Glasgow Math. J. **53** (2011) 401–410. © Glasgow Mathematical Journal Trust 2011.
doi:10.1017/S0017089511000036.

ON WEAKLY $\mathcal{M}$-SUPPLEMENTED SUBGROUPS OF SYLOW p-SUBGROUPS OF FINITE GROUPS*

LONG MIAO

School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, P. R. China
e-mail: miaolong714@vip.sohu.com

(Received 25 August 2009; revised 21 September 2010; accepted 2 November 2010)

Abstract. A subgroup H is called weakly $\mathcal{M}$-supplemented in a finite group G if there exists a subgroup B of G provided that (1) $G = HB$, and (2) if H_1/H_G is a maximal subgroup of H/H_G, then $H_1B = BH_1 < G$, where H_G is the largest normal subgroup of G contained in H. In this paper we will prove the following: Let G be a finite group and P be a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. Suppose that P has a non-trivial proper subgroup D such that all subgroups E of P with order $|D|$ and $2|D|$ (if P is a non-abelian 2-group, $|P : D| > 2$ and there exists $D_1 \trianglelefteq E \leq P$ with $2|D_1| = |D|$ and E/D_1 is cyclic of order 4) have p-nilpotent supplement or weak $\mathcal{M}$-supplement in G, then G is p-nilpotent.

2010 *Mathematics Subject Classification.* 20D10, 20D20.

1. Introduction. All groups considered in this paper are finite. Most of the notations are standard and can be found in [**3**] and [**9**].

It is well known that the relationship between the properties of primary subgroups and the structure of finite groups has been investigated extensively by many authors. For instance, in 1980 Srinivasan [**14**] proved that a finite group is supersolvable if every maximal subgroup of the Sylow subgroup is normal. By considering normal c-supplement of some primary subgroups, Wang [**16**] in 1996 obtained some new conditions for the solvability and supersolvability of a finite group. Furthermore, Guo and Shum [**7**] in 2003 considered the c-normal maximal subgroups and minimal subgroups of a Sylow p-subgroup of G, and got some new results about p-nilpotent groups. In 2004, Guo, Sun and Shum [**8**] showed that if there is a maximal subgroup M of a group G and a prime p for which every cyclic subgroup of p-power order in M is c-supplemented in G, then G is solvable. In 2005, Guo, Shum and Skiba [**6**] obtained some new properties of supersolvable groups by using conditionally permutable subgroups.

Recently, as an interesting application of these generalisations, Skiba [**12**, **13**] fixed in every noncyclic Sylow subgroup P of G a group D satisfying $1 < |D| < |P|$, and investigated the structure of G under the assumption that all subgroups H with $|H| = |D|$ are c-quasinormal or weakly s-permutable in G. Moreover, Guo [**4**] proposed the conception of $\mathcal{F}$-supplemented subgroup and obtained some new results about supersolvable and solvable groups. Guo and Skiba [**5**] introduced s-embedded and n-embedded subgroups, and obtained some new results about supersolvable groups. Miao and Lempken [**10**] presented the definition of $\mathcal{M}$-supplemented subgroup, and

*The author is supported by the grant of NSFC (Grant #10901133) and sponsored by Qing Lan Project and Natural Science Fund for colleges and universities in Jiangsu Province.

got some new information on the structure of finite groups. More recently, Miao and Lempken [**11**] generalised $\mathcal{M}$-supplemented and c-normal subgroups with weakly $\mathcal{M}$-supplemented subgroups, and obtained some new results about supersolvable groups.

As a continuation, we will extensively investigate the properties of the weakly $\mathcal{M}$-supplemented subgroups in a finite group G.

DEFINITION 1.1. A subgroup H of a group G is said to be weakly $\mathcal{M}$-supplemented in G if there exists a subgroup B of G such that (1) $G = HB$, and (2) if H_1/H_G is a maximal subgroup of H/H_G, then $H_1B = BH_1 < G$, where H_G is the largest normal subgroup of G contained in H; in this case, B is also called a weak $\mathcal{M}$-supplement of H in G.

Recall that a subgroup H is called $\mathcal{M}$-supplemented in a finite group G [**10**] if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for every maximal subgroup H_1 of H. Moreover, a subgroup H is called weakly s-permutable in G [**13**] if there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_{sG}$, where H_{sG} is the largest s-quasinormal subgroup of G contained in H.

It is clear that every $\mathcal{M}$-supplemented subgroup and every c-normal subgroup are weakly $\mathcal{M}$-supplemented. The following examples indicate that the weak $\mathcal{M}$-supplementation of subgroups can neither be deduced from Skiba's result nor from other related results.

EXAMPLE 1.2. Let $G = S_4$ and $H = \langle(1234)\rangle$ be a cyclic subgroup of order 4. Then $G = HA_4$, where A_4 is the alternating group of degree 4. Clearly, since $A_4 \trianglelefteq G$, we have A_4 permutes all maximal subgroups of H and hence H is weakly $\mathcal{M}$-supplemented in G. On the other hand, we have $H_{sG} = 1$. Otherwise, if H is s-quasinormal in G, then H is normal in G, a contradiction. If $H_{sG} = \langle(13)(24)\rangle$ is s-quasinormal in G, then $\langle(13)(24)\rangle$ is normal in G, a contradiction. Therefore H is not weakly s-permutable in G.

EXAMPLE 1.3. Let $G = S_4$ and H be a Sylow 2-subgroup of G. Clearly, H is weakly $\mathcal{M}$-supplemented in G and $G = HA_4$. Furthermore, H is not $\mathcal{M}$-supplemented in G.

2. Preliminaries. For the sake of convenience, we first list here some known results that will be useful in the sequel.

LEMMA 2.1 [**11**, Lemma 2.1]. *Let G be a group. Then,*

(1) *If H is weakly $\mathcal{M}$-supplemented in G, $H \leq M \leq G$, then H is weakly $\mathcal{M}$-supplemented in M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is weakly $\mathcal{M}$-supplemented in G if and only if H/N is weakly $\mathcal{M}$-supplemented in G/N.*

(3) *Let π be a set of primes. Let K be a normal π'-subgroup and H be a π-subgroup of G. If H is weakly $\mathcal{M}$-supplemented in G, then HK/K is weakly $\mathcal{M}$-supplemented in G/K.*

(4) *Let R be a solvable minimal normal subgroup of group G and R_1 be a maximal subgroup of R. If R_1 is weakly $\mathcal{M}$-supplemented in G, then R is a cyclic group of prime order.*

(5) *Let P be a p-subgroup of G, where p is a prime divisor of $|G|$. If P is weakly $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $|G : TB| = p$ for every maximal subgroup T of P containing P_G.*

LEMMA 2.2 [**10**, Lemma 2.11]. *Let p be the smallest prime divisor of $|G|$ and $P \in Syl_p(G)$. Then G is p-nilpotent if and only if P is $\mathcal{M}$-supplemented in G.*

LEMMA 2.3 [**17**, Theorem 4.1]. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal and all cyclic subgroups with order 4 of $F(H)$ are c-supplemented in G, then $G \in \mathcal{F}$.*

LEMMA 2.4 [**3**, Theorem 1.8.17]. *Let N be a non-trivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G that are contained in N.*

LEMMA 2.5 [**18**, Proposition 4.6]. *If H is a subgroup of G with $|G : H| = p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.*

LEMMA 2.6 [**2**, main theorem]. *Suppose a finite group G has a Hall π-subgroup, where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

LEMMA 2.7 [**9**, IV, Theorem 5.4]. *Suppose that G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then G is a group which is not nilpotent but whose proper subgroups are all nilpotent.*

LEMMA 2.8 [**3**, Theorem 3.4.11]. *Suppose that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then*

(1) *G has a normal Sylow p-subgroup P for some prime p and $G/P \cong Q$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*
(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
(3) *If P is non-abelian and $p \neq 2$, then the exponent of P is p.*
(4) *If P is non-abelian and $p = 2$, then the exponent of P is 4.*
(5) *If P is abelian, then P is of exponent p.*

LEMMA 2.9 [**3**, Lemma 3.6.10]. *Let K be a normal subgroup of G, and P be a p-subgroup of G, where p is a prime divisor of $|G|$. Then $N_{G/K}(PK/K) = N_G(P_1)K/K$, here P_1 is a Sylow p-subgroup of PK.*

3. Main results.

THEOREM 3.1. *Let G be a finite group and P be a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P has a p-nilpotent supplement or a weak $\mathcal{M}$-supplement in G, then G is p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of the smallest order.

Let P_1 be a maximal subgroup of P. By hypotheses, if P_1 has a p-nilpotent supplement in G, then there exists a p-nilpotent subgroup K of G such that $G = P_1K$. Furthermore, since K is p-nilpotent, we have $K \leq N_G(K_{p'})$, where $K_{p'}$ is a Hall p'-subgroup of K and also of G. Therefore, $G = P_1K = PN_G(K_{p'})$. Clearly, $P \nleq N_G(K_{p'})$ and $P \cap N_G(K_{p'}) \leq L_2 < L_1$, where L_1 is a maximal subgroup of P and L_2 is a maximal subgroup of L_1. Otherwise, if $P \cap N_G(K_{p'}) = L_1$, then $|G : N_G(K_{p'})| = |P : P \cap N_G(K_{p'})| = p$. By Lemma 2.5, we know that $N_G(K_{p'}) \trianglelefteq G$ and hence $K_{p'} \trianglelefteq G$, a contradiction. Furthermore, if L_1 has a p-nilpotent supplement in G, then there exists a p-nilpotent subgroup H in G such that $G = L_1N_G(H_{p'})$, where $H_{p'}$ is a

https://doi.org/10.1017/S0017089511000036 Published online by Cambridge University Press

Hall p'-subgroup of G. By Lemma 2.6, there exists an element g of L_1 such that $N_G(K_{p'}) = (N_G(H_{p'}))^g$. So we have $G = L_1 N_G(H_{p'}) = L_1(N_G(H_{p'}))^g = L_1 N_G(K_{p'})$ and $P = P \cap L_1 N_G(K_{p'}) = L_1(P \cap N_G(K_{p'})) = L_1$, a contradiction.

So we may assume that L_1 is weakly $\mathcal{M}$-supplemented in G. If L_1 is normal in G, then $|G/L_1|_p = p$ and hence G/L_1 is p-nilpotent by the Burnside p-nilpotent Theorem. Let L/L_1 be a normal p-complement of G/L_1. By the Schur–Zassenhaus Theorem, $L = [L_1]L_{p'}$ and $G = LN_G(L_{p'}) = L_1 N_G(L_{p'})$, with the similar discussion, we get the contradiction.

If $1 < (L_1)_G < L_1$ by the definition of a weakly $\mathcal{M}$-supplemented subgroup, then there exists a subgroup B of G such that $G = L_1 B$ and $TB < G$ for every maximal subgroup T of L_1 containing $(L_1)_G$. By Lemma 2.1(2), $G/(L_1)_G$ satisfies the condition of the theorem, the minimal choice of G implies that $G/(L_1)_G$ is p-nilpotent. The same arguments as above show that G is p-nilpotent, which also is a contradiction.

Next we may assume $(L_1)_G = 1$. By Lemma 2.1(5), $|G : TB| = p$ for every maximal subgroup T of L_1. Particularly, $|G : L_2 B| = p$ and hence $L_2 B \trianglelefteq G$ by Lemma 2.5. Clearly, $P \cap L_2 B = L_2(P \cap B)$ is a maximal subgroup of P. By hypotheses, if $L_2(P \cap B)$ has a p-nilpotent supplement in G, we get a contradiction. So we have that $L_2(P \cap B)$ is weakly $\mathcal{M}$-supplemented in G. Moreover, if $(L_2(P \cap B))_G \neq 1$, then we denote $(L_2(P \cap B))_G := S$ and G/S is p-nilpotent, since the hypotheses hold on G/S. G/S has a normal Hall p'-subgroup X/S and $X = [S]X_{p'}$, where $X_{p'}$ is also a Hall p'-subgroup of G. By the Frattini Argument we have $G = XN_G(X_{p'}) = SN_G(X_{p'}) = L_2(P \cap B)N_G(X_{p'})$. By Lemma 2.6, there exists an element x in $L_2(P \cap B)$ such that $N_G(K_{p'}) = (N_G(X_{p'}))^x$. So we have $G = L_2(P \cap B)N_G(X_{p'}) = L_2(P \cap B)(N_G(X_{p'}))^x = L_2(P \cap B)N_G(K_{p'})$ and $P = P \cap L_2(P \cap B)N_G(K_{p'}) = L_2(P \cap B)(P \cap N_G(K_{p'})) = L_2(P \cap B)$, a contradiction. Therefore, $(L_2(P \cap B))_G = 1$ and the Sylow p-subgroup $L_2(P \cap B)$ of $L_2 B$ is $\mathcal{M}$-supplemented in $L_2 B$ by Lemma 2.1(1). So $L_2 B$ is p-nilpotent by Lemma 2.2, a contradiction.

Therefore P_1 is weakly $\mathcal{M}$-supplemented in G. With the similar argument as above, G is p-nilpotent, a final contradiction. □

THEOREM 3.2. *Let G be a finite group and P be a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. If every minimal subgroup of P and every cyclic subgroup of order* 4 *have p-nilpotent supplement or weak $\mathcal{M}$-supplement in G, then G is p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of the minimal order.

Clearly, the hypotheses is inherited by all proper subgroups of G by Lemma 2.1(1). Thus, G is a minimal non-p-nilpotent group. Now Lemma 2.7 implies that G is a group which is not nilpotent but whose proper subgroups are all nilpotent. Then by Lemma 2.8, G has a normal Sylow p-subgroup P and $G = [P]Q$, where Q is a non-normal cyclic Sylow q-subgroup of G, and $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. We consider the following cases.

Case 1. $p \neq 2$. By Lemma 2.8, the exponent of P is p. Let E be a minimal subgroup of P. By hypotheses, E has a p-nilpotent supplement in G or is weakly $\mathcal{M}$-supplemented in G. Clearly, if E has a p-nilpotent supplement in G, then we have that G is p-nilpotent, a contradiction. Therefore E is weakly $\mathcal{M}$-supplemented in G. If E is non-normal in G, then E has a complement B in G. By Lemma 2.5, $B \trianglelefteq G$ and hence G is nilpotent, a contradiction. So every minimal subgroup of P is normal in G, we also get a contradiction.

Case 2. $p = 2$. If the exponent of P is 2, then with the similar discussion as case 1, we have the same contradiction. So we may assume P is of exponent 4 and so is non-abelian. Let A be a cyclic subgroup of P of order 4. By hypotheses, if A has a p-nilpotent supplement in G, then there exists a p-nilpotent subgroup L such that $G = AL$. Clearly, $L < G$ and hence L is nilpotent by Lemma 2.8. Next we consider $N_G(L_p)$, where L_p is a Sylow p-subgroup of L. If $L_p = 1$, then $P = A$ is cyclic, a contradiction. Since $L \leq N_G(L_p)$, we have that $|G : N_G(L_p)| = 2$ or $|G : N_G(L_p)| = 1$. If $|G : N_G(L_p)| = 2$, then $N_G(L_p) \trianglelefteq G$ by Lemma 2.5 and hence G is 2-nilpotent, a contradiction. If $|G : N_G(L_p)| = 1$, then $L_p \trianglelefteq G$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, we have $P = L_p$ or $L_p \leq \Phi(P)$. It is clear that $P = L_p$ is impossible. If $L_p \leq \Phi(P)$, then $P = AL_p = A$, a contradiction. So we may assume that A is weakly $\mathcal{M}$-supplemented in G. If A is normal in G, then $A\Phi(P)/\Phi(P) = P/\Phi(P)$ and $A = P$ is abelian by Lemma 2.8, a contradiction. If A is not normal in G, then $1 < A/\Phi(P) < P/\Phi(P)$. Since A is weakly $\mathcal{M}$-supplemented in G, there is a subgroup B of G such that $AB = G$ and $A_1B < G$ for every maximal subgroup A_1 contained A_G. Let $T = A_1B$. Then $G = AT = PT$, clearly, $\Phi(P) \leq T$ since $|G : T| = 2$. Since $P/\Phi(P)$ is minimal normal in $G/\Phi(P)$, $G/\Phi(P) = (P/\Phi(P))(T/\Phi(P)) = [P/\Phi(P)](T/\Phi(P))$ and hence $|P/\Phi(P)| = |G/\Phi(P) : T/\Phi(P)| = 2$. It follows that $P/\Phi(P)$ is cyclic of order 2, a contradiction.

The final contradiction completes our proof. □

THEOREM 3.3. *Let G be a finite group and P be a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. Suppose that P has a subgroup D such that $1 < D < P$, and all subgroups E of P with order $|D|$ and $2|D|$ (if P is a non-abelian 2-group, $|P : D| > 2$ and there exists $D_1 \trianglelefteq E \leq P$ with $2|D_1| = |D|$ and E/D_1 is cyclic of order 4) have p-nilpotent supplement or weak $\mathcal{M}$-supplement in G, then G is p-nilpotent.*

Proof. Assume that the Theorem is false and choose G to be a counterexample of minimal order.

By hypotheses, P has a subgroup D such that $1 < D < P$, and all subgroups E of P with order $|D|$ and order $2|D|$ (if P is a non-abelian 2-group, $|P : D| > 2$ and there exists $D_1 \trianglelefteq E \leq P$ with $2|D_1| = |D|$ and E/D_1 is cyclic of order 4) have p-nilpotent supplement or weak $\mathcal{M}$-supplement in G. Fix a subgroup E of P with order $|D|$. We will derive a contradiction in several steps.

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 2.1(3) guarantees that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus, $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. $|D| > p$. Suppose $|D| = p$. By Theorem 3.2, G is p-nilpotent, a contradiction.

Step 3. $|P : D| > p$. If $|P : D| = p$, then every maximal subgroup of P has a p-nilpotent supplement or a weak $\mathcal{M}$-supplement in G and hence G is p-nilpotent by Theorem 3.1, a contradiction.

Step 4. If there exists a minimal normal subgroup N of G contained in P, then $|N| \leq |D|$.

If $|N| > |D|$, then we may choose a subgroup E of P with order $|D|$ such that $E < N$. By hypotheses, if E has a p-nilpotent supplement in G, then there exists a p-nilpotent subgroup K of G such that $G = EK$. Clearly, $N \cap K \in \{1, N\}$, a contradiction.

So E is weakly $\mathcal{M}$-supplemented in G. Therefore there exists a subgroup B of G such that $G = EB$ and $E_1B < G$ for every maximal subgroup E_1 of E containing E_G. Since N is a minimal normal subgroup of G contained in P, we have $E_G = 1$ and $N \cap B = 1$ or N. If $N \cap B = 1$, then $N = E$, a contradiction. If $N \cap B = N$, then $B = G$, is also a contradiction.

Step 5. G/N is p-nilpotent.

If $|N| < |D|$, clearly, G/N satisfies the hypotheses by Lemma 2.1(2). Therefore G/N is p-nilpotent by the minimal choice of G. So we may assume $|N| = |D|$. Next we will show that every cyclic subgroup of P/N of order p and order 4 (if P is a non-abelian 2-group) have p-nilpotent supplement or weak $\mathcal{M}$-supplement in G/N.

Let $K \leq P$ and $|K/N| = p$. Clearly, N is not cyclic. Otherwise, N_1 char N and $N \trianglelefteq G$, where N_1 is the maximal subgroup of N, it follows that $N_1 \trianglelefteq G$, which is contrary to the minimality of N. So all subgroups containing N are not cyclic. Hence, there exists a maximal subgroup L of K such that $K = LN$ and $|D| = |L| = |N|$. If L has a p-nilpotent supplement in G, then $K/N = LN/N$ also has a p-nilpotent supplement in G/N. So we have L that is weakly $\mathcal{M}$-supplemented in G. If L is normal in G, then K/N is normal in G/N. If L is not normal in G, then there exists a subgroup B of G such that $G = LB$ and $TB < G$ for every maximal subgroup T of L containing L_G. By Lemma 2.1(5) $|G : TB| = p$ and hence $TB \trianglelefteq G$ by Lemma 2.5. By Lemma 2.1(1), TB satisfies the condition of the theorem. Therefore TB is p-nilpotent by the minimal choice of G and hence G is p-nilpotent, a contradiction.

If X/N is a cyclic group of order 4 and K/N is a maximal subgroup of X/N, then K is maximal in X and $|K/N| = 2$. Since X is not cyclic and X/N is cyclic, there exists a maximal subgroup L of X such that $N \nleq L$. Thus, $X = LN$ and $|L| = |K| = 2|D|$, $X/N = LN/N \cong L/L \cap N$ is cyclic of order 4. If L has a p-nilpotent supplement in G, then $X/N = LN/N$ also has a p-nilpotent supplement in G/N. By hypotheses, L is weakly $\mathcal{M}$-supplemented in G. If L is normal in G, then LN/N is also normal in G/N. So we may assume that L is not normal in G. There exists a subgroup C of G such that $G = LC$ and $TC < G$ for every maximal subgroup T of L containing L_G. By Lemma 2.1(5) $|G : TC| = 2$ and hence $TC \trianglelefteq G$ by Lemma 2.5. By Lemma 2.1(1), TC satisfies the condition of the theorem. Therefore TC is p-nilpotent by the minimal choice of G and, hence, G is p-nilpotent, a contradiction.

Step 6. $O_p(G) = 1$.

Suppose $O_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $O_p(G)$. By Step 5, G/N is p-nilpotent. Clearly, N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$ since the class of all p-nilpotent groups is a saturated formation. By Lemma 2.4, $O_p(G) = N$. There exists a maximal subgroup M of G such that $G = NM = NN_G(M_{p'})$, where $M_{p'}$ is the Hall p'-subgroup of M and also of G. If $M_p = M \cap P = 1$, then $N = P$, contrary to step 4. If $|D| \leq |M \cap P|$, then we may choose a subgroup E of $M \cap P$ with order $|D|$ and hence $E_G = 1$. By hypotheses, if E has a p-nilpotent supplement in G, then there exists a p-nilpotent subgroup K of G such that $G = EK$. On the other hand, there exists a maximal subgroup P_1 of P such that $E \leq P \cap M \leq P_1 < P$ and $G = EK = P_1K$. Since K is p-nilpotent, we have $G = P_1K = P_1N_G(K_{p'})$, where $K_{p'}$ is the Hall p'-subgroup of K and also of G. By Lemma 2.6, there exists an element g of P_1 such that $N_G(M_{p'}) = (N_G(K_{p'}))^g$. So $G = P_1N_G(K_{p'}) = P_1(N_G(K_{p'}))^g = P_1N_G(M_{p'})$ and $P = P \cap P_1N_G(M_{p'}) = P_1(P \cap N_G(M_{p'})) = P_1$, a contradiction. So we may assume E is weakly $\mathcal{M}$-supplemented in G. There exists a subgroup B of G such that $G = EB$ and $E_iB < G$ for every maximal subgroup E_i of E containing E_G. By Lemma 2.1(5), we have $|G : E_iB| = p$ and hence $E_iB \trianglelefteq G$. Since $|P : D| > p$, E_iB satisfies the condition of the theorem. The minimal choice of G implies that E_iB is p-nilpotent and hence G is p-nilpotent, a contradiction.

If $|M \cap P| < |D|$, then we may choose a subgroup E containing $M \cap P$ with order $|D|$ and get a contradiction with the similar argument as above.

Step 7. Final contradiction.

If all subgroups E of P with order $|D|$ and all cyclic subgroups of P of order $2|D|$ (if P is a non-abelian 2-group, $|P : D| > 2$ and there exists $D_1 \trianglelefteq E \leq P$ with $2|D_1| = |D|$ and E/D_1 is cyclic of order 4) have p-nilpotent supplement in G, then all maximal subgroups of P have p-nilpotent supplement in G and hence G is p-nilpotent, a contradiction. There exists at least a subgroup E of P with order $|D|$, which is weakly $\mathcal{M}$-supplemented in G. Since $O_p(G) = 1$ by step 6, E is not normal in G and hence there exists a subgroup B of G such that $G = EB$ and $E_iB < G$ for every maximal subgroup E_i of E containing E_G. By Lemma 2.1(5), we have $|G : E_iB| = p$ and hence $E_iB \trianglelefteq G$ by Lemma 2.5. Since $|P : D| > p$, E_iB satisfies the hypotheses and E_iB is p-nilpotent by the minimal choice of G. Put $R = E_iB$. Then $R_{p'}$ char R and $R \trianglelefteq G$. Therefore $R_{p'} \trianglelefteq G$ and G is p-nilpotent, a contradiction.

The final contradiction completes our proof. □

THEOREM 3.4. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is weakly $\mathcal{M}$-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the assertion is false and choose G to be a counterexample of the minimal order. Furthermore, we have

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemmas 2.1(3) and 2.9, $G/O_{p'}(G)$ satisfies the condition of the theorem, the minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and hence G is p-nilpotent, a contradiction.

(2) If S is a proper subgroup of G containing P, then S is p-nilpotent.

Clearly, $N_S(P) \leq N_G(P)$ and hence $N_S(P)$ is p-nilpotent. Applying Lemma 2.1(1), we find that S satisfies the hypotheses of our theorem. Now, by the minimality of G, S is p-nilpotent.

(3) $G = PQ$, where Q is the Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson [**15**, Corollary 1], there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $H < K \leq P$. Since $N_G(P) \leq N_G(H)$ and $N_G(H)$ is not p-nilpotent, we have $N_G(P) < N_G(H)$. Then by (2), we have $N_G(H) = G$. This leads to $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P such that $O_p(G) < K \leq P$. Now by Thompson [**15**, Corollary 1], again $G/O_p(G)$ is p-nilpotent and, therefore, G is p-solvable. Since G is p-solvable, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $PQ = QP$ is a subgroup of G by Gorenstein [**1**, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2). This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ by Guo [**3**, Theorem 1.8.18] since $O_{p'}(G) = 1$, a contradiction. Thus, we have proven that $G = PQ$.

(4) Conclusion.

Since $O_p(G) \neq 1$, we may take a minimal normal subgroup L of G with $L \leq O_p(G)$. Clearly, G/L satisfies the condition of the theorem. Now, the minimality of G implies

https://doi.org/10.1017/S0017089511000036 Published online by Cambridge University Press

that G/L is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we may assume that L is the unique minimal normal subgroup of G contained in $O_p(G)$ and $L \nleq \Phi(G)$. Thus, by Lemma 2.4, we have $O_p(G) = L$ is an elementary abelian p-group. Furthermore, there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Hence, $P = P \cap LM = L(P \cap M)$ and $P \cap M = P^*$ is a Sylow p-subgroup of M. If $P^* = 1$, then $P = L$, and therefore $G = N_G(L) = N_G(P)$ is p-nilpotent, which is a contradiction. So we may assume $P^* \neq 1$. Pick a maximal subgroup P_1 of P with $P^* \leq P_1$. If $P^* = P_1$, then $|L| = p$. If $p < q$, then LQ is p-nilpotent and therefore $Q \leq C_G(L) = C_G(O_p(G))$, which contradicts $C_G(O_p(G)) \leq O_p(G)$. On the other hand, if $q < p$, then $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to some subgroup of $Aut(N)$, since $C_G(N) = C_G(O_p(G)) = O_p(G) = N$. Therefore Q is a cyclic group. Since Q is cyclic and $q < p$, G is q-nilpotent and therefore P is normal in G. Hence, $N_G(P) = G$ is p-nilpotent, a contradiction.

So we may assume $P^* < P_1$. By hypotheses, P_1 is weakly $\mathcal{M}$-supplemented in G. There exists a subgroup B such that $G = P_1B$ and $TB < G$ for every maximal subgroup $(P_1)_G \leq T$. If $(P_1)_G \neq 1$, then we have $L \leq (P_1)_G \leq P_1$, a contradiction. So we have $(P_1)_G = 1$. By Lemma 2.1(5), $|G : TB| = p$ for every maximal subgroup T of P_1. Particularly, there exists at least a maximal subgroup T of P_1 such that $L \nleq TB$. We may choose a maximal subgroup T of P_1 such that $P^* \leq T$. Clearly, $L \nleq TB$. Otherwise, $L \leq TB$ and $TB = LTB = PB = G$, a contradiction. Therefore, $|L| = p$ and we may get a contradiction with the similar discussion as above.

The final contradiction completes our proof. □

COROLLARY 3.5 [**7**, Theorem 3.1]. *Let p be an odd prime dividing G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent..*

THEOREM 3.6. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and suppose that P has a subgroup D such that $1 < D < P$, and every subgroup E of P with order $|D|$ is weakly $\mathcal{M}$-supplemented in G, then G is p-nilpotent.*

Proof. Assume that the assertion is false and choose G to be a counterexample of the minimal order. Furthermore, we have

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemma 2.1(3), $G/O_{p'}(G)$ satisfies the condition of the theorem, the minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and hence G is p-nilpotent, a contradiction.

(2) If S is a proper subgroup of G containing P, then S is p-nilpotent.

Clearly, $N_S(P) \leq N_G(P)$ and hence $N_S(P)$ is p-nilpotent. Applying Lemmas 2.1(1) and 2.9, S satisfies the hypotheses of our theorem. Then the minimal choice of G implies that S is p-nilpotent.

(3) $G = PQ$, where Q is the Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson [**15**, Corollary 1] there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $H < K \leq P$. Since $N_G(P) \leq N_G(H)$ and $N_G(H)$ is not p-nilpotent, we have $N_G(P) < N_G(H)$. Then by (2), we have $N_G(H) = G$. This leads to $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P such that $O_p(G) < K \leq P$. Now by Thompson [**15**, Corollary 1], again we see that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. Since G is p-solvable, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $PQ = QP$ is a subgroup of G by Gorenstein [**1**, Theorem 6.3.5]. If $PQ < G$, then PQ is p-nilpotent by (2). This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ by Guo [**3**, Theorem 1.8.18] since $O_{p'}(G) = 1$, a contradiction. Thus, we have proven that $G = PQ$.

(4) $|D| > p$.

Suppose $|D| = p$. By hypotheses, every minimal subgroup of P is weakly $\mathcal{M}$-supplemented in G; in fact, in this case every minimal subgroup of P is also c-supplemented in G; by (1) and (3) we have $F(G) = O_p(G)$. It follows that G is super solvable by Lemma 2.3. If $p < q$, then G is p-nilpotent by Theorem 3.2, a contradiction. If $p > q$, then G is q-nilpotent and hence G has a normal Sylow p-subgroup P. Therefore $G = N_G(P)$ is p-nilpotent, also a contradiction.

(5) $|P : D| > p$.

If $|P : D| = p$, then every maximal subgroup of P is weakly $\mathcal{M}$-supplemented in G and hence G is p-nilpotent by Theorem 3.4.

(6) $O_p(G) = N$ is a unique minimal normal subgroup of G and $C_G(N) = N$.

If $|N| > |D|$, by hypotheses we may choose a subgroup E of P with order $|D|$ such that $E < N$. Since E is weakly $\mathcal{M}$-supplemented in G, there exists a subgroup B of G such that $G = EB$ and $TB < G$ for every maximal subgroup T of E. Since N is a minimal normal subgroup of G, we have $N \cap B = 1$ or N. If $N \cap B = 1$, then $N = E$, a contradiction. If $N \cap B = N$, then $B = G$, which is also a contradiction.

If $|N| < |D|$, clearly G/N satisfies the hypotheses of the Lemma by Lemma 2.1(2). Therefore G/N is p-nilpotent by the minimal choice of G. So we may assume $|N| = |D|$. Let $K \leq P$ and $|K/N| = p$. Clearly, N is not cyclic. Otherwise, N_1 char N and $N \trianglelefteq G$, where N_1 is the maximal subgroup of N, it follows that $N_1 \trianglelefteq G$, contrary to the minimality of N. So all subgroups containing N are not cyclic. Hence, there exists a maximal subgroup L of K such that $K = LN$ and $|D| = |L| = |N|$. If L is normal in G, then K/N is normal in G/N. If L is non-normal in G, then there exists a subgroup B of G such that $G = LB$ and $TB < G$ for every maximal subgroup T of L containing L_G. If $NB = G$, then $G = NTB$ and hence $|N| = |G : TB| = p$, this is contrary to (4). So we have $NB < G$ and $G/N = (LN/N)(BN/N)$. Therefore G/N is p-nilpotent.

Clearly, N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$ since the class of all p-nilpotent groups is a saturated formation. By Lemma 2.4, $O_p(G) = N$.

(7) Final contradiction.

There exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. If $M_p = M \cap P = 1$, then $N = P$, a contradiction. Let P_1 be a maximal subgroup of P containing M_p. Clearly, $P_1 = P_1 \cap NM_p = M_p(P_1 \cap N)$. If $P_1 \cap N = 1$, then $|N| = p$. If $p < q$, then NQ is p-nilpotent and therefore $Q \leq C_G(N) = C_G(O_p(G))$, which contradicts $C_G(O_p(G)) \leq O_p(G)$. On the other hand, if $q < p$, then, since $C_G(N) = C_G(O_p(G)) = O_p(G) = N$, $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $Aut(N)$ and therefore M, and particular Q is a cyclic group. Since Q is a cyclic group and $q < p$, G is q-nilpotent and therefore P is normal in G. Hence, $N_G(P) = G$ is p-nilpotent, a contradiction.

So we may assume $L = P_1 \cap N \neq 1$. By (6), $|N| \leq |D|$. Choose a subgroup E of P_1 containing L with $|E| = |D|$. Clearly, $N \nleq E$ and $E = E \cap P_1 = E \cap LM_p = L(E \cap M_p)$. By hypotheses, E is weakly $\mathcal{M}$-supplemented in G and $E_G = 1$. There exists a subgroup B of G such that $G = EB$ and $TB < G$ for every maximal subgroup

https://doi.org/10.1017/S0017089511000036 Published online by Cambridge University Press

T of E, since $E_G = 1$. Furthermore, we may choose a maximal subgroup P_2 of E such that $E \cap M_p \leq P_2$. Therefore $P_2 = P_2 \cap (P_1 \cap N)(E \cap M_p) = (E \cap M_p)(P_2 \cap P_1 \cap N) = (E \cap M_p)(P_2 \cap N)$. Then we may choose $T = P_2$. If $N \leq P_2B$, then $P_2B = NP_2B = EB = G$, a contradiction. So $N \cap P_2B = 1$ and $|G : P_2B| = |N| = p$. With a similar discussion as above, we get a final contradiction. □

ACKNOWLEDGEMENTS. The author thanks the referee for his/her careful reading of the manuscript and for his/her suggestions that have helped to improve the original version.

REFERENCES

1. D. Gorenstein, *Finite groups* (Harper & Row, New York, 1968).

2. F. Gross, Conjugacy of odd order Hall subgroups, *Bull. Lond. Math. Soc.* **19** (1987), 311–319.

3. W. Guo, *The theory of classes of groups* (Science Press-Kluwer Academic, New York, 2000).

4. W. Guo, On $\mathcal{F}$-supplemented subgroups of finite groups, *Manuscripta Math.* **127** (2008), 139–150.

5. W. Guo and A. N. Skiba, Finite groups with given s-embedded and n-embedded subgroups, *J. Algebra* **321** (2009), 2843–2860.

6. W. Guo, K. P. Shum and A. N. Skiba, Conditionally permutable subgroups and supersolubility of finite groups, *Southeast Asian Bull. Math.* **29**(3) (2005), 493–510.

7. X. Guo and K. P. Shum, On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups, *Arch. Math.* **80** (2003), 561–569.

8. X. Guo, X. Sun and K. P. Shum, On the solvability of certain c-supplemented finite groups, *Southeast Asian Bull. Math.* **28**(6) (2004), 1029–1040.

9. B. Huppert, *Endliche gruppen I* (Springer-Verlag, New York, 1983).

10. L. Miao and W. Lempken, On $\mathcal{M}$-supplemented subgroups of finite groups, *J. Group Theor.* **12**(2) (2009), 271–287.

11. L. Miao and W. Lempken, On weakly $\mathcal{M}$-supplemented primary subgroups of finite groups, *Turk. J. Math*. **34**(4) (2010), 489–500.

12. A. N. Skiba and O. V. Titov, Finite groups with c-quasinormal subgroups, *Sib. Math. J.* **48**(3) (2007), 544–554.

13. A. N. Skiba, On weakly s-permutable subgroups of finite groups, *J. Algebra* **315** (2007), 192–209.

14. S. Srinivasan, Two sufficient conditions for supersolvability of finite groups, *Israel J. Math.* **35**(3) (1980), 210–214.

15. J. G. Thompson, Normal p-complement for finite groups, *J. Algebra* **1** (1964), 43–46.

16. Y. Wang, c-normality of groups and its properties, *J. Algebra* **180** (1996), 954–965.

17. Y. Wang, H. Wei and Y. Li, A generalization of Kramer's theorem and its applications, *Bull. Aust. Math. Soc.* **65** (2002), 467–475.

18. M. Xu, *An introduction to finite groups* (Science Press, Beijing, 1999) (in Chinese).