Notes

107.28 Sums, and sums of squares

1. *Introduction*

Apart from a change in notation, the problem discussed in [1] is to solve the two real simultaneous equations

$$x + y + z = 3d, \tag{1}$$

$$x^2 + y^2 + z^2 = R^2, \tag{2}$$

where $R \geqslant 0$, and, in particular, to find the integer solutions of these equations. Now the author of [1] says that "it is not at all a pleasant exercise to find solutions (still less integer solutions) of this pair of equations", and this is certainly true if by 'finding solutions' we mean obtaining an explicit formula for the solutions. On the other hand, in any given case we can easily find all integer solutions by a numerical search on a computer. As is so often the case, there is a middle path between these two extremes, namely to use mathematics to obtain a greater understanding of the problem even if ultimately we use the computer to undertake a more refined search, and this is the path we take in this paper.

We shall work in $\mathbb{R}^3$, we let $\mathbf{0} = (0, 0, 0)$, and $\mathbf{x} = (x, y, z)$, and we shall use $\|\cdot\|$ for the distance in $\mathbb{R}^3$. As noted in [1], the set of solutions of the simultaneous equations (1) and (2) is the set $S \cap \Pi$, where S is the sphere given by $\|\mathbf{x} - \mathbf{0}\| = R$, and Π is the plane which passes through the point $\mathbf{d}$, where $\mathbf{d} = (d, d, d)$, and which has normal $(1, 1, 1)$. Since the point $\mathbf{d}$ has distance $d\sqrt{3}$ from $\mathbf{0}$, it is clear that Π and S are disjoint when $d\sqrt{3} > R$, and if this is so then there are no solutions. Alternatively, we can use the Cauchy-Schwarz inequality to show that if a solution (x, y, z) exists, then

$$(3d)^2 = (x + y + z)^2 \leqslant \left(1^2 + 1^2 + 1^2\right)\left(x^2 + y^2 + z^2\right) = 3R^2.$$

We may now assume that $3d^2 \leqslant R^2$ and, with this assumption, the set of solutions is a circle C with centre $\mathbf{d}$ and radius r, where $R^2 = 3d^2 + r^2$ (and $r \geqslant 0$); see Figure 1. We shall now obtain a geometric solution in the following way. We project the circle C vertically into the horizontal co-ordinate plane and, as the projection is an ellipse E, we can parametrize the points on E and then 'lift' the result back to obtain a parametrization of C.

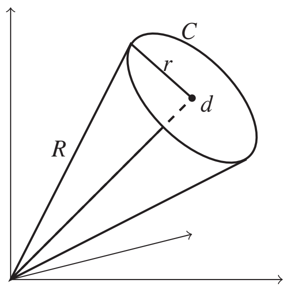

FIGURE 1: The circle C

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Every point on the plane Π has the form $\mathbf{d} + (a, b, -a - b)$, and this point lies on C if, and only if, $r^2 = \|(a, b, -a - b)\|^2$ or, equivalently

$$3(a + b)^2 + (a - b)^2 = 2r^2.$$

This implies that $(a + b, a - b)$ lies on an ellipse in $\mathbb{R}^2$, and if we use the familiar parametrisation of an ellipse we find that, for some real θ,

$$a + b = \sqrt{\frac{2}{3}}r\cos\theta, \qquad a - b = \sqrt{2}r\sin\theta.$$

This gives

$$a = \frac{r}{\sqrt{6}}\left(\cos\theta + \sqrt{3}\sin\theta\right), \qquad b = \frac{r}{\sqrt{6}}\left(\cos\theta - \sqrt{3}\sin\theta\right),$$

so the circle C is parametrized, for all real θ, by

$$x = d + \frac{r\cos\theta}{\sqrt{6}} + \frac{r\sin\theta}{\sqrt{2}}, \quad y = d + \frac{r\cos\theta}{\sqrt{6}} - \frac{r\sin\theta}{\sqrt{2}}, \quad z = d - \frac{2r\cos\theta}{\sqrt{6}}. \tag{3}$$

Now

$$r\cos\theta = \frac{\sqrt{6}(d - z)}{2}, \qquad r\sin\theta = \pm\tfrac{1}{2}\sqrt{4r^2 - 6(d - z)^2},$$

and if we substitute these values in (3) we obtain

$$x = \frac{3d - z + \varepsilon\sqrt{2r^2 - 3(d - z)^2}}{2}, \quad y = \frac{3d - z - \varepsilon\sqrt{2r^2 - 3(d - z)^2}}{2}. \tag{4}$$

where ε can be 1 or -1. The choice of ε here relates to the fact (which is discussed in [1]) that if (x, y, z) is any solution, then any triple obtained by permuting the entries in (x, y, z) is also a solution. This is obvious algebraically, and it follows from the geometry because Π and S are invariant under the reflection across each of the planes $x = y$, $y = z$ and $x = z$.

We give one more geometric fact which is of interest. The circle C lies in the closed first octant if and only if the points $(3d, 0, 0)$, $(0, 3d, 0)$ and $(0, 0, 3d)$ lie on, or are outside, C. Thus if $\|(3d, 0, 0) - \mathbf{d}\|^2 \geqslant r^2$, equivalently $3|d| \geqslant R$, then any solution to the problem will have non-negative entries. If $3|d| < R$ then some entries may be negative.

2. *Integer solutions*

We shall now comment on the search for integer solutions of (1) and (2). First, we note that (3) implies that if (x, y, z) is a solution, then $|z| \leqslant |d| + 2r/\sqrt{6}$. Thus if z is an integer, then $z \in \{-M, \ldots, 0, \ldots, M\}$, where M is the integer part of $|d| + 2r/\sqrt{6}$. If we let z range over this set, then the corresponding values of x and y are given by (4), and it is a simple matter of excluding those cases in which x or y is not an integer. We end with two examples to illustrate these ideas.

Example 2.1: In this example, we start the integer triple (0, 21, 21) and find all integer solutions on the circle C that passes through this point (this example is discussed in [1]). This means that we are looking for the integer solutions of

$$x + y + z = 42, \qquad x^2 + y^2 + z^2 = 882, \tag{5}$$

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and, using the notation introduced above, we have $d = 14$, $R^2 = 882$, $r = 7\sqrt{6}$ and $M = 28$. Now (4) shows that for any solution (x, y, z) of (5), we have $z = 14 - 14\cos\theta$ so that $|z| \leqslant 28$. However, as $9d^2 > R^2$, all entries of all solutions will be non-negative, so that $0 \leqslant z \leqslant 28$. It follows that if (x, y, z) is an integer solution to (5) then $z \in \{0, \ldots, 28\}$,

$$x = \frac{42 - z \pm \sqrt{3z(28 - z)}}{2}, \qquad y = 42 - x - z.$$

If we now compute the values of (x, y, z) for $z = 0, 1, \ldots, 28$, we find that the only integer solutions of (5) are (0, 21, 21), (7, 7, 28), (3, 12, 27), (1, 16, 25) and all triples obtained by permuting the entries.

Example 2.2: The simultaneous equations

$$x + y + z = -1, \qquad x^2 + y^2 + z^2 = 21 \tag{6}$$

provide an example in which some of the entries are negative. The discussion above shows that if (x, y, z) is an integer solution of these equations, then $z \in \{-4, \ldots, 4\}$ and, in addition,

$$x = \frac{-(1 + z) + \varepsilon\sqrt{41 - 2z - 3z^2}}{2}.$$

A computation now shows that the only integer solutions to (6) are $(1, 2, -4)$ and all triples obtained by permuting the entries.

References

1. G. J. O. Jameson, Equal sums, sums of squares and sums of cubes, *Math. Gaz.* **106** (565) (March 2022) pp. 54-60.

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107.29 A cautionary example relating to the interpretation of numerical computations

1. *An example case*

The availability of software tools [1, 2, 3, 4, 5, 6] and multi-function calculators has unquestionably had an impact on how students interpret numerical solutions to mathematical problems. However, these tools can also lead to an overly-casual attitude about how to interpret the effects of numerical precision. For example, students quickly learn to interpret a numerical result of 0.99999999 as being exactly 1, or 3.54e-16 as being zero. The problem, of course, is that they may become so habituated to disregarding low-order terms that they fail to recognise results that are *nearly* an integer – *but are not*. The following trig expression (in radians) provides a cautionary example:

$$\arcsin(1 + \sin(11)) - \sin(11). \tag{1}$$