E-UNITARY INVERSE SEMIGROUPS OVER SEMILATTICES

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1. Introduction. An inverse semigroup is called E-unitary if the equations $ea = e = e^2$ together imply $a^2 = a$. In a previous paper [**4**], the author showed that any E-unitary inverse semigroup is isomorphic to a semigroup constructed from a triple $(G, \mathscr{X}, \mathscr{Y})$ consisting of a down-directed partially ordered set $\mathscr{X}$, an ideal and subsemilattice $\mathscr{Y}$ of $\mathscr{X}$ and a group G acting on $\mathscr{X}$, on the left, by order automorphisms in such a way that $\mathscr{X} = G\mathscr{Y}$. This semigroup is denoted by $P(G, \mathscr{X}, \mathscr{Y})$; it consists of all pairs $(a, g) \in \mathscr{Y} \times G$ such that $g^{-1}a \in \mathscr{Y}$, under the multiplication

$$(a, g)(b, h) = (a \wedge gb, gh).$$

The aim of this paper is to give necessary and sufficient conditions on an inverse semigroup in order that it should be isomorphic to some $P(G, \mathscr{X}, \mathscr{Y})$ with $\mathscr{X}$ a semilattice. As well, we consider those congruences ρ on an inverse semigroup $P(G, \mathscr{X}, \mathscr{Y})$ for which the quotient has the form $P(H, \mathscr{U}, \mathscr{V})$ for some triple H, $\mathscr{U}$, $\mathscr{V}$ as above, with $\mathscr{U}$ a semilattice.

We shall assume familiarity with the construction and properties of $P(G, \mathscr{X}, \mathscr{Y})$ from [**3**], [**4**]. Undefined notation and terminology is that of Clifford and Preston [**1**]. In particular, when we are considering a partial order on an inverse semigroup, the partial order being referred to is the natural partial order; it is defined by

$$a \leq b \quad \text{if and only if} \quad a = eb \quad \text{for some} \quad e^2 = e \in S.$$

Throughout the paper, when the terminology "triple $(G, \mathscr{X}, \mathscr{Y})$" is used, it means that $\mathscr{X}$ is a down-directed partially ordered set with $\mathscr{Y}$ an ideal and subsemilattice of $\mathscr{X}$, and that G is a group acting on $\mathscr{X}$ by order automorphisms in such a way that $\mathscr{X} = G\mathscr{Y}$.

DEFINITION 1.1. Let S be an inverse semigroup. Then we say that S is an *E-unitary inverse semigroup over a semilattice* if $S \cong P(G, \mathscr{X}, \mathscr{Y})$ for some triple $(G, \mathscr{X}, \mathscr{Y})$ with $\mathscr{X}$ a semilattice.

In terms of Definition 1.1, the aim of this paper is therefore to characterize E-unitary inverse semigroups over a semilattice.

2. The general case

DEFINITION 2.1. Let S be a partially ordered set and let $\theta: S \to T$ be a mapping of S into a set T. Then θ is an *m-map* if, for each $t \in T$, the set $\{s \in S : s\theta = t\}$ has a maximum member.

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Let S be an inverse semigroup. Then Munn [7] has shown that the relation σ on S defined by

$$(a, b) \in \sigma \quad \text{if and only if} \quad ea = eb \quad \text{for some} \quad e^2 = e \in S$$

is the smallest congruence ρ on S for which S/ρ is a group.

The following results about σ will be used without comment in several places in the remainder of the paper.

Lemma 2.2. (A) *Let S be an inverse semigroup and let e, f be idempotents in S. Let σ be the minimum group congruence on S. Then*

$$\sigma_{e,f} = \sigma \cap (eSf \times eSf)$$

is the minimum group congruence on eSf. Similarly

$$\sigma_e = \sigma \cap (Se \times Se)$$

is the minimum group congruence on Se.

(B) *If I is a non-empty ideal of S then $\sigma \cap (I \times I)$ is the minimum group congruence on I.*

(C) *Let $(G, \mathscr{X}, \mathscr{Y})$ be a triple and let S be an inverse subsemigroup of $P(G, \mathscr{X}, \mathscr{Y})$. Let $a = (u, g)$, $b = (v, h)$ belong to S. Then*

$$(a, b) \in \sigma \quad \textit{if and only if} \quad g = h.$$

Proof. (A) Let $\gamma_{e,f}$ denote the restriction of $\gamma = \sigma^\natural$ to a homomorphism of eSf into $G = S/\sigma$. Then, since for each $a \in S$,

$$a\gamma = e\gamma a\gamma f\gamma = (eaf)\gamma,$$

because $e\gamma = f\gamma = 1$ (the identity of G), $\gamma_{e,f}$ is a homomorphism of eSf onto G. Thus $\sigma_{e,f}$ is a group congruence on eSf.

On the other hand, suppose that ρ is a group congruence on eSf and let $(a, b) \in \sigma_{e,f}$. Then $au = bu$ for some idempotent $u \in S$. This implies

$$a(euf) = (au)ef = (bu)ef = b(euf)$$

since idempotents commute. But, since $euf \in eSf$ and ρ is a group congruence on eSf, it follows from these equalities that $(a, b) \in \rho$. Hence $\sigma_{e,f} \subseteq \rho$. In the same way it can be shown that σ_e is the minimum group congruence on Se.

(B) The proof of this is similar.

(C) Suppose that $(a, b) \in \sigma$. Then $ae = be$ for some idempotent $e = (f, 1) \in S$; thus $g = h$. On the other hand, suppose that $g = h$. Let $e = b^{-1}aa^{-1}b \in S$. Then $ae = be$ and so $(a, b) \in \sigma$.

Let S be an inverse semigroup and let e, f be idempotents of S. Then we shall follow the notation introduced in Lemma 2.2 and denote by $\gamma_{e,f}$ the restriction of $\sigma^\natural$ to eSf and by γ_e the restriction of $\sigma^\natural$ to Se; each is a homomorphism onto $G = S/\sigma$.

Theorem 2.3. *Let $(G, \mathscr{X}, \mathscr{Y})$ be a triple and set $S = P(G, \mathscr{X}, \mathscr{Y})$. Then $\mathscr{X}$ is a semilattice if and only if, for each pair of idempotents $e, f \in S$, $\gamma_{e,f} : eSf \to S/\sigma$ is an m-map.*

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Proof. Since $\mathscr{X}=G\mathscr{Y}$ and G acts by order automorphisms, it is easy to see that $\mathscr{X}$ is a semilattice if and only if $a\wedge e$ exists for each $a\in\mathscr{X}$, $e\in\mathscr{Y}$.

Suppose that $\mathscr{X}$ is a semilattice. Let $e=(u,1)$, $f=(v,1)$ with $u, v\in\mathscr{Y}$, and pick $g\in G$. Then $s\in eSf$ is such that $s\sigma^{\natural}=g$ if and only if $s=(b,g)$ for some $b\leq u$ with $g^{-1}b\leq v$. If this is the case, then $b\leq u$, gv and, by hypothesis, $u\wedge gv$ exists. Hence $b\leq u\wedge gv$ so that $(b,g)\leq(u\wedge gv,g)\in eSf$; but $(u\wedge gv,g)\sigma^{\natural}=g$. Thus

$$(u\wedge gv,g)=\max\{s\in eSf: s\sigma^{\natural}=g\}$$

and, since g was arbitrarily chosen in G, $\gamma_{e,f}$ is consequently an m-map.

Conversely, suppose that each $\gamma_{e,f}$ is an m-map and let $a\in\mathscr{X}$, $e\in\mathscr{Y}$; then $a=gf$ for some $g\in G$, $f\in\mathscr{Y}$. By hypothesis, the set

$$\{s\in(e,1)S(f,1): s\sigma^{\natural}=g\}$$

has a maximum member (c,g). Since $(c,g)\in(e,1)S(f,1)$, $c\leq e$, $g^{-1}c\leq f$, so that $c\leq e, a$. On the other hand, if $b\leq e$, a then $(b,g)\in(e,1)S(f,1)$. But $(b,g)\sigma^{\natural}=g$, so that $(b,g)\leq(c,g)$; that is $b\leq c$. Hence $c=e\wedge a$ exists and $\mathscr{X}$ is consequently a semilattice.

COROLLARY 2.4. *An inverse semigroup S is an E-unitary inverse semigroup over a semilattice if and only if S is E-unitary and each $\gamma_{e,f}$ is an m-map.*

DEFINITION 2.5 [**6**]. An inverse semigroup S is *F-inverse* if and only if $\sigma^{\natural}: S\to S/\sigma$ is an m-map.

McFadden and O'Carroll [**6**] showed that an F-inverse semigroup has an identity. On the other hand, it is shown in [**4**] that an inverse monoid is E-unitary over a semilattice if and only if it is F-inverse. This result is expressed in the context of this paper by the next theorem.

THEOREM 2.6. *Let S be an inverse semigroup. Then the following statements are equivalent:*

(i) *S is F-inverse;*

(ii) *S has an identity and each $\gamma_{e,f}: eSf\to S/\sigma$ is an m-map;*

(iii) *$S\cong P(G,\mathscr{X},\mathscr{Y})$ for some triple $(G,\mathscr{X},\mathscr{Y})$ with $\mathscr{X}$ a semilattice and $\mathscr{Y}$ a principal ideal of $\mathscr{X}$.*

Proof. (i)$\Rightarrow$(ii). As pointed out above, McFadden and O'Carroll [**6**] have shown that any F-inverse semigroup has an identity; the identity is the element $e=\max\{s\in S: s\sigma^{\natural}=1\}$, where 1 denotes the identity of S/σ.

Let u, v be idempotents of S and, for $g\in G$, let $h=\max\{s\in S: s\sigma^{\natural}=g\}$. Then $uhv\in uSv$ and $(uhv)\sigma^{\natural}=g$. If $s\in uSv$ is such that $s\sigma^{\natural}=g$ then $s\leq h$ and so $s=usv\leq uhv$. Hence $uhv=\max\{s\in uSv: s\sigma^{\natural}=g\}$; it follows that $\gamma_{u,v}$ is an m-map.

(ii)$\Rightarrow$(iii). Since S has an identity, it follows from Corollary 2.4 that we need only verify that S is E-unitary. Suppose that $fa=f=f^2$ for some $a\in S$. Then $a\sigma^{\natural}=1$ so that $a\leq\max\{s\in S: s\sigma_{e,e}=1\}$, where e is the identity of S. But e is a maximal element of $S=eSe$ so that $e=\max\{s\in S: s\sigma^{\natural}=1\}$; thus $a\leq e$. This implies $a=aa^{-1}e=aa^{-1}$, so that a is idempotent. Hence S is E-unitary.

(iii) ⇒ (i). Suppose $S=P(G,\mathscr{X},\mathscr{Y})$ with $\mathscr{X}$ a semilattice and $\mathscr{Y}$ a principal ideal of $\mathscr{X}$, and let e be the maximum element of $\mathscr{Y}$. Then, as in the proof of Theorem 2.3, $(e\wedge ge, g)$ is the maximum element s of S with $s\sigma^{\natural}=g$. Hence, S is F-inverse.

Theorem 2.6 shows that, in the presence of an identity, the condition

$$\text{each } \gamma_{e,f} \text{ is an } m\text{-map}$$

ensures that S is E-unitary. This is not the case in general.

Example 2.7. Let M_2 be the Brandt semigroup $\mathscr{M}^0(\{1\},\{1,2\},\{1,2\},\Delta)$. Then M_2 has the multiplication table

	0	a	a^{-1}	e	f
0	0	0	0	0	0
a	0	0	e	0	a
a^{-1}	0	f	0	a^{-1}	0
e	0	a	0	e	0
f	0	0	a^{-1}	0	f

with $a=(1,2)$, $a^{-1}=(2,1)$, $e=(1,1)$, $f=(2,2)$.

In M_2, $eSe=\{e,0\}$, $eSf=\{a,0\}$, $fSe=\{a^{-1},0\}$, $fSf=\{f,0\}$ and all other uSv with $u^2=u$, $v^2=v$ are $\{0\}$. Hence each $\gamma_{u,v}$ is an m-map. But $S=M_2$ is not E-unitary.

In a sense, M_2 is the only counterexample to the hypothesis:

$$\text{if each } \gamma_{e,f} \text{ is an } m\text{-map, then } S \text{ is } E\text{-unitary.}$$

Before verifying this, we prove a lemma.

Definition 2.8 [9]. Let S be an inverse semigroup. Then S is *E-reflexive* if and only if, for $a,b\in S$, ab is idempotent if and only if ba is idempotent.

Lemma 2.9. *Let S be an inverse semigroup. Then the following statements are equivalent:*

(i) *there exists $a\in S$ such that $a^2<a$;*

(ii) *S contains an isomorphic copy of M_2;*

(iii) *S is not E-reflexive.*

Proof. (i) ⇒ (ii). Suppose $a^2<a$. Then $a^2=a^2a^{-2}a$ so that $a^3=a^2a^{-2}a\cdot a=a^2$. This implies that a^2 is idempotent and $a^2=a^{-2}$. Consider the subsemigroup T of S generated by a and a^{-1}. Then, since T is a homomorphic image of the free inverse semigroup on one generator, it follows from [2] that each element of T is of the form $a^ra^{-s}a^t$ with r, $t\le s$. Because $a^2=a^3=a^{-3}=a^{-2}$, one sees that T has at most five members, a, a^{-1}, $e=aa^{-1}$, $f=a^{-1}a$ and $0=a^2$. Indeed, all five are distinct since, otherwise, a would belong to some subgroup of S and this would contradict $a^2<a$. Thus T has the multiplication table in Example 2.7. That is, $T\cong M_2$.

(ii) ⇒ (iii). Suppose $M_2\subseteq S$ and let $a=(1,2)$, $b=(1,1)$. Then $ab=0$ is idempotent but $ba=a$ is not.

(iii) ⇒ (i) Let $c,d\in S$ be such that cd is idempotent but $a=dc$ is not. Then $a^2=a^3$ so that $a^2<a$.

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THEOREM 2.10. *Let S be an inverse semigroup. Then S is an E-unitary inverse semigroup over a semilattice if and only if S is E-reflexive and each $\gamma_{e,f}: eSf \to S/\sigma$ is an m-map.*

Proof. Suppose that S is E-reflexive and that each $\gamma_{e,f}$ is an m-map. Let $ea = e = e^2$. Then, since a is a maximal member of $aa^{-1}Sa^{-1}a$,

$$a = \max\{b \in aa^{-1}Sa^{-1}a : b\sigma^{\natural} = 1\}.$$

Thus $a^2 \leq a$ so that, by Lemma 2.9, $a^2 = a$. Hence S is E-unitary and so, by Corollary 2.4, $S \cong P(G, \mathscr{X}, \mathscr{Y})$ where $\mathscr{X}$ is a semilattice.

Conversely, let $S \cong P(G, \mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ is a semilattice. Then, since M_2 is not E-unitary, S does not contain M_2. Thus, by Lemma 2.9, S is E-reflexive. Further, Theorem 2.3 shows that each $\gamma_{e,f}$ is an m-map.

It is an easy matter to see that if an inverse semigroup T is E-reflexive and each $\gamma_{e,f}: eTf \to T/\sigma$ is an m-map, then the same is true for each ideal of T. In particular, if an inverse semigroup S is embedded as an ideal in an F-inverse semigroup T then S is E-unitary over a semilattice. Example 2.13 shows that the converse need not be the case.

LEMMA 2.11. *Let $(G, \mathscr{U}, \mathscr{V})$ be a triple and let $\mathscr{Y}$ be an ideal of $\mathscr{V}$, thus of $\mathscr{U}$, and set $\mathscr{X} = G\mathscr{Y}$. Suppose that $\mathscr{X} \cap \mathscr{V} = \mathscr{Y}$. Then $(G, \mathscr{X}, \mathscr{Y})$ is a triple and $P(G, \mathscr{X}, \mathscr{Y})$ is an ideal of $P(G, \mathscr{U}, \mathscr{V})$; if $\mathscr{U}$ is a semilattice, so is $\mathscr{X}$.*

Conversely, if S is an ideal of $P(G, \mathscr{U}, \mathscr{V})$ then $\mathscr{Y} = \{a \in \mathscr{V} : (a, 1) \in S\}$ is an ideal of $\mathscr{V}$ such that $G\mathscr{Y} \cap \mathscr{V} = \mathscr{Y}$. Further, $S = P(G, G\mathscr{Y}, \mathscr{Y})$.

Proof. This is straightforward.

THEOREM 2.12. *Let S be an inverse semigroup. Then the following statements are equivalent:*

(1) *each $\gamma_e : Se \to S/\sigma$ is an m-map, for $e^2 = e \in S$;*
(2) *the translational hull $\Omega(S)$ of S is F-inverse;*
(3) *S can be embedded as an ideal in an F-inverse semigroup.*

Proof. (1)$\Rightarrow$(2). Suppose that (1) holds. We first show that S is E-unitary. Suppose that $ea = e = e^2$. Then $a \in Sa^{-1}a$ is such that $a\sigma^{\natural} = 1$. By hypothesis, the set $\{s \in Sa^{-1}a : s\sigma^{\natural} = 1\}$ has a maximum member t; thus $a, a^{-1}a \leq t$. But $a, a^{-1}a$ are maximal in $Sa^{-1}a$, from which it follows that $t = a = a^{-1}a$. Thus $a^2 = a$ and so S is E-unitary.

We may therefore suppose that $S = P(G, \mathscr{X}, \mathscr{Y})$ for some triple $(G, \mathscr{X}, \mathscr{Y})$. Let $\mathscr{X}^*$ and $\mathscr{Y}^*$ denote the set of all non-empty order ideals of $\mathscr{X}$ and $\mathscr{Y}$, respectively, under inclusion, and let G act on $\mathscr{X}^*$ by $gA = \{ga : a \in A\}$ for each $A \in \mathscr{X}^*$. Let $\mathfrak{X}^* = G\mathscr{Y}^*$. Then $(G, \mathfrak{X}^*, \mathscr{Y}^*)$ is a triple and we may regard S as being embedded in $P(G, \mathfrak{X}^*, \mathscr{Y}^*)$ by $(a, g) \mapsto (\bar{a}, g)$, where, for $a \in \mathscr{X}$, $\bar{a} = \{x \in \mathscr{X} : x \leq a\}$. Assume that this has been done. Then it is shown in [5, Section 3] that $\Omega(S)$ is isomorphic to the idealizer of S in $P(G, \mathfrak{X}^*, \mathscr{Y}^*)$. Further, it is shown there that $\hat{S} = P(G, \mathfrak{X}^*, \mathscr{Y}^*)$ is an F-inverse semigroup in which, for each $g \in G$, (I_g, g) is the maximum element t of $\hat{S}$ with $t\sigma^{\natural} = g$; here $I_g = \mathscr{Y} \cap g\mathscr{Y}$.

It follows from these remarks and Lemma 2.2 that in order to show that $\Omega(S)$ is F-inverse, it suffices to show that (I_g, g) is in $\Omega(S)$ for each $g \in G$. It is shown in [**5**, Theorem 3.9] that $\Omega(S)$ consists of all pairs $(A, g) \in P(G, \mathscr{X}^*, \mathscr{Y}^*)$ such that, for each $e \in \mathscr{Y}$, $\{x \in A: x \leq e\}$ has a maximum member.

Let $e \in \mathscr{Y}$, $g \in G$. Then, by hypothesis, the set $\{s \in S(e, 1): s\sigma^\natural = g^{-1}\}$ has a maximum member $t = (g^{-1}c, g^{-1})$ for some $c \in \mathscr{Y}$. Since $t \in S(e, 1)$, $t^{-1}t \leq (e, 1)$; that is, $c \leq e$. Hence $c \in \{x \in \mathscr{Y} \cap g\mathscr{Y}: x \leq e\}$. On the other hand, if $x \in \mathscr{Y} \cap g\mathscr{Y}$, $x \leq e$ then $(g^{-1}x, g^{-1}) \in S(e, 1)$ and $(g^{-1}x, g^{-1})\sigma^\natural = g^{-1}$. This means that $(g^{-1}x, g^{-1}) \leq (g^{-1}c, g^{-1})$ so that $g^{-1}x \leq g^{-1}c$ and so $x \leq c$. It follows that $c = \max\{x \in I_g : x \leq e\}$. Hence $(I_g, g) \in \Omega(S)$ and then $(I_g, g) = \max\{s \in \Omega(S) : s\sigma = g\}$. Consequently $\Omega(S)$ is F-inverse.

(2) ⇒ (3) is immediate, since S is an ideal of $\Omega(S)$.

(3) ⇒ (1). Suppose that S is an ideal of an F-inverse semigroup T. By Theorem 2.6 we may assume $T = P(G, \mathscr{U}, \mathscr{V})$, where $\mathscr{U}$ is a semilattice and $\mathscr{V}$ is a principal ideal with greatest element v. Let $e = (f, 1) \in S$ and pick $g \in G$. Then $g^{-1}v \wedge f \leq f$ so that $g^{-1}v \wedge f \in \mathscr{Y}$, where $\mathscr{Y} = \{u \in \mathscr{U} : (u, 1) \in S\}$, and $g(g^{-1}v \wedge f) \geq v$ so that $g(g^{-1}v \wedge f) \in \mathscr{V} \cap G\mathscr{Y} = \mathscr{Y}$, by Lemma 2.11. Now by Lemma 2.11, $S = P(G, G\mathscr{Y}, \mathscr{Y})$. Hence $(g^{-1}v \wedge f, g^{-1}) \in S$ and so, consequently, $(g^{-1}v \wedge f, g^{-1})^{-1} = (v \wedge gf, g) \in S$. Indeed, since $g^{-1}(gf \wedge v) \leq f$, $(v \wedge gf, g) \in Se$ and $(v \wedge gf, g)\sigma^\natural = g$.

On the other hand, suppose that $(a, g) \in Se$. Then $g^{-1}a \leq f$ and $a \in \mathscr{Y} \subseteq \mathscr{V}$ so that $a \leq v$. Hence $a \leq v \wedge gf$ and consequently $(a, g) \leq (v \wedge gf, g)$. It follows that $(v \wedge gf, g) = \max\{s \in Se : s\sigma^\natural = g\}$; hence σ_e is an m-map.

EXAMPLE 2.13. Let $\mathbb{Q}^+$ be the set of positive rationals under the reverse of the usual ordering and let $\mathbb{Y} = \{x \in \mathbb{Q}^+ : x^2 > 2\}$. Then the group G of positive rationals acts on $\mathbb{Q}^+$ by multiplication. Let $S = P(G, \mathbb{Q}^+, \mathbb{Y})$ and let $e = (2, 1) \in S$, $g = \frac{1}{3} \in G$. Then $(f, g) \in Se$ for all $f \in \mathbb{Y}$. Thus, as ordered sets,

$$\{s \in Se : s\sigma^\natural = g\} \cong \mathbb{Y}.$$

But $\mathbb{Y}$ has no maximum member ($\sqrt{2}$ is irrational) so that σ_e is not an m-map. Hence S cannot be embedded as an ideal in an F-inverse semigroup.

From the characterization of $\Omega(S)$ in [**5**, Theorem 3.9], one can show that $\Omega(S) = S^1$.

3. Some special cases. In this section, we consider some special cases in which it is possible to improve on the result in Theorem 2.10.

PROPOSITION 3.1. *Let S be an inverse semigroup and suppose that the semilattice of idempotents of S is up-directed. Then S is an E-unitary inverse semigroup over a semilattice if and only if eSe is F-inverse for each $e^2 = e \in S$.*

Proof. Suppose that S is E-unitary over a semilattice and let $e^2 = e \in S$. Then, by Theorem 2.3, $\gamma_{e,e} : eSe \to G = S/\sigma$ is an m-map. Since eSe is an inverse semigroup, it follows, from Lemma 2.2(A), that eSe is F-inverse. On the other hand, suppose that each eSe is F-inverse. We show first that S cannot contain M_2. Thus, by Lemma 2.9, S is E-reflexive.

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For any $a \in S$, there exists $e^2 = e$ such that $e \geq aa^{-1}$, $a^{-1}a$. Then a, a^{-1} belong to eSe which is E-unitary, being F-inverse. Since M_2 is not E-unitary, it follows that $T = \langle a, a^{-1} \rangle$ is not isomorphic to M_2. Hence S does not contain any copy of M_2.

Next, let u, v be idempotents and let $e^2 = e \geq u, v$. Then $uSv \subseteq eSe$ and, by hypothesis, eSe is F-inverse. Let $g \in G = S/\sigma$. Then, as in the proof of Theorem 2.6,

$$u(\max\{s \in eSe : s\sigma^{\natural} = g\})v = \max\{s \in uSv : s\sigma^{\natural} = g\}.$$

Hence $\gamma_{u,v}$ is an m-map for all idempotents $u, v \in S$.

It now follows from Theorem 2.10 that S is E-unitary over a semilattice.

The results in the next two propositions are similar to Proposition 3.1. However, they depend on the algebraic structure of the semigroup S rather than on the order structure of the idempotents of S.

PROPOSITION 3.2. *Let S be an inverse semigroup which is a semilattice of groups. Then S is an E-unitary inverse semigroup over a semilattice if and only if eSe is F-inverse for each $e^2 = e \in S$.*

Proof. As with Proposition 3.1, the condition is necessary. On the other hand, suppose that eSe is F-inverse for each $e^2 = e \in S$. Then, since S cannot contain M_2, we need only show that each $\gamma_{u,v}$ is an m-map. But $uSv = uvSuv$ since idempotents are central, and then, by definition, $\sigma_{u,v} = \sigma_{uv,uv}$, so that the hypothesis in the statement of the proposition implies that each $\gamma_{u,v}$ is an m-map.

PROPOSITION 3.3. *Let S be a simple E-unitary inverse semigroup. Then $S \cong P(G, \mathscr{X}, \mathscr{Y})$ for some triple $(G, \mathscr{X}, \mathscr{Y})$, with $\mathscr{X}$ a semilattice, if and only if, for some idempotent $e \in S$, eSe is F-inverse.*

Proof. We need only show that the condition is sufficient. Suppose eSe is F-inverse for some idempotent e and let u, v be idempotents of S. Since S is simple, there exist b, c such that $bb^{-1} = u$, $b^{-1}b \leq e$, $cc^{-1} = v$, $c^{-1}c \leq e$. Suppose that $x \in uSv$ is such that $x\sigma^{\natural} = g \in G$. Then $b^{-1}xc$ is in eSe since

$$b^{-1}xcc^{-1}x^{-1}b \leq b^{-1}b \leq e \quad \text{and} \quad c^{-1}x^{-1}bb^{-1}xc \leq c^{-1}c \leq e.$$

Further $(b^{-1}xc)\sigma^{\natural} = (b^{-1}\sigma^{\natural})g(c\sigma^{\natural}) = h$, say, so that, by the hypothesis on eSe, $b^{-1}xc \leq z = \max\{x \in eSe : x\sigma^{\natural} = h\}$. Then $x = bb^{-1}xcc^{-1} \leq bzc^{-1}$, and $bzc^{-1} \in uSv$ is such that $(bzc^{-1})\sigma^{\natural} = g$. Hence $bzc^{-1} = \max\{s \in uSv : s\sigma^{\natural} = g\}$, and so $\gamma_{u,v}$ is an m-map.

The requirement, in Proposition 3.3, that S should be E-unitary is necessary as the following example shows.

EXAMPLE 3.4. Let E be the ω-tree with the Hasse diagram shown in Fig. 1 and let $S = T_E$ be the inverse semigroup of order isomorphisms between principal ideals of E. Then E is uniform so that S is bisimple.

For each $a \in E$, denote by ε_a the identity mapping on the principal ideal $\bar{a} = \{x \in E : x \leq a\}$. Then

$$\varepsilon_0 T_E \varepsilon_0 = \{\alpha \in T_E : \Delta\alpha \cup \nabla\alpha \subseteq \bar{0}\}.$$

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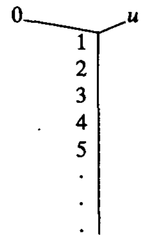

Figure 1

Thus $\varepsilon_0 S \varepsilon_0$ is the semigroup of isomorphisms between principal ideals of $\{0, 1, \ldots\}$. Consequently $\varepsilon_0 S \varepsilon_0 \cong B$, where B is the bicyclic semigroup. McFadden and O'Carroll **[6]** have shown that B is F-inverse.

On the other hand, let $\alpha \in T_E$ be defined by $\Delta\alpha = \bar{0}$ and

$$x\alpha = \begin{cases} u & \text{if} \quad x = 0, \\ x & \text{if} \quad x > 0. \end{cases}$$

Then $\alpha^2 = \varepsilon_1 < \alpha$ so that S contains a copy of M_2. Hence S is not E-unitary.

The last result is different in type from the earlier ones in this section since on this occasion we impose a restriction on the idempotents of S and deduce a result about the maps $\gamma_{e,f}$.

Definition 3.5. A *locally finite tree* is a partially ordered set X in which the following three conditions are satisfied:

(i) X is down-directed;

(ii) if a, $b \le c$ where a, b, $c \in X$, then $a \le b$ or $b \le a$;

(iii) the set $\{x \in X : a \le x \le b\}$ is finite for all a, $b \in X$ with $a \le b$.

Proposition 3.6. *Let S be an inverse semigroup whose idempotents form a locally finite tree. Then the following conditions on S are equivalent:*

(i) *S is E-unitary;*

(ii) *S is E-unitary over a semilattice;*

(iii) *$\Omega(S)$ is F-inverse.*

Proof. (i) $\Rightarrow$ (ii). Suppose that S is E-unitary. Then $S \cong P(G, \mathscr{X}, \mathscr{Y})$ for some triple $(G, \mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ is down-directed and G acts on $\mathscr{X}$ in such a way that $\mathscr{X} = G\mathscr{Y}$. Since $P(G, \mathscr{X}, \mathscr{Y})$ has semilattice of idempotents $\mathscr{Y} \times 1$, $\mathscr{Y}$ is a locally finite tree. In fact, since $\mathscr{X} = G\mathscr{Y}$ and $\mathscr{X}$ is down-directed, it is easy to see that $\mathscr{X}$ is a locally finite tree. Hence $\mathscr{X}$ is a semilattice.

(ii) $\Rightarrow$ (iii). Again, we may suppose that $S = P(G, \mathscr{X}, \mathscr{Y})$ and, as in the proof of Theorem 2.12, it suffices to show that, for each $g \in G$, $e \in \mathscr{Y}$, the set $\{x \in I_g : x \le e\}$ has a maximum member.

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For each $x \in I_g$ such that $x \leq e$, the set $\{y \in I_g : x \leq y \leq e\}$ has a maximum member m_x since $\mathscr{Y}$ is locally finite. Fix such an x and let $z \in I_g$ with $z \leq e$. Then either $z \leq x$ or $x < z$. In the first case $z \leq x \leq m_x$ while, in the second, $x < z \leq e$, $z \in I_g$ imply $z \leq m_x$. Hence $m_x = \max\{y \in I_g : y \leq e\}$. This shows that $(I_g, g) \in \Omega(S)$, so that $\Omega(S)$ is F-inverse.

(iii) $\Rightarrow$ (i). This is immediate.

4. Congruences. In this section, we characterize those congruences on an E-unitary inverse semigroup over a semilattice whose quotient is also of this type.

DEFINITION 4.1 [**11**]. Let S be an inverse semigroup with semilattice of idempotents E. Then a *normal partition on* E is an equivalence π on E such that

(i) if $e \, \pi \, f$, $u \in E$ then $eu \, \pi \, fu$;
(ii) if $e \, \pi \, f$ then $a^{-1}ea \, \pi \, a^{-1}fa$ for all $a \in S$.

If ρ is a congruence on S then $\pi_\rho = \rho \cap (E \times E)$ is a normal partition on E and is called the normal partition *induced by* ρ.

If π is a normal partition on E then Reilly and Scheiblich [**11**] show that the smallest congruence ρ_π on S which induces π is given by the following rule: $(a, b) \in \rho_\pi$ if and only if $ea = eb$ for some $e^2 = e \in S$ such that $aa^{-1} \, \pi \, e \, \pi \, bb^{-1}$.

The result in the next lemma is due to Reilly and Munn [**10**]. They derived it by giving an explicit construction for S/ρ_π in the form $P(G, \mathscr{X}, \mathscr{Y})$. We shall derive it as a direct consequence of Corollary 2.4.

LEMMA 4.2. *Let S be an E-unitary inverse semigroup over a semilattice and let π be a normal partition on the idempotents of S. Then S/ρ_π is an E-unitary inverse semigroup over a semilattice.*

Proof. We verify that the conditions of Corollary 2.4 hold for $T = S/\rho_\pi$. First, suppose that $(e, ea) \in \rho_\pi$ for some $e^2 = e$. Then $fe = fea$ for some idempotent f. Since S is E-unitary, this implies that $a^2 = a$. Hence T is E-unitary.

Next, let $\bar{X}$ be a σ-class of T and let $\bar{e}$, $\bar{f}$ be idempotents of T, say with $\bar{e} = e\rho_\pi$, $\bar{f} = f\rho_\pi$, $e^2 = e$, $f^2 = f$. Suppose $\bar{x} \in \bar{X} \cap \bar{e}T\bar{f}$ and let $x \in eSf$ be such that $x\rho_\pi = \bar{x}$. By hypothesis, $x \leq z$, where $z = \max\{s \in eSf : s\gamma_{e,f} = x\gamma_{e,f}\}$. Hence $\bar{x} \leq \bar{z} = z\rho_\pi$. But, since $x\gamma_{e,f} = z\gamma_{e,f}$, $uz = ux$ for some $u^2 = u \in S$; thus $\bar{x}\sigma\bar{z}$. By the choice of e, f, $\bar{z} \in \bar{X} \cap \bar{e}T\bar{f}$. Hence $\bar{z}$ is the maximum element of $\bar{X} \cap \bar{e}T\bar{f}$. It follows that $\gamma_{\bar{e},\bar{f}}$ is an m-map for each pair $\bar{e}, \bar{f}$ of idempotents of T. Therefore, by Corollary 2.4, T is an E-unitary inverse semigroup over a semilattice.

THEOREM 4.3 [**10**]. *Let S be an inverse semigroup. Then S is an idempotent-separating homomorphic image of an E-unitary inverse semigroup over a semilattice.*

Proof. Let $\theta : FI_X \to S$ be a homomorphism from a free inverse semigroup onto S. Then (cf. [**8**]) FI_X^1 is F-inverse and FI_X is an ideal of FI_X^1. Hence, by Lemma 2.11, FI_X is E-unitary over a semilattice.

Let $\pi = \theta \circ \theta^{-1} \cap (E \times E)$ where E is the set of idempotents of FI_X. Then π is a normal partition and S is an idempotent-separating homomorphic image of S/ρ_π. The result now follows immediately from Lemma 4.2.

The following corollary strengthens Theorem 4.2 of [**4**].

Corollary 4.4. *Let E be a semilattice. Then E can be embedded as an ideal in a semilattice F with the following property: each isomorphism between principal ideals of F can be extended to an automorphism of F.*

Proof. This follows from Theorem 4.3, using the argument in [**4**, Theorem 4.2].

We now turn to consider the idempotent-separating congruences on an inverse semigroup $P = P(G, \mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ is a semilattice. The next result is related to some of those in [**10**], and the proof is omitted.

Lemma 4.5. *Let $(G, \mathscr{X}, \mathscr{Y})$ be a triple and let N be a normal subgroup of G such that*

$$a \in \mathscr{Y},\ na \in \mathscr{Y} \text{ imply } a = na \text{ for each } a \in \mathscr{Y},\ n \in N. \qquad (*)$$

Then the relation ρ_N defined by

$$(a, g)\, \rho_N\, (b, h) \quad \text{if and only if} \quad a = b \quad \text{and} \quad g^{-1}h \in N$$

is an idempotent-separating congruence on $S = P(G, \mathscr{X}, \mathscr{Y})$ such that S/ρ_N is E-unitary.

Conversely, suppose that ρ is an idempotent-separating congruence on S such that S/ρ is E-unitary. Then

$$N = \{g \in G : (a, g)\ \rho\ (a, 1) \text{ for some } a \in \mathscr{Y}\}$$

is a normal subgroup of G which satisfies condition $(*)$. Further $\rho = \rho_N$.

The next proposition gives necessary and sufficient conditions on N in order that $T = S/\rho_N$ should be E-unitary over a semilattice. In the proof of the result, we shall denote the elements of T by $[a, g]$ where $(a, g) \in S$. If $[a, g] = (a, g)\rho_N^\natural \in T$, it is easy to see that the mapping $[a, g] \to Ng$ is a homomorphism of T onto G/N which induces σ. Hence we may identify T/σ with G/N by means of this mapping.

Proposition 4.6. *Let $S = P(G, \mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ is a semilattice, and let N be a normal subgroup of G which satisfies condition $(*)$. Then $T = S/\rho_N$ is E-unitary over a semilattice if and only if, for each $a, b \in \mathscr{X}$,*

$$\{a \wedge nb : n \in N\}$$

has a maximum member.

Proof. First note that, by Lemma 4.5, T is E-unitary. Suppose that, for all $a, b \in \mathscr{X}$, $\{a \wedge nb : n \in N\}$ has a maximum member. Let u, v be idempotents of T and let $X = Ng \in T/\sigma$; then $u = [e, 1]$, $v = [f, 1]$ for some $e, f \in \mathscr{Y}$. By hypothesis, $\bar{e} = \max\{e \wedge ngf : n \in N\}$ exists; say $\bar{e} = e \wedge n_1 gf$. Then $[\bar{e}, n_1 g] \in uTv$ and $[\bar{e}, n_1 g]\sigma = Ng$.

On the other hand, suppose that $[a, h] \in uTv$ is such that $[a, h]\sigma^\natural = Ng$. Then, because ρ_N is idempotent-separating, $a \le e$, $h^{-1}a \le f$ and, further, $h = ng$ for some $n \in N$. Thus

$a \le e \wedge ngf \le \bar{e}$ so that $[a, h] \le [\bar{e}, n_1 g]$. Hence

$$[\bar{e}, n_1 g] = \max\{t \in uTv : t\sigma^{\natural} = Ng\}.$$

Thus, by Corollary 2.4, T is E-unitary over a semilattice.

Conversely, suppose that T is E-unitary over a semilattice and let $a = ge$, $b = hf$, where $e, f \in \mathscr{Y}$ and $g, h \in G$. By the hypothesis that T is over a semilattice,

$$x = \max\{t \in [e, 1]T[f, 1] : t\sigma^{\natural} = Ng^{-1}h\}$$

exists. Then $x = [u, k]$, where $u \le e$, $k^{-1}u \le f$ and $Nk = Ng^{-1}h$; thus $k = g^{-1}n_1 h$ and $u \le e \wedge g^{-1}n_1 hf$ for some $n_1 \in N$, so that $gu \le a \wedge n_1 b$.

On the other hand, for any $n \in N$, $t = [e \wedge g^{-1}nhf, g^{-1}nh]$ is in $[e, 1]T[f, 1]$ and $t\sigma^{\natural} = Ng^{-1}h$. Hence $t \le x$ which, since ρ_N is idempotent-separating, implies $e \wedge g^{-1}nhf \le u$; that is, $a \wedge nb \le gu$. But, as we have seen, $gu \le a \wedge n_1 b$, so we must have

$$gu = a \wedge n_1 b = \max\{a \wedge nb : n \in N\}.$$

COROLLARY 4.7. *Let $S = P(G, \mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ is a semilattice, and let N be a normal subgroup of G such that $na = a$ for each $n \in N$, $a \in \mathscr{X}$. Then S/ρ_N is E-unitary over a semilattice.*

It was shown in [5] that every idempotent-separating congruence on $S = P(G, \mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ is a semilattice, can be extended to an idempotent-separating congruence on $P(G, \mathscr{X}, \mathscr{X})$. It is easy to see that ρ_N can be extended to an E-unitary congruence if and only if $N \subseteq \{g \in G : ga = a \quad \text{for all} \quad a \in \mathscr{X}\}$.

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