CONVERGENCE IN PARTIALLY ORDERED GROUPS

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Abstract

Relative uniform limits need not be unique in a non-archimedean partially ordered group, and order convergence need not imply metric convergence in a Banach lattice. We define a new type of convergence on partially ordered groups (R-convergence), which implies both the previous ones, and does not have these defects. Further R-convergence is equivalent to relative uniform convergence on divisible directed integrally closed partially ordered groups, and to order convergence on fully ordered groups.

1. Introduction

The open-interval topology is usually not considered a satisfactory one for partially ordered groups (abbreviated to pogroups), because it is discrete for non-fully ordered l-groups. We shall indicate, however, that it is of interest if we proceed as follows.

Given a pogroup we consider the family of restrictions of this order, which satisfy an interpolation property and are related in a certain sense to the original order. Each of the open-interval topologies for this family of orders makes the group a Hausdorff topological group. We say a net is R-convergent if it is convergent in one of these open-interval topologies. R-limits are unique, and in an l-group the operations $+$, $\vee$ and $\wedge$ are continuous with respect to taking R-limits. R-convergence is related to relative uniform convergence and order convergence as indicated above, and also it implies convergence in the interval topology. We also determine the relation between R-convergence and the inductive limit of the family of open-interval topologies.

2. Preliminaries

Let $(G, \leqq)$ be a pogroup. Taking the intervals $(a, b) = \{x : a < x < b\}$, where $a, b \in G$, $a < b$, as a sub-base, we define the *open-interval topology* $\mathscr{U}$ on G. Denote by $S^{-\mathscr{U}}$ or S^{-} the closure of $S \subseteq G$ in $(G, \mathscr{U})$. We say $(G, \leqq)$ is *integrally closed* if $nx \leqq y$ for $n = 1, 2, 3, \ldots$ implies that $x \leqq 0$. $(G, \leqq)$ is *archimedean* if $nx \leqq y$ for all integers n implies that $x = 0$. We say $x \in G$ is *pseudopositive* if $x \ngeq 0$ and $x + p > 0$ for all $p > 0$. If x and $-x$ are both pseudopositive, then x is

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said to be a *pseudozero*. If $(G, \leqq)$ has no pseudozeros we write $0<'x$ to mean '$0<x$ or x is pseudopositive'. Then $(G, \leqq')$ is a pogroup and we call $\leqq'$ the *associated order*. Write $(\leqq')'$ (if it exists) as $\leqq''$.

We say $(G, \leqq)$ satisfies the *tight Riesz* (m, n) *interpolation property* (abbreviated $TR(m, n)$) (3) if, given $a_1, \ldots, a_m, b_1, \ldots, b_n \in G$ with $a_i<b_j$ for $i = 1, \ldots, m, j = 1, \ldots, n$, there exists $c \in G$ such that $a_i<c<b_j$ for $i = 1, \ldots, m$, $j = 1, \ldots, n$. We say $(G, \leqq)$ is *dense* if it is $TR(1, 1)$. Clearly $TR(m, n)$ is equivalent to $TR(n, m)$ and $TR(2, n)$ implies $TR(m, n)$ for $m \geqq 2$. However $TR(1, 2)$ does not imply $TR(2, 2)$ (3).

If for every $a>0$, and every positive integer n there exists $b>0$, such that $a>nb$, we say $(G, \leqq)$ contains *small elements*. We say G is *divisible* if given $x \in G$, n a positive integer, there exists $y \in G$ such that $ny = x$. For any other undefined terms see Birkhoff's book (2).

Definition. Let $(G, \preccurlyeq)$ be a non-trivially ordered pogroup. Then $\leqq$ is a *compatible tight Riesz order* (abbreviated CTRO) on $(G, \preccurlyeq)$ if $(G, \leqq)$ is a non-trivially ordered pogroup, satisfies $TR(1, 2)$ and $\leqq' = \preccurlyeq$.

Lemma 1. *Let $(G, \leqq)$ be $TR(1, 2)$. Then $(G, \leqq, \mathscr{U})$ is a topological group, which is Hausdorff if and only if $(G, \leqq)$ has no pseudozeros. Also $(G, \leqq)$ contains small elements and the family $\{(-a, a): a>0\}$ form a base for $\mathscr{U}$ at 0. If $(G, \leqq)$ has no pseudozeros and $a<b$, then $(a, b)^- = \{x: a\leqq' x \leqq' b\}$.*

The proof is essentially the same as for abelian $TR(2, 2)$ groups in (6).

Note. If $(G, \leqq)$ is dense and has no pseudozeros then $\leqq'' = \leqq'$. So in particular, if $\leqq$ is a CTRO on $(G, \preccurlyeq)$ then $(G, \preccurlyeq)$ has no pseudopositives. If $(G, \leqq)$ is an l-group, and $x+p>0$ for all $p>0$, then $0 = \bigwedge_{p>0} p \geqq -x$. If $(G, \leqq)$ is an integrally closed divisible isolated pogroup and $x+p/n>0$ for $n = 1, 2, \ldots$, and $p>0$, then $p>n(-x)$, so $x \geqq 0$. Hence in either case $\leqq' = \leqq$.

Lemma 2. *There is a one-one correspondence between CTROs on $(G, \preccurlyeq)$ and sets T with the properties:*

(i) *T is l-directed (i.e. $a, b \in T$ implies that there exists $c \in T$ with $a, b \succcurlyeq c$), T is an upper class (i.e. $a \in T$ and $x \succcurlyeq a$ implies that $x \in T$) and $\emptyset \neq T \subset G^+$,*

(ii) $T = T+T$,

(iii) $\bigwedge_{\preccurlyeq} T = 0$,

(iv) *T is normal (i.e. $-x+T+x = T$ for all $x \in G$).*

In fact the set of strictly positive elements of a CTRO satisfies conditions (i)-(iv) and vice versa.

Proof. The proof which can be adapted from that of Theorem 2 in (9) dealing with the abelian l-group case, will be omitted. The non-abelian l-group case has been considered in (7).

Note. In (9) we defined a CTRO, $\leqq$ on an abelian non-trivial l-group $(G, \preccurlyeq)$ to be $TR(2, 2)$ with $\leqq' = \preccurlyeq$ and $(G, \leqq)$ directed. We showed in (9) that conditions (i)-(iii) of Lemma 2 above are necessary and sufficient for a CTRO on an abelian l-group. So the definitions are consistent, the one here generalises the previous one.

We will denote a CTRO by T or $\leqq$ as the need arises, and the family of all CTROs by $\mathscr{C}$. In (9) we give sufficient conditions for an abelian l-group to have a CTRO; in particular divisibility is such a condition.

Lemma 3. *If* $(G, \preccurlyeq)$ *has a CTRO then* $(\mathscr{C}, \subseteq)$ *is l-directed. Also* $T_1 \subseteq T_2$ *if and only if* $\mathscr{U}_1 \subseteq \mathscr{U}_2$.

Proof. If $T_1, T_2 \in \mathscr{C}$, let $T_3 = T_1 + T_2$. If $t_i \in T_i$ for $i = 1, 2$, then $t_1 + t_2 \succcurlyeq t_i$, for $i = 1, 2$. So by Lemma 2 (i), $T_3 \subseteq T_i$ for $i = 1, 2$. Since T_1 and T_2 are normal, T_3 is the subsemigroup generated by T_1 and T_2. The proof of Theorem 6 in (9) can be adapted to show that $T_3 \in \mathscr{C}$.

Suppose that $T_1, T_2 \in \mathscr{C}$ and $T_1 \subseteq T_2$. Let $0 <_1 a$, then by Lemma 1, to show that $\mathscr{U}_1 \subseteq \mathscr{U}_2$ it will suffice to show that $\{x: -a <_1 x <_1 a\}$ is a $\mathscr{U}_2$-neighbourhood of 0. Now there exists $b \in G$, such that $0 <_1 b <_1 a$, hence $0 <_2 b$. If $y <_2 b$, then $y \prec b$, by Lemma 2 (i), and so $y <_1 a$, also by Lemma 2 (i). So $\{y: -b <_2 y <_2 b\} \subseteq \{x: -a <_1 x <_1 a\}$.

Conversely, let $\mathscr{U}_1 \subseteq \mathscr{U}_2$. If $a >_1 0$, then for some $b >_2 0$, $-b <_2 x <_2 b$ implies that $-a <_1 x <_1 a$. Now there exists $c \in G$ such that $b >_2 c >_2 0 >_2 -b$. So $0 <_2 c <_1 a$. Hence $0 <_2 c \prec a$, and so $0 <_2 a$. Hence $T_1 \subseteq T_2$.

3. R-convergence

Let $(G, \preccurlyeq)$ be a non-trivially ordered pogroup.

Definition. The net $\{x_\alpha\}$ R-converges to x, i.e. R-lim $x_\alpha = x$, if either $\{x_\alpha\}$ converges to x in the open-interval topology of some CTRO on $(G, \preccurlyeq')$ or there exists α_0 such that $x_\alpha = x$ for all $\alpha \geqq \alpha_0$.

If $(G, \preccurlyeq')$ has no CTROS, or if $(G, \preccurlyeq)$ has pseudozeros and so $\preccurlyeq'$ is not defined, then R-convergence is convergence in the discrete topology. If $\{x_\alpha\}$ converges to x in the open-interval topology of a CTRO T we will write this as T-lim $x_\alpha = x$.

Theorem 1. *Let* $\{x_\alpha\}$ *and* $\{y_\beta\}$ *be nets in G then*

(i) R-lim $x_\alpha = x$ *and* R-lim $x_\alpha = y$ *implies* $x = y$,
(ii) $x_\alpha = x$ *for all* α *implies* R-lim $x_\alpha = x$,
(iii) *If* $\{x_\gamma\}$ *is a subnet of* $\{x_\alpha\}$ *and* R-lim $x_\alpha = x$ *then* R-lim $x_\gamma = x$,
(iv) R-lim $x_\alpha = x$ *and* R-lim $y_\beta = y$ *implies R-lim* $(x_\alpha + y_\beta) = x + y$,
(v) *if* $(G, \preccurlyeq)$ *is an l-group,* R-lim $y_\alpha = 0$ *and* $|x_\alpha| \preccurlyeq |y_\alpha|$ *for all* α *then* R-lim $x_\alpha = 0$,

(vi) *if* $(G, \preccurlyeq)$ *is an l-group then* R-lim $x_\alpha = x$ *and* R-lim $y_\beta = y$ *implies that* R-lim $(x_\alpha \vee y_\beta) = x \vee y$, *and* R-lim $(x_\alpha \wedge y_\beta) = x \wedge y$,

(vii) *if* $(G, \preccurlyeq)$ *is a directed partially ordered real vector space and* $\preccurlyeq' = \preccurlyeq$, *then* lim $\lambda_n = \lambda$ *and* R-lim $x_n = x$ *implies* R-lim $\lambda_n x_n = \lambda x$, *if and only if*, $(G, \preccurlyeq)$ *is integrally closed.*

Proof. (i)-(v). These follow from Lemmas 1 and 3, and the definition of R-limit.

(vi) The proof of this for T-lim in the context of $(G, \preccurlyeq)$, an abelian l-group, is given in (6). The T-lim result here is proved similarly, and the rest follows from Lemma 3.

(vii) Let $(G, \preccurlyeq)$ be a directed integrally closed partially ordered vector space and so $\preccurlyeq' = \preccurlyeq$, with lim $\lambda_n = \lambda$ and R-lim $x_n = x$. There exists $y \succ 0$, x, $-x$, since $(G, \preccurlyeq)$ is directed. Let $T_1 = \{z : \mu z \succcurlyeq y \text{ for some } \mu > 0\}$; so $y/n \in T_1$ for all positive integers n, and since $(G, \preccurlyeq)$ is integrally closed, $\wedge T_1 = 0$. So by Lemma 2, T_1 is a CTRO. Now

$$\lambda x - \lambda_n x_n = (\lambda - \lambda_n)x + \lambda_n(x - x_n)$$

and T-lim $x_n = x$ for some CTRO $T(\leqq)$. Also T_1-lim $(\lambda - \lambda_n)x = 0$. So by (iv) it only remains to show that R-lim $\lambda_n(x - x_n) = 0$. There exists $m \geqq |\lambda_n|$ for all n. So given $t \in T$ there exists $u \in T$ such that $mu < t$, by Lemma 1. There exists k such that

$$-u < x - x_n < u \text{ for all } n \geqq k.$$

Hence $-t < -mu \prec \lambda_n(x - x_n) \prec mu < t$ for all $n \geqq k$. So T-lim $\lambda_n(x - x_n) = 0$.

Now suppose that $(G, \preccurlyeq)$ is a directed partially ordered vector space which is not integrally closed and $\preccurlyeq' = \preccurlyeq$. Then there exist $a, b \in G$ such that $a \not\preccurlyeq 0$, $b \succ 0$ and $na \preccurlyeq b$ for all positive integers n. Let $\lambda_n = 1/n$, $x_n = b$, then R-lim $x_n = b$ and lim $\lambda_n = 0$. Suppose that T-lim $b/n = x$ for some CTRO T, $x \in G$. Then T-lim $b/(2n) = x$ by (iii), and also T-lim $b/(2n) = x/2$. So by (i), $x = 0$. Hence $\wedge b/n = \wedge T = 0$. But $a \preccurlyeq b/n$ for all $n > 0$ and $a \not\preccurlyeq 0$. Hence $\{b/n\}$ has no R-limit.

Note. It follows from Theorem 1 (i), (ii), (iii) that $(G, R\text{-lim})$ is a *limit space* in the sense of Birkhoff (1). Also from Theorem 1 (ii)-(v) it follows that our R-convergence is a type of *Riesz convergence* as defined by Leader (5).

Definition. The net $\{x_\alpha\}$ 0-converges to x, i.e. 0-lim $x_\alpha = x$, if there exists an increasing net $\{y_\beta\}$ and a decreasing net $\{z_\gamma\}$ such that:

(i) $\vee y_\beta = x = \wedge z_\gamma$

(ii) given any β, γ there exists α_0 such that $y_\beta \preccurlyeq x_\alpha \preccurlyeq z_\gamma$ for all $\alpha \geqq \alpha_0$.

This is essentially Vulikh's definition (8), which allows the three index sets to be different. One disadvantage of insisting on the same index set for all three nets, as is done by many authors, is that then 0-lim $\{x_\alpha\}_{\alpha \in A}$ may not exist, but 0-lim $\{x_\alpha\}_{\alpha \in A \text{ and } \alpha \geqq \alpha_0}$ may exist for some $\alpha_0 \in A$. We note that if $(G, \preccurlyeq)$ is directed then the two definitions agree for sequences.

Order-convergence appears to be satisfactory if $(G, \preccurlyeq)$ is fully ordered. In particular if $G = R$ then 0-convergence coincides with the usual metric convergence. However, in the Banach lattice $C[0, 1]$ 0-convergence does not imply metric convergence. In fact, if $f_n(x) = x^n$, $x \in [0, 1]$, then 0-lim $f_n = 0$, but $\|f_n\| = 1$.

Definition. The sequence $\{x_n\}$ *ru-converges* to x, i.e. *ru*-lim $x_n = x$, if there exists a sequence of non-negative integers $\{\lambda_n\}$ and $u \succ 0$ such that $\lim \lambda_n = \infty$ and $-u \preccurlyeq \lambda_n(x - x_n) \preccurlyeq u$ for all n.

This is essentially Leader's definition (**5**); he in fact defines *ru*-convergence on *l*-groups. Our definition agrees with the usual one (**2**) for directed partially ordered vector spaces.

In a Banach lattic *ru*-convergence implies metric convergence, and the two are equivalent if the Banach lattice has a strong unit (**2**). However, *ru*-limits need not be unique in non-archimedean pogroups. In fact, it can easily be shown that in a divisible isolated pogroup they are unique if and only if the pogroup is archimedean. For example, in RoR (lexicographic order), if $x_n = (0, 1)$ for all n then *ru*-lim $x_n = (0, 1)$ and *ru*-lim $x_n = (0, 0)$.

Definition. Let $\mathscr{F}$ be the collection of all intersections of finite unions of sets of the form $\{x : a \preccurlyeq x \preccurlyeq b\}$. A set $S \subseteq G$ is closed in the *interval topology* if $S \cap F \in \mathscr{F}$ for all $F \in \mathscr{F}$.

This is Birkhoff's modification of the Frink interval topology, to sets "possibly without universal bounds" (**2**).

Theorem 2. *Let $\{x_\alpha\}$ be a net in G then*

(i) *0-convergence in $(G, \preccurlyeq)$ is non-trivial only if $\preccurlyeq' = \preccurlyeq$. If $\preccurlyeq' = \preccurlyeq$ then R-lim $x_\alpha = x$ implies 0-lim $x_\alpha = x$,*
(ii) *if $(G, \preccurlyeq)$ is fully ordered then R-convergence and 0-convergence are equivalent,*
(iii) *if $\preccurlyeq' = \preccurlyeq$ then R-lim $x_n = x$ implies ru-lim $x_n = x$,*
(iv) *if $(G, \preccurlyeq)$ is divisible isolated and directed and $\preccurlyeq' = \preccurlyeq$ then R-convergence and ru-convergence are equivalent if and only if $(G, \preccurlyeq)$ is integrally closed,*
(v) *if $\preccurlyeq' = \preccurlyeq$ then R-lim $x_\alpha = x$ implies that $\{x_\alpha\}$ converges to x in the interval topology.*

Proof. (i) Suppose that $\preccurlyeq' = \preccurlyeq$ and T-lim $x_\alpha = x$ for some CTRO T. Let $y_{-t} = x - t$, $z_{-t} = x + t$, i.e. $(-T, \preccurlyeq)$ is the index set for these two nets. By Lemma 2, $\vee y_{-t} = \wedge z_{-t} = x$, and in fact 0-lim $x_\alpha = x$.

Now suppose that $(G, \preccurlyeq)$ contains a pseudopositive, q say. If 0-convergence is non-trivial, i.e. if there exists a net, not eventually constant, which is 0-convergent, then $\wedge S = 0$, where S is some non-empty subset of the strictly positive cone of $(G, \preccurlyeq)$. Now $s \succ -q$ for all $s \in S$, so $0 \succcurlyeq -q$. However $q \not\succcurlyeq 0$, so we have the required contradiction.

(ii) Trivial.

(iii) Suppose that $\preccurlyeq' = \preccurlyeq$ and T-lim $x_n = x$ for some CTRO T. Let $u \in T$ and define λ_n by

$$\lambda_n = \min\{\max[m \geqq 0 : -u \preccurlyeq m(x-x_n) \preccurlyeq u], n\}.$$

By the existence of small elements, lim $\lambda_n = \infty$. So ru-lim $x_n = x$.

(iv) Let $(G, \preccurlyeq)$ be divisible isolated directed and integrally closed. Then by the note following Lemma 1, $\preccurlyeq' = \preccurlyeq$. Suppose that $-u \preccurlyeq \lambda_n x_n \preccurlyeq u$ for all n, with $u \succ 0$ and lim $\lambda_n = \infty$. Let $T = \{t : kt \succcurlyeq u$ for some positive integer $k\}$. Then T is a CTRO (cf. proof of Theorem 1 (vii)). If $t \in T$ then for some positive integer m, $mt \succcurlyeq u$, so

$$-t < -\frac{t}{2} \preccurlyeq -\frac{u}{2m} \preccurlyeq \frac{\lambda_n}{2m} x_n \preccurlyeq \frac{u}{2m} \preccurlyeq \frac{t}{2} < t \quad \text{for all } n.$$

Since lim $\lambda_n = \infty$ and $(G, \preccurlyeq)$ is isolated, it follows that $-t < x_n < t$ for all n sufficiently large. So T-lim $x_n = 0$.

If $(G, \preccurlyeq)$ is divisible directed and not integrally closed and $\preccurlyeq' = \preccurlyeq$, then with the notation of the second part of the proof of Theorem 1 (vii) ru-lim $b/n = 0$ whereas R-lim b/n does not exist.

(v) Suppose that $\preccurlyeq' = \preccurlyeq$. We show that if $T(\leqq)$ is a CTRO and S is closed in the interval topology then S is T-closed. Let $\{x_\alpha\} \subseteq S$ with T-lim $x_\alpha = x$, we show that $x \in S$. Let $t \in T$ then for some α_0, $x_\alpha \in S \cap \{y : x-t \preccurlyeq y \preccurlyeq x+t\}$ for all $\alpha \geqq \alpha_0$. So in fact it will suffice to show that $\{x : a \preccurlyeq x \preccurlyeq b\}$ is a T-closed set. If $\{y_\beta\} \subseteq \{x : a \preccurlyeq x \preccurlyeq b\}$ and T-lim $y_\beta = y$, then $z > y$ implies that $z > y_\beta$ for some β, since $\{x : x < z\}$ is a T-neighbourhood of y. So $z > y$ implies that $z > a$, hence $y \succcurlyeq a$. Similarly it can be shown that $y \preccurlyeq b$. Hence $\{x : a \preccurlyeq x \preccurlyeq b\}$ is T-closed.

It is well known that 0-convergence and ru-convergence need not be convergence with respect to a topology. The following theorem determines when R-convergence is convergence with respect to a topology.

Theorem 3. *If $\mathscr{C}$, the set of all CTROs on $(G, \preccurlyeq')$, is non-empty, then the following are equivalent:*

(i) *R-convergence is convergence with respect to a topology,*
(ii) *R-convergence is convergence in the inductive limit of the CTROs,*
(iii) *$(\mathscr{C}, \subseteq)$ contains a smallest element.*

Proof. (i)⇒(ii). Write $T_1 \preccurlyeq T_2$ if $T_1 \supseteq T_2$. Then $(\mathscr{C}, \preccurlyeq)$ is u-directed (the dual of l-directed), by Lemma 3. If $T_\alpha, T_\beta \in \mathscr{C}$, define

$$\theta_{\alpha\beta}:(G, \mathscr{U}_\alpha)\to(G, \mathscr{U}_\beta),$$

by $\theta_{\alpha\beta}(x) = x$ for all $x \in G$. Then if $T_\alpha \preccurlyeq T_\beta$, $\theta_{\alpha\beta}$ is continuous by Lemma 3. Also $\overrightarrow{\lim} (G, \mathscr{U}_\alpha)$ is G with the strongest topology which is at least as weak as all the $\mathscr{U}_\alpha$ topologies. Hence (i) implies (ii).

(i)⇒(iii). If $T_i \preccurlyeq T_j$ write $i \preccurlyeq j$, and let $\mathscr{C} = \{T_i : i \in I\}$. Then $\{i, \preccurlyeq\}_{i \in I}$ and $\{-t, \preccurlyeq\}_{t \in T_i}$ are u-directed by Lemmas 3 and 2, respectively. If $i \in I$ and $t \in T_i$, let $y(i, -t) = t$, then

$$R-\lim_{i \in I} R-\lim_{-t \in -T_i} y(i, -t) = 0.$$

If R-convergence is convergence with respect to a topology then by a theorem on iterated limits (4),

$$R-\lim_{\Gamma} z(i, \ldots, -t, \ldots) = 0,$$

where

$$z(i, \ldots, -t, \ldots) = y(i, -t) \text{ and } \Gamma = I \times \prod_{i \in I} (-T_i).$$

So for some

$$T_0 \in \mathscr{C},\ T_0 - \lim_{\Gamma} z(i, \ldots, -t, \ldots) = 0.$$

Let $t \in T_0$, then there exists $i_0 \in I$ such that for each $i \succcurlyeq i_0$, $t > u_i$, where u_i is some element of T_i, and $\leqq$ refers to T_0. So by Lemma 2 (i), $t \in T_i$ for all $i \succcurlyeq i_0$. If $j \in I$, then there exists $i \in I$ such that $i \succcurlyeq j, i_0$, since $(I, \preccurlyeq)$ is u-directed. So $t \in T_i$ and $T_i \subseteq T_j$, hence $t \in T_j$, and so $T_0 \subseteq T_j$ for all $j \in I$. Thus T_0 is the smallest element of $(C, \subseteq)$.

(ii)⇒(i), and (iii)⇒(i) are obvious.

In Lemma 4 of (9) it is shown that if $(G, \preccurlyeq)$ is a divisible integrally closed l-group then $(G, \preccurlyeq)$ has a smallest CTRO if and only if $(G, \preccurlyeq)$ has a strong unit.

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