ON SUPPORTS OF SEMIGROUPS OF MEASURES

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Let S denote a compact semitopological semigroup (i.e. the multiplication is separately continuous) and P(S) the set of probability measures on S. Then P(S) is a compact semitopological semigroup under convolution and the weak * topology (4). Let Γ be a subsemigroup of P(S) and $S(\Gamma) = \bigcup_{\mu \in \Gamma} \sup \mu$ where supp μ is the support of $\mu \in P(S)$. In the case in which S is commutative it was shown by Glicksberg in (4) that $S(\Gamma)$ is an algebraic group in S if Γ is an algebraic group. For a general semigroup S, Pym (7) considered $\Gamma = \{\eta\}$, η being an idempotent, and established that $S(\Gamma)$ is a topologically simple subsemigroup of S, i.e. every ideal of $S(\Gamma)$ is dense in $S(\Gamma)$. In this note we prove that if Γ is a simple subsemigroup of P(S) (a semigroup is simple if it contains no proper ideal) which contains an idempotent then $S(\Gamma)$ is a topologically simple subsemigroup of S. We also give an example to show that our conclusion (hence also Pym's) is best possible in the sense that $S(\Gamma)$ is not simple in general.

Next, if S is jointly continuous and supp $\Gamma = \overline{S}(\Gamma)$ (the bar denotes closure), we can then obtain that supp Γ must be simple if Γ is assumed simple but not necessarily containing an idempotent. In other words, (Theorem 4) for a jointly continuous semigroup S, the support of a simple subsemigroup of P(S) is a compact simple subsemigroup of S. This fills a lacuna in the literature. In fact it was first stated by Lin as a corollary of the following (5, Theorem 3): "If Σ is a closed subsemigroup of P(S), then the minimal ideal $K(\text{supp }\Sigma)$ of supp Σ is the support of the minimal ideal $K(\Sigma)$ of Σ ." Unfortunately the proof of this statement given in (5) contains a gap since the set

$$\{\mu \in \Sigma : \text{ supp } \mu \cap K(\text{supp } \Sigma) \neq \emptyset\}$$

is not shown to be non-empty although essential use is made of the fact that it is an ideal of Σ . Here we give a direct proof of the "corollary" and deduce the "theorem". (We note that Theorem 3 of (5) was used to derive several more results in (5), quoted in (8), and employed in the proof of Theorem 2 of (2).)

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Proposition 1. Suppose S is a compact semitopological semigroup. If Γ is a simple subsemigroup of P(S) and Γ contains an idempotent η , then $S(\Gamma)$ is a topologically simple subsemigroup of S.

Proof. Let I be an ideal of $S(\Gamma)$ and let $\Omega = \{\mu \in \Gamma : \text{supp } \mu \cap I \neq \emptyset\}$. It is easily seen that Ω is non-empty and an ideal of Γ by virtue of (4, Lemma 4.1). This gives $\Omega = \Gamma$, whence supp $\eta \cap I \neq \emptyset$. Since supp η is topologically simple, it follows that supp $\eta \subset \overline{I}$. Hence for any measure $\mu \in \Gamma = \Gamma \eta \Gamma$, we have supp $\mu = \overline{\sup \alpha \sup \eta \sup \overline{\eta} \sup \overline{\beta} \subset \overline{I}}$ for measures $\alpha, \beta \in \Gamma$. Consequently $S(\Gamma) \subset \overline{I}$, i.e. $S(\Gamma)$ is topologically simple.

Example 2. In the preceding proposition, $S(\Gamma)$ need not be simple. For instance, take the semigroup S given in (1, IV.7.1), i.e. $S = I \times I \times I$ with the usual topology, where I = [0, 1]. Consider the separately continuous function $f: I \times I \rightarrow I$ defined by $f(x, z) = 2xz/(x^2+z^2)$ for x, z not both zero, and f(0, 0) = 0. Then S endowed with the multiplication

$$(x, y, z)(x', y', z') = (x, f(x, z'), z')$$

is a compact semitopological semigroup. Its minimal ideal

$$K(S) = \{(x, f(x, z), z): x, z \in I\}$$

is not closed since $(0, 1, 0) \in \overline{K}(S) \setminus K(S)$. Now for any $a \in K(S)$ and $\mu \in P(S)$ we obtain the relation $\delta(a)\mu\delta(a) = \delta(a)$ ($\delta(a)$ denotes the unit point mass at a), since

$$\delta(a)\mu\delta(a)(f) = \int f(ata)d\mu(t) = \int f(a)d\mu(t) = f(a) = \delta(a)(f)$$

for all $f \in C(S)$. It follows that the measure $\eta = \mu \delta(a)\mu$ is idempotent. If $\Gamma = \{\eta\}$ and μ is a measure with S as its support (e.g. the restriction to S of the Lebesgue measure on \mathbb{R}^3), then $S(\Gamma) = \text{supp } \eta = \overline{SaS} = \overline{K}(S)$ which is not simple.

Example 3. With the hypotheses of Proposition 1, supp Γ may not be topologically simple. For instance, take the one-point compactification $S = \mathbb{R} \cup \{\infty\}$ of \mathbb{R} which is the additive group of real numbers with the operation extended by $x + \infty = \infty + x = \infty$ for $x \in S$. It follows that S is a compact semitopological semigroup (1, IV.1.1.1.1(a)). Let $\Gamma = \{\delta(x): x \in \mathbb{R}\}$. Then $S(\Gamma) = \mathbb{R}$ and so supp $\Gamma = S$ which is clearly not topologically simple.

In the remainder of this paper, let S be a compact jointly continuous semi-group. We have the next simple lemma which extends Lemma 1.3.11 of (6).

Lemma 4. Suppose A is a subsemigroup of S. Then A is topologically simple if and only if \overline{A} is simple.

Proof. Let \overline{A} be simple and take an ideal I of A. It is obvious that \overline{I} is an ideal of \overline{A} and so $\overline{I} = \overline{A} \supset A$, i.e. A is topologically simple. To show the converse, we model on the proof of (1, II.3.6). In fact for any $a \in A$ we see that $\overline{A}a\overline{A} \cap A \neq \emptyset$ is an ideal of A and so $\overline{A}a\overline{A} \supset A$. Whence $\overline{A}a\overline{A} = \overline{A}$. Now we check that $\overline{A}x\overline{A} = \overline{A}$ for $x \in \overline{A}$. Clearly there exists a net (x_a) in A such that $x_a \to x$ and $\overline{A}x_a\overline{A} = \overline{A}$. Let $b \in \overline{A}$ and we obtain two nets (c_a) and (d_a) in \overline{A}

such that $b = c_{\alpha}x_{\alpha}d_{\alpha}$. By passing to subnets if necessary, we have $c_{\alpha} \rightarrow c$ and $d_{\alpha} \rightarrow d$, giving $b = cxd \in \overline{A}x\overline{A}$. It follows that $\overline{A} \subset \overline{A}x\overline{A}$. Thus \overline{A} is simple.

- **Theorem 5.** (1) If Γ is a simple subsemigroup of P(S), then supp Γ is a compact simple subsemigroup of S.
 - (2) If Σ is a closed subsemigroup of P(S), then supp $K(\Sigma) = K(\text{supp }\Sigma)$.
- **Proof.** (1) Because supp $\overline{\Gamma} = \sup \Gamma$ (cf. (3), p. 55), we consider the compact semigroup $\overline{\Gamma}$ which contains an idempotent η (6, Theorem 1.1.10). Moreover, $\overline{\Gamma}$ is simple by Lemma 4. Then it follows from Proposition 1 that $S(\overline{\Gamma})$ is topologically simple and so supp $\overline{\Gamma}$ is simple by Lemma 4 again. That supp Γ is simple follows.
- (2) Clearly supp $K(\Sigma)$ is an ideal of the semigroup supp Σ and hence contains $K(\text{supp }\Sigma)$. The result is now immediate.

Remark. In the theorem above, it is clear that $S(\Gamma)$ is a topologically simple semigroup, but we have been unable to determine whether this semigroup is always simple.

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