(z) - 49E_{\psi,1}^{(3)}(z) + 42\eta_{7^3\cdot 1^3}(z)\}$
$8^2\cdot 4\cdot 2\cdot 1^2$	$8A$	$8+$	$-\frac{2}{3}E_{1,\varphi}^{(3)}(z) + \frac{16}{3}E_{\varphi,1}^{(3)}(z) - \frac{14}{3}\eta_{8^2\cdot 4\cdot 2\cdot 1^2}(z)$
$6^2\cdot 3^2\cdot 2^2\cdot 1^2$	$6A$	$6+$	$\frac{1}{50}\{E_4(z) + 4E_4(2z) + 9E_4(3z) + 36E_4(6z)$ $-240\eta_{6^2\cdot 3^2\cdot 2^2\cdot 1^2}(z)\}$
$11^2\cdot 1^2$	$11A$	$11+$	$\frac{12}{5}\{G_2(z) - 11G_2(11z) - \eta_{11^2\cdot 1^2}(z)\}$
$15\cdot 5\cdot 3\cdot 1$	$15A$	$15+$	$\frac{3}{2}\{G_2(z) + 3G_2(3z) - 5G_2(5z) - 15G_2(15z) - \eta_{15\cdot 5\cdot 3\cdot 1}(z)\}$
$14\cdot 7\cdot 2\cdot 1$	$14A$	$14+$	$\frac{4}{3}\{G_2(z) + 2G_2(2z) - 7G_2(7z) - 14G_2(14z) - \eta_{14\cdot 7\cdot 2\cdot 1}(z)\}$
$23\cdot 1$	$23A$	$23+$	$\frac{2}{3}\{E_{1,\varphi}^{(1)}(z) - \eta_{23\cdot 1}(z)\}$
12^2	$24E$	$24\|6+$	$4E_{1,\chi}^{(1)}(6z)$
6^4	$12D$	$12\|3+$	$8\{G_2(3z) - 4G_2(12z)\}$
4^6	$8B$	$8\|2+$	$-4E_{1,\chi}^{(3)}(2z) + 16E_{\chi,1}^{(3)}(2z)$
3^8	$3C$	$3\|3$	$E_4(3z)$
2^{12}	$4A$	$4+$	$-\frac{1}{63}E_6(z) + \frac{64}{63}E_6(4z) - 8\eta_{2^{12}}(z)$
$10^2\cdot 2^2$	$20A$	$20+$	$\frac{4}{3}\{G_2(z) - 4G_2(4z) + 5G_2(5z) - 20G_2(20z) - \eta_{10^2\cdot 2^2}(z)\}$
$21\cdot 3$	$21C$	$21\|3+$	$2E_{1,\psi}^{(1)}(3z)$
$4^4\cdot 2^4$	$4B$	$4\|2+$	$\frac{1}{5}\{E_4(2z) + 4E_4(4z)\}$
$12\cdot 6\cdot 4\cdot 2$	$12C$	$12\|2+$	$6\{G_2(2z) - 2G_2(4z) + 3G_2(6z) - 6G_2(12z)\}$

Table II

m	1^{24}	$2^8\cdot 1^8$	$3^6\cdot 1^6$	$5^4\cdot 1^4$	$4^4\cdot 2^2\cdot 1^4$
a_0	1	1	1	1	1
a_1	0	0	0	0	0
a_2	196560	4320	756	120	260
a_3	16773120	61440	4032	240	960
a_4	398034000	522720	20412	600	3060
a_5	4629381120	2211840	60480	1440	8704
a_6	34417656000	8960640	139860	2400	16320
a_7	187489935360	23224320	326592	3120	28800
a_8	814879774800	67154400	652428	5400	53300
a_9	2975551488000	135168000	1020096	7200	87040

Table II (continued)

m	$7^3\cdot 1^3$	$8^2\cdot 4\cdot 2\cdot 1^2$	$6^2\cdot 3^2\cdot 2^2\cdot 1^2$	$11^2\cdot 1^2$	$15\cdot 5\cdot 3\cdot 1$	$14\cdot 7\cdot 2\cdot 1$	$23\cdot 1$
a_0	1	1	1	1	1	1	1
a_1	0	0	0	0	0	0	0
a_2	42	30	72	12	6	8	2
a_3	56	56	192	12	12	8	2
a_4	84	66	504	12	12	16	2
a_5	168	144	576	12	0	8	0
a_6	280	188	2280	24	30	24	2
a_7	336	584	1728	24	12	0	0
a_8	462	378	4248	36	18	40	2
a_9	336	448	4800	36	36	16	2

Table II (continued)

m	12^2	6^4	4^6	3^8	2^{12}	$10^2\cdot 2^2$	$21\cdot 3$	$4^4\cdot 2^4$	$12\cdot 6\cdot 4\cdot 2$
a_0	1	1	1	1	1	1	1	1	1
a_1	0	0	0	0	0	0	0	0	1
a_2	0	0	12	0	264	4	0	48	6
a_3	0	8	0	240	2048	8	2	0	0
a_4	0	0	60	0	7944	4	0	624	6
a_5	0	0	0	0	24576	16	0	0	0
a_6	4	24	160	2160	64416	16	4	1344	42
a_7	0	0	0	0	135168	8	0	0	0
a_8	0	0	252	0	253704	4	0	5232	6
a_9	0	32	0	6720	475136	16	0	0	0

Table III

m	1^{24}	$2^8\cdot1^8$	$3^6\cdot1^6$	$5^4\cdot1^4$	$4^4\cdot2^2\cdot1^4$	$7^3\cdot1^3$	$8^2\cdot4\cdot2\cdot1^2$
c_1	1	1	1	1	1	1	1
c_2	-24	-8	-6	-4	-4	-3	-2
c_3	252	12	9	2	0	0	-2
c_4	-1472	64	4	8	16	5	4
c_5	4830	-210	6	-5	-14	0	0
c_6	-6048	-96	-54	-8	0	0	4
c_7	-16744	1016	-40	6	0	-7	0
c_8	84480	-512	168	0	-64	-3	-8
c_9	-113643	-2043	81	-23	81	9	-5
c_{10}	-115920	1680	-36	20	56	0	0

Table III (continued)

m	$6^2\cdot3^2\cdot2^2\cdot1^2$	$11^2\cdot1^2$	$15\cdot5\cdot3\cdot1$	$14\cdot7\cdot2\cdot1$	$23\cdot1$	12^2	6^4
c_1	1	1	1	1	1	1	1
c_2	-2	-2	-1	-1	-1	0	0
c_3	-3	-1	-1	-2	-1	0	0
c_4	4	2	-1	1	0	0	0
c_5	6	1	1	0	0	0	0
c_6	6	2	1	2	1	0	0
c_7	-16	-2	0	1	0	0	-4
c_8	-8	0	3	-1	1	0	0
c_9	9	-2	1	1	0	0	0
c_{10}	-12	-2	-1	0	0	0	0

Table III (continued)

m	4^6	3^8	2^{12}	$10^2\cdot2^2$	$21\cdot3$	$4^4\cdot2^4$	$12\cdot6\cdot4\cdot2$
c_1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0
c_3	0	0	-12	-2	0	-4	-1
c_4	0	-8	0	0	-1	0	0
c_5	-6	0	54	-1	0	-2	-2
c_6	0	0	0	0	0	0	0
c_7	0	20	-88	2	-1	24	0
c_8	0	0	0	0	0	0	0
c_9	9	0	-99	1	0	-11	1
c_{10}	0	0	0	0	0	0	0

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