# 7
# Examples and applications

To expose the value of the method in the foregoing chapters, it is instructive to apply it to a number of important and well known physical problems. Through these examples we shall see how a unified methodology makes the solution of a great many disparate systems essentially routine. The uniform approach does not necessarily convey with it any automatic physical understanding, but then no approach does. What we learn from this section is how many problems can be reduced to the basics of 'cause followed by effect', or, here, 'source followed by field'.

## 7.1 Free particles

Solving Newton's law $F = ma$ using a Green function approach is hardly to be recommended for any practical purpose; in fact, it is a very inefficient way of solving the problem. However, it is useful to demonstrate how the Green function method can be used to generate the solution to this problem. This simple test of the theory helps to familiarize us with the working of the method in practice. The action for a one-dimensional-particle system is

$$S = \int \mathrm{d}t \,\left\{ -\frac{1}{2} m \dot{x}^2 - F x \right\}. \tag{7.1}$$

The variation of the action leads to

$$\delta S = \int \mathrm{d}t \,\{ m\ddot{x} - F \}\, \delta x + \Delta(m\dot{x})\delta x = 0, \tag{7.2}$$

which gives us the equation of motion

$$F = m\ddot{x} \tag{7.3}$$

and the continuity condition

$$\Delta(m\dot{x}) = 0, \tag{7.4}$$



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which is the conservation of momentum. The equation of motion can be written in the form of 'operator acting on field equals source',

$$\mathcal{D}x = J, \tag{7.5}$$

by rearranging

$$\partial_t^2 \, x(t) = F/m. \tag{7.6}$$

Clearly, we can integrate this equation directly with a proper initial condition $x(t_0) = x_0$, $\dot{x}(t_0) = v$, to give

$$x(t) - x_0 = \frac{F}{2m}(t - t_0) + v(t - t_0). \tag{7.7}$$

But let us instead try to use the Green function method to solve the problem. There are two ways to do this: the first is quite pointless and indicates a limitation of the Green function approach, mentioned in section 5.2.4. The second approach demonstrates a way around the limitation and allows us to see the causality principle at work.

*Method 1* The operator on the left hand side of eqn. (7.6) is $\partial_t^2$, so we define a Green function

$$\partial_t^2 \, G(t, t') = \delta(t, t'). \tag{7.8}$$

As usual, we expect to find an integral expression by Fourier transforming the above equation:

$$G(t - t') = \int \frac{\mathrm{d}\omega}{2\pi} \, \frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{-\omega^2}. \tag{7.9}$$

This expression presents us with a problem, however: it has a non-simple pole, which must be eliminated somehow. One thing we can do is to re-write the integral as follows:

$$\begin{aligned} G(t - t') &= \int \mathrm{d}\tilde{t} \int \mathrm{d}\tilde{t} \int \frac{\mathrm{d}\omega}{2\pi} \, \mathrm{e}^{-\mathrm{i}\omega\tilde{t}}, \\ &= \int \mathrm{d}\tilde{t} \int \mathrm{d}\tilde{t} \; \delta(\tilde{t}), \end{aligned} \tag{7.10}$$

where $\tilde{t} = t - t'$. It should be immediately clear that this is just telling us to replace the Green function with a double integration (which is how one would

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normally solve the equation). We obtain two extra, unspecified integrals:

$$\begin{aligned} x(t) &= \int \mathrm{d}t'\ G(t,t')F/m \\ &= \int \mathrm{d}\tilde{t}\mathrm{d}\tilde{t}\mathrm{d}t'\ \delta(t-t')F/m \\ &= \int \mathrm{d}\tilde{t}\mathrm{d}\tilde{t}\ F/m \\ &= \int \mathrm{d}\tilde{t}\left[F/m(t-t')+v\right] \\ &= \frac{F}{2m}(t-t_0)^2 + v(t-t_0) + x_0. \end{aligned} \tag{7.11}$$

So, the result is the same as that obtained by direct integration and for the same reason: the Green function method merely adds one extra (unnecessary) integration and re-directs us to integrate the equation directly. The problem here was that the denominator contained a non-simple pole. We can get around this difficulty by integrating it in two steps.

*Method 2* Suppose we define a Green function for the linear differential operator

$$\partial_t\ g(t,t') = \delta(t,t'). \tag{7.12}$$

From section A.2, in Appendix A, we immediately recognize this function as the Heaviside step function. (We could take the Fourier transform, but this would only lead to an integral representation of the step function.) The solution has advanced and retarded forms

$$\begin{aligned} g_{\mathrm{r}}(t,t') &= \theta(t-t') \\ g_{\mathrm{a}}(t,t') &= -\theta(t'-t). \end{aligned} \tag{7.13}$$

Now we have an integrable function, which allows us to solve the equation in two steps:

$$\begin{aligned} \partial_t\, x(t) &= \int \mathrm{d}t'\ g_{\mathrm{r}}(t,t')\ F/m \\ &= \frac{F}{m}(t-t') + \partial_t\, x(t') \qquad\qquad (t > t'). \end{aligned} \tag{7.14}$$

Then, applying the Green function again,

$$\begin{aligned} x(t) &= \int \mathrm{d}t'\ g_{\mathrm{r}}(t-t')\left[\frac{F}{m}(t-t') + \partial_t\, x(t')\right] \\ &= \frac{F}{2m}(t-t_0)^2 + v(t-t_0) + x_0. \end{aligned} \tag{7.15}$$

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Again we obtain the usual solution, but this time we see explicitly the causality inferred by a linear derivative. The step function tells us that the solution only exists for a causal relationship between force $F$ and response $x(t)$.

### 7.1.1 Velocity distributions

In a field of many particles, there is usually a distribution of velocities or momenta within the field. In a particle field this refers to the momenta $p_i$ of individual localizable particles. In other kinds of field there is a corresponding distribution of wavenumbers $k_i$ of the wave modes which make up the field. The action describes the dynamics of a generic particle, but it does not capture the macroscopic state of the field. The macrostate is usually described in terms of the numbers of components (particles or modes) with a given momentum or energy (the vector nature of momentum is not important in an isotropic plasma).

The distribution function $f$ is defined so that its integral with respect to the distribution parameter gives the number density or particles per unit volume. We use a subscript to denote the control variable:

$$\begin{aligned} N &= \int \mathrm{d}^n\mathbf{k}\; f_k(\mathbf{k}) \\ &= \int \mathrm{d}^n\mathbf{p}\; f_p(\mathbf{p}) \\ &= \int \mathrm{d}^n\mathbf{v}\; f_v(\mathbf{v}). \end{aligned} \tag{7.16}$$

This distribution expresses averages of the field. For example, the average energy is the weighted average of the energies of the different momenta:

$$\langle E \rangle = \frac{1}{N} \int \mathrm{d}^n\mathbf{k}\, f_k(\mathbf{k}) E(k). \tag{7.17}$$

## 7.2 Fields of bound particles

A field of particles, sometimes called a plasma when charged, is an *effective* field, formed from the continuum approximation of discrete particles. Its purpose is to capture some of the bulk dynamics of material systems; it should not be confused with the deeper description of the atoms and their sub-atomic components in terms of fundamental fields which might focus on quite different properties, not relevant for the atoms in a bulk context. The starting point for classical analyses of atomic systems coupled to an electromagnetic field is the idea that matter consists of billiard-ball atoms with some number density $\rho_N$, and that the wavelength of radiation is long enough to be insensitive to the particle nature of the atoms. The only important fact is that there are many particles whose combined effect in space is to act like a smooth field. When

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perturbed by radiation (which we shall represent as an abstract source $J_i$) the particles are displaced by a spatial vector $s^i$ where $i = 1, 2, \ldots, n$. The action for this system may be written

$$S_{\text{eff}} = \frac{1}{\sigma_x} \int (\mathrm{d}x) \left\{ -\frac{1}{2} m \dot{s}^2 + \frac{1}{2} \kappa s^2 - m\gamma s \dot{s} - J^i s_i \right\}. \tag{7.18}$$

This requires some explanation. The factor of the spatial volume of the total system, $\sigma_x$, reflects the fact that this is an effective average formulation. Dividing by a total scale always indicates an averaging procedure. As an alternative to using this explicit value, we could use the average density, $\rho = m/\sigma_x$, and other parameter densities to express the action in appropriate dimensions. The first term is a kinetic energy term, which will describe the acceleration of particles in response to the forcing term $J^i$. The second term is a harmonic oscillator term, which assumes that the particles are bound to a fixed position $s_i = 0$, just as electrons are bound to atoms or ions are bound in a lattice. The effective spring constant of the harmonic interaction is $\kappa$. Because $s^i(x)$ represents the displacement of the particles from their equilibrium position, we use the symbol $s^i$ rather than $x^i$, since it is not the position which is important, but the deviation from equilibrium position. The dimensions of $s^i(x)$ are position divided by the square-root of the density because of the volume integral in the action, and $s^i(x)$ is a function of $x^\mu$ because the displacement could vary from place to place and from time to time in the system. The final term in eqn. (7.18) is a term which will provide a phenomenological damping term for oscillations, as though the system were leaky or had friction. As we have already discussed in section 4.2, this kind of term is not well posed unless there is some kind of boundary in the system which can leak energy. The term is actually a total derivative. Nevertheless, since this is not a microscopic fundamental theory, it is possible to make sense of this as an effective theory by 'fiddling' with the action. This actually forces us to confront the reason why such terms cannot exist in fundamental theories, and is justifiable so long as we are clear about the meaning of the procedure.

The variation of the action is given, after partial integration, by

$$\begin{aligned} \delta S = &\frac{1}{\sigma_x} \int (\mathrm{d}x) \left\{ m\ddot{s}_i + \kappa s_i - m\gamma \dot{s}_i + m\gamma \dot{s}_i - J_i \right\} \delta s^i \\ &+ \frac{1}{\sigma_x} \int \mathrm{d}\sigma \left[ m\dot{s}_i + m\gamma s_i \right] \delta s^i. \end{aligned} \tag{7.19}$$

The terms containing $\gamma$ clearly cancel, leaving only a surface term. But suppose we divide the source into two parts:

$$J^i = J^i_\gamma + J^i_s, \tag{7.20}$$

where $J^i_\gamma$ is postulated to satisfy the equation

$$-m\gamma \dot{s}^i = J^i_\gamma. \tag{7.21}$$

This then has the effect of preventing the frictional terms from completely disappearing. Clearly this is a fiddle, since we could have simply introduced a source in the first place, with a velocity-dependent nature. However, this is precisely the point. If we introduce a source or sink for the energy of the system, then it is possible to violate the conservational properties of the action by claiming some behaviour for $J^i$ which is not actually determined by the action principle. The lesson is this: *if we specify the behaviour of a field rather than deriving it from the action principle, we break the closure of the system and conservation laws.* What this tells us is that dissipation in a system has to come from an external agent; it does not arise from a closed mechanical theory, and hence this description of dissipation is purely phenomenological. Taking eqn. (7.21) as given, we have the equation of motion for the particles

$$m\ddot{s}^i - m\gamma\dot{s}^i + \kappa s = J_s^i, \tag{7.22}$$

with continuity condition

$$\Delta\,(m\dot{s} + m\gamma s) = 0. \tag{7.23}$$

It is usual to define the natural frequency $\omega_0^2 = \kappa/m$ and write

$$(\partial_t^2 - \gamma\partial_t + \omega_0^2)s^i(x) = \frac{J_s^i}{m}. \tag{7.24}$$

If we consider a plane wave solution of the form

$$s(x) = \int (\mathrm{d}k)\ \mathrm{e}^{\mathrm{i}(k_i x^i - \omega t)} s(k), \tag{7.25}$$

then we may write

$$(-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2)s^i(k) = \frac{J_s^i(k)}{m}. \tag{7.26}$$

From this we see that the Green function $G_{ij}(x, x')$ for $s^i(x)$ is

$$G_{ij}(x, x') = \delta_{ij}\int (\mathrm{d}k)\ \frac{\mathrm{e}^{\mathrm{i}(k_i x^i - \omega t)}}{(-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2)}. \tag{7.27}$$

As long as the integral contains both positive and negative frequencies, this function is real and satisfies retarded boundary conditions. It is often referred to as the susceptibility, $\chi_{ij}$. In a quantum mechanical treatment, $\hbar\omega_0 = E_2 - E_1$ is the difference between two energy levels.

Notice that the energy density

$$P^i E_i = \int E_i(x) G_{ij}(x, x') E_j(x')\ (\mathrm{d}x') \tag{7.28}$$

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cannot be expressed in terms of a retarded Green function, since the above expression requires a spacetime symmetrical Green function. The Feynman Green function is then required. This indicates that the energy of the field is associated with a statistical balance of virtual processes of emission and absorption, rather than simply being a process of emission. In general, the interaction with matter introduces an imaginary part into the expression for the energy, since the Green function idealizes the statistical processes by treating them as steady state, with no back-reaction. It thus implicitly assumes the existence of an external source whose behaviour is unaffected by the response of our system. The energy density reduces to $E^2$ in the absence of material interactions and the result is then purely real.

## 7.3 Interaction between matter and radiation

Classical field theory is normally only good enough to describe non-interacting field theories. A complete description of interactions requires the quantum theory. The exception to this rule is the case of an external source. In electromagnetism we are fortunate in having a system in which the coupling between matter and radiation takes on the form of a linear external source $J^\mu$, so there are many systems which behave in an essentially classical manner.

### *7.3.1 Maxwell's equations*

The interaction between matter and radiation begins with the relativistically invariant Maxwell action

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu \right\}. \tag{7.29}$$

The variation of the action,

$$\begin{aligned} \delta S &= \int (\mathrm{d}x) \left\{ (\partial^\mu \delta A^\nu) F_{\mu\nu} - J^\mu \delta A_\mu \right\} \\ &= \int (\mathrm{d}x) \left\{ \delta A^\nu (-\partial^\mu F_{\mu\nu}) - J^\mu \delta A_\mu \right\} + \int \mathrm{d}\sigma^\mu \left\{ \delta A^\nu F_{\mu\nu} \right\} \\ &= 0, \end{aligned} \tag{7.30}$$

leads immediately to the field equations for the electromagnetic field interacting with charges in an ambient vacuum:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu. \tag{7.31}$$

The spatial continuity conditions are

$$\Delta F_{i\mu} = 0, \tag{7.32}$$

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or

$$\Delta E_i = 0$$
$$\Delta B_i = 0. \tag{7.33}$$

### 7.3.2 Electromagnetic waves

In the Lorentz gauge, $\partial^\mu A_\mu = 0$, Maxwell's equations (7.31) reduce to

$$-\Box\, A_\mu = J_\mu. \tag{7.34}$$

The solution to this equation is a linear combination of a particular integral with non-zero $J_\mu$ and a complementary function with $J_\mu = 0$. The free-field equation,

$$-\Box\, A_\mu = 0, \tag{7.35}$$

is solved straightforwardly by taking the Fourier transform:

$$A_\mu(x) = \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\, \mathrm{e}^{\mathrm{i}k_\mu x^\mu} A_\mu(k). \tag{7.36}$$

Substituting into the field equation, we obtain the constraint

$$\chi(k) = k^2 = k^\mu k_\mu = \left(-\frac{\omega^2}{c^2} + k^i k_i\right) = 0. \tag{7.37}$$

This is the result we found in eqn. (2.52), obtained only slightly differently. The retarded and Feynman Green functions for the field clearly satisfy

$$-\Box\, D_{\mu\nu}(x, x') = g_{\mu\nu}\, c\delta(x, x'). \tag{7.38}$$

Thus, the solution to the field in the presence of the source is, by analogy with eqn. (5.41),

$$\begin{aligned} A_\mu(x) &= \int (\mathrm{d}x') D_{\mu\nu}(x, x') J^\nu(x') \\ &= \int (\mathrm{d}k)\, \mathrm{e}^{\mathrm{i}k_\mu (x-x')^\mu} \left[\frac{1}{k^2} + X(k)\delta(k^2)\right] J(x'), \end{aligned} \tag{7.39}$$

where $X(k)$ is an arbitrary and undetermined function. In order to determine this function, we need to make some additional assumptions and impose some additional constraints on the system.

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### 7.3.3 Dispersion and the Faraday effect

When linearly polarized electromagnetic waves propagate through a magnetized medium, in the direction of the applied magnetization, the plane of polarization becomes rotated in an anti-clockwise sense about the axis of propagation by an amount proportional to $z$, where $z$ is the distance travelled through the medium. The angle of rotation

$$\psi = VBz, \tag{7.40}$$

where $B$ is the magnetic field and $V$ is Verdet's constant, the value of which depends upon the dielectric properties of the material. This phenomenon is important in astronomy in connection with the polarization of light from distant stars. It is also related to optical activity and the Zeeman effect.

Classical descriptions of this effect usually involve a decomposition of the electric field vector into two contra-rotating vectors which are then shown to rotate with different angular velocities. The sum of these two vectors represents the rotation of the polarization plane. An alternative description can be formulated in complex coordinates to produce the same result more quickly and without prior assumptions about the system.

Let us now combine some of the above themes in order to use the action method to solve the Faraday system. Suppose we have a particle field, $s^i(x)$, of atoms with number density $\rho_N$, which measures the displacement of optically active electrons $-e$ from their equilibrium positions, and a magnetic field $B = B_3$, which points along the direction of motion for the radiation. In the simplest approximation, we can represent the electrons as being charges on springs with spring constant $\kappa$. As they move, they generate an electric current density

$$J_i = -e\rho_N \dot{s}_i. \tag{7.41}$$

Since the Faraday effect is about the rotation of radiation's polarization vector (which is always perpendicular to the direction of motion $x_3$), we need only $s^i$ for $i = 1, 2$. The action then can be written

$$S = \frac{1}{\sigma_x}\int (\mathrm{d}x)\left\{-\frac{1}{2}m(\partial_t s)(\partial_t s) + \frac{1}{2}eB\epsilon_{ij}s^i(\partial_t s) + ks^i s_i - J^i s_i\right\}. \tag{7.42}$$

Here, $J^i$ is an external source which we identify with the radiation field

$$J^i(x) = -eE^i(x) = -\frac{e}{c}F^{0i}(x). \tag{7.43}$$

As is often the case with matter–radiation interactions, the relativistically invariant electromagnetic field is split into $E^i, B^i$ by the non-relativistically

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invariant matter field $s^i$. The field equations are now obtained by varying the action with respect to $\delta s^i$:

$$\delta S = \int (\mathrm{d}x) \left\{ m\ddot{s}_i + eB\epsilon_{ij}\dot{s}^j + \kappa s_i - J_i \right\} \delta s^i + \int \mathrm{d}\sigma \left[ m\dot{s}_i + eB\epsilon_{ij}s^j \right] \delta s^i. \tag{7.44}$$

Thus, the field equations are

$$m\ddot{s}_i + eB\epsilon_{ij}\dot{s}^j + \kappa s_i = J_i = -eE_i, \tag{7.45}$$

and continuity of the field requires

$$\begin{aligned} \Delta(m\dot{s}_i) &= 0 \\ \Delta(eB\epsilon_{ij}s^j) &= 0. \end{aligned} \tag{7.46}$$

The first of these is simply the conservation of momentum for the electrons. The latter tells us that any sudden jumps in the magnitude of magnetic field must be compensated for by a sudden jump in the amplitude of the transverse displacement.

If we compare the action and the field equations with the example in section 7.2, it appears as though the magnetic field has the form of a dissipative term. In fact this is not the case. Magnetic fields do no work on particles. The crucial point is the presence of the anti-symmetric matrix $\epsilon_{ij}$ which makes the term well defined and non-zero.

Dividing eqn. (7.45) through by the mass, we can defined the Green function for the $s^i(x)$ field:

$$\left[ \left( \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega_0^2 \right) \delta_{ij} + \frac{eB}{m}\epsilon_{ij} \right] G_{jk}(x, x') = \delta_{ik}(x, x'), \tag{7.47}$$

where $\omega_0^2 = \kappa/m$, so that the formal solution for the field is

$$s_i(x) = \int (\mathrm{d}x') G_{ij}(x, x') J^j(x'). \tag{7.48}$$

Since we are interested in coupling this equation to an equation for the radiation field $J_i$, we can go no further. Instead, we turn to the equation of motion (7.34) for the radiation. Because of the gauge freedom, we may use a gauge in which $A_0 = 0$, this simplifies the equation to

$$\begin{aligned} -\Box\, A_i &= \mu_0 J_i^e \\ E_i &= -\partial_t A_i. \end{aligned} \tag{7.49}$$

Thus, using the Green function $D_{ij}(x, x')$,

$$-\Box\, D_{ij}(x, x') = \delta_{ij}\ c\delta(x, x'), \tag{7.50}$$

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for $A_i(x)$, we may write the solution for the electric field formally as

$$E_i(x) = -\mu_0 \partial_t \int (\mathrm{d}x') D_{ij}(x, x')(-e\rho_N \dot{s}_j(x')) = -J_i/e. \tag{7.51}$$

This result can now be used in eqn. (7.45), giving

$$\begin{aligned}\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega_0^2\right)\delta_{ij} + \frac{eB\omega}{m}\epsilon_{ij}\right] s_{\mathrm{j}}(x) = \\ -\frac{e^2}{m}\rho_N \mu_0\, \partial_t \int (\mathrm{d}x') D_{jk}(x, x')\dot{s}^k.\end{aligned} \tag{7.52}$$

Operating from the left with $-\overset{x}{\Box}$, we have

$$(-\Box)\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \omega_0^2\right)\delta_{ij} + \frac{eB\omega}{m}\epsilon_{ij}\right] s_{\mathrm{j}}(x) = -\frac{e^2}{m}\rho_N \mu_0 \ddot{s}_i. \tag{7.53}$$

This is a matrix equation, with a symmetric part proportional to $\delta_{ij}$ and an anti-symmetric part proportional to $\epsilon_{ij}$. If we take plane wave solutions moving along the $x_3 = z$ axis,

$$\begin{aligned} s^i(x) &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\, \mathrm{e}^{\mathrm{i}(k_z z - \omega t)} s^i(k)\delta(\chi) \\ E^i(x) &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\, \mathrm{e}^{\mathrm{i}(k_z z - \omega t)} E^i(k)\delta(\chi), \end{aligned} \tag{7.54}$$

for the dispersion relation $\chi$ implied by eqn. (7.53), eqn. (7.53) implies that the wavenumber $k_z$ must be a matrix in order to find a solution. This is what will lead to the rotation of the polarization plane for $E_i$. Substituting the above form for $s^i(x)$ into eqn. (7.53) leads to the replacements $\partial_z \to \mathrm{i}k_z$ and $\partial_t \to -\mathrm{i}\omega$. Thus the dispersion relation is

$$\chi = \left(k_z^2 - \frac{\omega^2}{c^2}\right)\left[(-\omega^2 + \omega_0^2)\delta_{ij} + \frac{eB\omega}{m}\epsilon_{ij}\right] - \frac{e^2}{m}\rho_N \mu_0 \omega^2\, \delta_{ij} = 0, \tag{7.55}$$

or re-arranging,

$$k_{z\,ij}^2 = \frac{\omega^2}{c^2}\left[\delta_{ij} + \frac{\frac{e^2}{m\epsilon_0}\rho_N\left[(\omega^2 + \omega_0^2)\delta_{ij} + \frac{eB\omega}{m}\epsilon_{ij}\right]}{(-\omega^2 + \omega_0^2)^2 - (\frac{eB\omega}{m})^2}\right]. \tag{7.56}$$

This only makes sense if the wavenumber $k_z$ is itself a matrix with a symmetric and anti-symmetric part:

$$k_{z\,ij} = \overline{k}\delta_{ij} + \tilde{k}\epsilon_{ij}. \tag{7.57}$$

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It is the anti-symmetric part which leads to a rotation of the plane of polarization. In fact, $k_z$ has split into a generator of linear translation $\bar{k}$ plus a generator or rotations $\tilde{k}$ about the $z$ axis:

$$k_z = \bar{k} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tilde{k} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{7.58}$$

The exponential of the second term is

$$\begin{pmatrix} \cos(\tilde{k}z) & \sin(\tilde{k}z) \\ -\sin(\tilde{k}z) & \cos(\tilde{k}z) \end{pmatrix}, \tag{7.59}$$

so $\tilde{k}$ is the rate of rotation. Using a binomial approximation for small $B$, we can write simply

$$\tilde{k}_{z\,ij} = \frac{\frac{e^3 B}{2m\epsilon_0}\rho_N}{(-\omega^2 + \omega_0^2)^2 - (\frac{eB\omega}{m})^2}. \tag{7.60}$$

Verdet's constant is defined by the phenomenological relation,

$$\tilde{k}z = VBz, \tag{7.61}$$

so we have

$$V = \frac{Ne^3\omega^2}{2m^2 c\epsilon_0 |(\omega_o^2 - \omega^2)^2 - (eB\omega/m)^2|}. \tag{7.62}$$

### *7.3.4 Radiation from moving charges in $n = 3$: retardation*

The derivation of the electromagnetic field emanating from a charged particle in motion is one of the classic topics of electrodynamics. It is an important demonstration of the Green function method for two reasons. First of all, the method of Green functions leads quickly to the answer using only straightforward algebraic steps. Prior to the Green function method, geometrical analyses were carried out with great difficulty. The second reason for looking at this example here is that it brings to bear the causal or retarded nature of the physical field, i.e. the property that the field can only be generated by charge disturbances in the past. This retardation property quickly leads to algebraic pitfalls, since the dynamical variables become defined recursively in term of their own motion in a strange-loop. Unravelling these loops demonstrates important lessons.

We begin by choosing the Lorentz gauge for the photon propagator with $\alpha = 1$. This choice will give the result for the vector potential in a form which is most commonly stated in the literature. Our aim is to compute the vector potential $A_\mu(x)$, and thence the field strength $F_{\mu\nu}$, for a particle at position $\mathbf{x}_p(t)$ which is in motion with speed $\mathbf{v} = \partial_t \mathbf{x}_p(t)$. The current distribution for a point particle is singular, and may be written

$$J^\mu = qc\beta^\mu\, \delta^n(\mathbf{x} - \mathbf{x}_p(t)). \tag{7.63}$$

The vector potential is therefore, in terms of the retarded propagator,

$$\begin{aligned} A_\mu(x) &= \mu_0 \int (\mathrm{d}x')\; G_\mathrm{r}(x, x') J_\mu(x') \\ &= \frac{q}{4\pi\epsilon_0 c} \int (\mathrm{d}x')\; \beta^\mu(t') \frac{\delta\left(c(t' - t_\mathrm{ret})\right)}{|\mathbf{x} - \mathbf{x}'|} \delta\left(\mathbf{x}' - \mathbf{x}_p(t)\right), \end{aligned} \tag{7.64}$$

where the retarded time is defined by $t_\mathrm{ret} = t - |\mathbf{x} - \mathbf{x}'|/c$. Performing the integral over $x^{0'}$ in the presence of the delta function sets $t' \to t_\mathrm{ret}$:

$$A_\mu(x) = \frac{q}{4\pi\epsilon_0 c} \int \mathrm{d}\sigma_{x'} \frac{\beta_\mu(t_\mathrm{ret})\delta\left(\mathbf{x}' - \mathbf{x}_p(t_\mathrm{ret})\right)}{|\mathbf{x} - \mathbf{x}'|}. \tag{7.65}$$

Here $\mathbf{x}$ is a free continuous coordinate parameter, which varies over all space around the charge, and $\mathbf{x}_p(t_\mathrm{ret})$ is the retarded trajectory of the charge $q$. We may now perform the remaining integral. Here it is convenient to change variables. Let

$$\int \mathrm{d}\sigma_{x'} \delta(\mathbf{x}' - \mathbf{x}_p(t_\mathrm{ret})) = \int \mathrm{d}\sigma_\mathrm{r} \delta(\mathbf{r}) |J|, \tag{7.66}$$

where $J = \det J_{ij}$ and

$$\begin{aligned} J_{ij}^{-1} &= \partial_i' r_j = \partial_i'(x' - x_p(t_\mathrm{ret}))_j \\ &= g_{ij} - \frac{\partial x_p^i}{\partial t_\mathrm{ret}} \frac{\partial t_\mathrm{ret}}{\partial x^{i'}}, \end{aligned} \tag{7.67}$$

is the Jacobian of the transformation. At this stage, $t_\mathrm{ret}$ is given by $t_\mathrm{ret} = t - |\mathbf{x} - \mathbf{x}'|/c$, i.e. it does not depend implicitly on itself. After the integration we are about to perform, it will. We complete the integration by evaluating the Jacobian:

$$\begin{aligned} \partial_i' t_\mathrm{ret} &= \frac{\hat{r}_i}{c} \\ J_{ij}^{-1} &= g_{ij} - \frac{v_j}{c} \hat{r}_i \\ \det J_{ij}^{-1} &= (1 - \beta^i \hat{r}_i)\Big|_{t_\mathrm{ret}}. \end{aligned} \tag{7.68}$$

The last line uses the fact that $r_i$ depends only on $x_i'$, not on $x_j$ for $i \neq j$. In this

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instance, the determinant becomes $1 + \mathrm{Tr}(J_{ij}^{-1})$, giving

$$
\begin{aligned}
A_\mu(x) &= \frac{q}{4\pi\epsilon_0 c}\int \mathrm{d}\sigma_\mathrm{r}\frac{\beta^\mu(t_\mathrm{ret})\delta(\mathbf{r})}{|\mathbf{x}-\mathbf{x}_p(t_\mathrm{ret})-\mathbf{r}|}\\
&= \frac{q\beta_\mu(t_\mathrm{ret})}{4\pi\epsilon_0 c\kappa|\mathbf{x}-\mathbf{x}_p|},
\end{aligned}
\tag{7.69}
$$

where $\kappa \equiv (1-\beta\cdot\hat{\mathbf{r}})$, and all quantities (including $\kappa$ itself) are evaluated at $t_\mathrm{ret}$. If we define the light ray $r^\mu$ as the vector from $\mathbf{x}_p$ to $\mathbf{x}$, then $r^\mu = (r, \mathbf{r})$ and $r = |\mathbf{r}|$, since, for a ray of light, $r = c\Delta t = c\times r/c$. Finally, noting that $r\kappa = -r^\mu\beta_\mu$, we have the Liénard–Wiechert potential in the Lorentz gauge,

$$
A_\mu(x) = \frac{-q}{4\pi\epsilon_0 c}\left(\frac{\beta_\mu}{r^\mu\beta_\mu}\right)_{t_\mathrm{ret}}.
\tag{7.70}
$$

To proceed with the evaluation of the field strength $F_{\mu\nu}$, or equivalently the electric and magnetic fields, it is useful to derive a number of relations which conceal subtleties associated with the fact that the retarded time now depends on the position evaluated at the retarded time. In other words, the retarded time $t_\mathrm{ret}$ satisfies an implicit equation

$$
t_\mathrm{ret} = t - \frac{|\mathbf{x}-\mathbf{x}_p(t_\mathrm{ret})|}{c} = t - \frac{r}{c}.
\tag{7.71}
$$

The derivation of these relations is the only complication to this otherwise purely algebraic procedure. Differentiating eqn. (7.71) with respect to $t_\mathrm{ret}$, we obtain

$$
\begin{aligned}
1 &= \frac{\partial t}{\partial t_\mathrm{ret}} + \hat{r}^i\beta_i(t_\mathrm{ret})\\
\partial_t(t_\mathrm{ret}) &= \kappa^{-1}\Big|_{t_\mathrm{ret}}.
\end{aligned}
\tag{7.72}
$$

Moreover,

$$
(\partial_i t_\mathrm{ret}) = -\frac{1}{c}(\partial_i r),
\tag{7.73}
$$

$$
\begin{aligned}
(\partial_i r) &= \partial_i\sqrt{r^j r_j}\\
&= \hat{r}^j(\partial_i r_j),
\end{aligned}
\tag{7.74}
$$

$$
\begin{aligned}
(\partial_i r_j) &= g_{ij} - \frac{\partial x_{pj}}{\partial t_\mathrm{ret}}(\partial_i t_\mathrm{ret})\\
&= g_{ij} + \beta_j(\partial_i r)
\end{aligned}
\tag{7.75}
$$

$$
(\partial_i r_j) = g_{ij} + \beta_j\hat{r}^k(\partial_i r_k).
\tag{7.76}
$$

The last line cannot be simplified further; however, on substituting eqn. (7.76) into eqn. (7.74), it is straightforward to show that

$$
(\partial_i r)(1-(\hat{\mathbf{r}}\cdot\beta)^2) = \hat{r}_i(1+\hat{\mathbf{r}}\cdot\beta),
\tag{7.77}
$$

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and thus

$$(\partial_i r) = \frac{\hat{r}_i}{\kappa}. \tag{7.78}$$

This may now be substituted back into eqn. (7.75) to give

$$(\partial_i r_j) = g_{ij} + \frac{\beta_j \hat{r}_i}{\kappa}. \tag{7.79}$$

Continuing in the same fashion, one derives the following relations:

$$\begin{aligned}
(\partial_0 r) &= -\frac{\hat{r}^i \beta_i}{\kappa} \\
(\partial_0 r^i) &= -\frac{\beta^i}{\kappa} \\
(\partial_i r) &= \frac{\hat{r}_i}{\kappa} \\
(\partial_i r_j) &= g_{ij} + \frac{\beta_j \hat{r}_i}{\kappa}. \\
(\partial_0 \beta_i) &= \frac{\alpha_i}{\kappa} \\
(\partial_i \beta_j) &= -\frac{\hat{r}_{\mathrm{i}} \alpha_j}{\kappa} \\
\partial_0 (r\kappa) &= \frac{1}{\kappa} \left( \beta^2 - \hat{\mathbf{r}} \cdot \beta - \alpha \cdot \mathbf{r} \right) \\
\partial_i (r\kappa) &= \frac{\hat{r}_i}{\kappa} \left( 1 - \beta^2 + \alpha \cdot \mathbf{r} \right) - \beta_i,
\end{aligned} \tag{7.80}$$

where we have defined $\alpha_\mu = \partial_0 \beta_\mu = (0, \dot{\mathbf{v}}/c^2)$. The field strength tensor may now be evaluated. From eqn. (7.70) one has

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\
&= \frac{q}{4\pi\epsilon_0 c} \left[ \frac{\partial_\mu \beta_\nu - \partial_\nu \beta_\mu}{r\kappa} - \frac{(\beta_\nu \partial_\mu - \beta_\mu \partial_\nu)(r\kappa)}{r^2 \kappa^2} \right].
\end{aligned} \tag{7.81}$$

And, noting that $\beta_0 = -1$ is a constant, we identify the three-dimensional electric and magnetic field vectors:

$$\begin{aligned}
E_i &= c F_{i0} \\
&= \frac{-q}{4\pi\epsilon_0} \left[ \frac{\partial_0 \beta_i}{r\kappa} - \frac{(\beta_{\mathrm{i}} \partial_0 - \beta_0 \partial_i)(r\kappa)}{r^2 \kappa^2} \right] \\
&= \frac{-q}{4\pi\epsilon_0} \left[ \frac{\alpha_i}{\kappa^2 r} + \frac{(\beta_i - \hat{r}_i)}{r^2 \kappa^3} \left( \alpha \cdot \mathbf{r} + (1 - \beta^2) \right) \right].
\end{aligned} \tag{7.82}$$

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$$
\begin{aligned}
B_i &= \frac{1}{2}\epsilon_{ijk}F_{jk} \\
&= \frac{q}{4\pi\epsilon_0 c}\epsilon_{ijk}\left[\frac{\partial_j\beta_k}{r\kappa} - \frac{(\beta_k\partial_j)(r\kappa)}{r^2\kappa^2}\right] \\
&= \frac{-q}{4\pi\epsilon_0 c}\epsilon_{ijk}\hat{r}_j\left[\frac{\alpha_k}{\kappa} + \frac{\beta_k}{r^2\kappa^3}\left(\alpha\cdot\mathbf{r} + (1-\beta^2)\right)\right] \qquad (7.83) \\
&= \frac{1}{c}\epsilon_{ijk}\hat{r}_j E_k \\
&= \frac{1}{c}(\hat{\mathbf{r}}\times\mathbf{E})_i. \qquad (7.84)
\end{aligned}
$$

From these relations, it is clear that the magnetic field is perpendicular to both the light ray $\mathbf{r}$ and the electric field. The electric field can be written as a sum of two parts, usually called the radiation field and the near field:

$$
E_{i\,\mathrm{rad}} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{\alpha_i}{\kappa^2 r} + \frac{(\beta_i - \hat{r}_i)(\alpha\cdot\hat{\mathbf{r}})}{\kappa^3 r}\right] \qquad (7.85)
$$

$$
E_{i\,\mathrm{near}} = \frac{q}{4\pi\epsilon_0 c}\left[\frac{(\beta_i - \hat{r}_i)(1-\beta^2)}{r^2\kappa^3}\right]. \qquad (7.86)
$$

The near field falls off more quickly than the long-range radiation field. The radiation field is also perpendicular to the light ray $\hat{\mathbf{r}}$. Thus, the far-field electric and magnetic vectors are completely transverse to the direction of propagation, but the near-field electric components are not completely transverse except at very high velocities $\beta \sim 1$. Note that all of the vectors in the above expressions are assumed to be evaluated at the retarded time.

Owing to their special relationship, the magnitude of the magnetic and electric components are equal up to dimensional factors:

$$
|\mathbf{E}|^2 = c^2|\mathbf{B}|^2. \qquad (7.87)
$$

Finally, the rate of work or power expended by the field is given by Poynting's vector,

$$
\begin{aligned}
S_i &= \epsilon_{ijk}E_j H_k \\
&= (\mu_0 c)^{-1}\epsilon_{ijk}E_j(\hat{\mathbf{r}}\times\mathbf{E})_k \\
&= \epsilon_0 c\epsilon_{ijk}E_j(\epsilon_{klm}\hat{r}_l E_m) \\
\mathbf{S} &= -\epsilon_0 c(\mathbf{E}\cdot\mathbf{E})\hat{\mathbf{r}}. \qquad (7.88)
\end{aligned}
$$

## 7.4 Resonance phenomena and dampening fields

In the interaction between matter and radiation, bound state transitions lead to resonances, or phenomena in which the strength of the response to a radiation field is amplified for certain frequencies. Classically these special frequencies are the normal modes of vibration for spring-like systems with natural frequency $\omega_0$; quantum mechanically they are transitions between bound state energy levels with a definite energy spacing $\omega_0 = (E_2 - E - 1)/\hbar$. The examples which follow are all cases of one mathematical phenomenon which manifests itself in several different physical scenarios. We see how the unified approach reveals these similarities.

### 7.4.1 Cherenkov radiation

The radiation emitted by charged particles which move in a medium where the speed of light is less than the speed of the particles themselves is called Cherenkov radiation. The effect was observed by Cherenkov [25] and given a theoretical explanation by Tamm and Frank [127] within the framework of classical electrodynamics. The power spectrum of the radiation may be calculated with extraordinary simplicity using covariant field theory [122].

Using the covariant formulation in a material medium from section 21.2.4 and adapting the expression in eqn. (5.118) for the Maxwell field, we have the Feynman Green function in the Lorentz–Feynman $\alpha = 1$ gauge, given by

$$D_{\mathrm{F}}(x, x') = \frac{-\mathrm{i}}{4\pi^2 c^2 |\mathbf{x} - \mathbf{x}'|} \int_0^\infty \mathrm{d}\omega \; \sin\left(\frac{n\omega}{c}|\mathbf{x} - \mathbf{x}'|\right) \mathrm{e}^{-\mathrm{i}\omega|t-t'|}, \tag{7.89}$$

where $n$ is the refractive index of the medium. Note that this index is assumed to be constant here, which is not the case in media of interest. One should really consider $n = n(\omega)$. However, the expressions generated by this form will always be correct in $\omega$ space for each value of $\omega$, since the standard textbook assumption is to ignore transient behaviour ($t$-dependence) of the medium. We may therefore write the dissipation term as

$$W = \mu_0 \mu_{\mathrm{r}} \int (\mathrm{d}x)(\mathrm{d}x') \hat{J}^\mu(x) \hat{D}_{\mathrm{F}\mu\nu}(x, x') \hat{J}^\nu(x'), \tag{7.90}$$

and we are interested in the power spectrum which is defined by

$$\int \mathrm{d}\omega \frac{P(\omega)}{\omega} = \frac{2}{\hbar} \mathrm{Im} \frac{\mathrm{d}W}{\mathrm{d}t}. \tag{7.91}$$

Substituting in expressions for $\hat{J}_\mu$, we obtain

$$\mathrm{Im} W = -\frac{1}{8\pi^2} \int \mathrm{d}\omega (\mathrm{d}x)(\mathrm{d}x') \frac{\mu_0 \mu_{\mathrm{r}} \sin(\frac{n\omega}{c}|\mathbf{x} - \mathbf{x}'|)}{c^2 |\mathbf{x} - \mathbf{x}'|} \times \cos(\omega|t - t'|) \hat{J}^\mu \hat{J}_\mu, \tag{7.92}$$

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from which we obtain

$$P(\omega) = -\frac{\omega\mu_0\mu_{\rm r}}{4\pi^2 n^2}\int \mathrm{d}\sigma_x(\mathrm{d}x')\frac{\sin(\frac{n\omega}{c}|\mathbf{x}-\mathbf{x}'|)}{n^2|\mathbf{x}-\mathbf{x}'|}\cos(\omega|t-t'|) \times \left[\rho(x)\rho(x') - \frac{n^2}{c^2}J^i(x)J_i(x')\right]. \tag{7.93}$$

The current distribution for charged particles moving at constant velocity is

$$\begin{aligned} \rho &= q\delta(\mathbf{x}-\mathbf{v}t) \\ J^i &= qv^i\delta(\mathbf{x}-\mathbf{v}t); \end{aligned} \tag{7.94}$$

thus we have

$$\begin{aligned} P(\omega, t) &= \frac{q^2}{4\pi^2}\frac{\mu_0\mu_{\rm r}\omega\beta}{c}\left(1 - \frac{1}{n(\omega)^2\beta^2}\right)\int_{-\infty}^{\infty}\sin(n\beta\omega\tau)\cos(\omega\tau)\mathrm{d}\tau \\ &= \begin{cases} 0 & n\beta < 1 \\ \frac{q^2}{4\pi}\frac{\mu_0\mu_{\rm r}\omega\beta}{c}\left(1 - \frac{1}{n^2\beta^2}\right) & n\beta > 1. \end{cases} \end{aligned} \tag{7.95}$$

This is the power spectrum for Cherenkov radiation, showing the threshold behaviour at $n\beta = 1$. We have derived the Cherenkov resonance condition for charges interacting with electromagnetic radiation. The Cherenkov effect is more general than this, however. It applies to any interaction in which particles interact with waves, either transverse or longitudinal.

### *7.4.2 Cyclotron radiation*

Cyclotron, or synchrotron, radiation is emitted by particles accelerated by a homogeneous magnetic field. Its analysis proceeds in the same manner as that for Cherenkov radiation, but with a particle distribution executing circular rather than linear motion. For the current, one requires

$$\begin{aligned} \rho &= q\delta(\mathbf{x}-\mathbf{x}_0) \\ J^i &= qv^i\delta(\mathbf{x}-\mathbf{x}_0), \end{aligned} \tag{7.96}$$

where $\mathbf{x}_0$ is the position of the charged particle. Since the electromagnetic field is not self-interacting, the Green function for the radiation field is not affected by the electromagnetic field in the absence of a material medium. In the presence of a polarizable medium, there is an effect, but it is small. (See the discussion of Faraday rotation.)

The force on charges is

$$\begin{aligned} F_i = \frac{\mathrm{d}p_i}{\mathrm{d}t} &= q(\mathbf{v}\times\mathbf{B})_i \\ &= qF_{ij}v^j, \end{aligned} \tag{7.97}$$

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and, since this is always perpendicular to the motion, no work is done; thus the energy is a constant of the motion:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0. \tag{7.98}$$

The generic equation of circular motion is

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = (\omega \times \mathbf{v})_i, \tag{7.99}$$

which, in this case, may be written as

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \frac{q}{m}\sqrt{1-\beta^2}(\mathbf{v} \times \mathbf{B})_i, \tag{7.100}$$

where $p_i = mv_i/\sqrt{1-\beta^2}$ and $\beta_i = v_i/c$. Thus, the angular frequency of orbit is the Larmor frequency,

$$\omega_\mathrm{i} = -\frac{qB_i}{m}\sqrt{1-\beta^2} = -\frac{qB_ic^2}{E}, \tag{7.101}$$

which reduces to the cyclotron frequency, $\omega_\mathrm{c} \simeq eB/m$, in the non-relativistic limit $\beta_i \to 0$. The radius of revolution is correspondingly

$$R = \frac{|v|}{\omega} = \frac{mc\beta}{|q|B\sqrt{1-\beta^2}}. \tag{7.102}$$

The primary difficulty in analysing this problem is a technical one associated with the circular functions. Taking boundary conditions such that the particle position is given by

$$\begin{aligned} x_1(t) &= R\cos(\omega t) \\ x_2(t) &= R\sin(\omega t) \\ x_3(t) &= 0, \end{aligned} \tag{7.103}$$

one finds the velocity

$$\begin{aligned} v_1(t) &= -R\omega\sin(\omega t) \\ v_2(t) &= R\omega\cos(\omega t) \\ v_3(t) &= 0. \end{aligned} \tag{7.104}$$

This may be substituted into the current in order to evaluate the power spectrum. This is now more difficult: one can use an integral representation of the delta function, such as the Fourier transform; this inevitably leads to exponentials of sines and cosines, or Bessel functions. We shall not pursue these details of evaluation here. See ref. [121] for further study of this topic.

### 7.4.3 Landau damping

Landau damping is the name given to the dissipative mixing of momenta in any particle field or plasma which interacts with a wave. The phenomenon of Landau damping is quite general and crops up in many guises, but it is normally referred to in the context of the interaction of a plasma with electromagnetic waves. In a collisionless plasma (no scattering by self-interaction), there is still scattering by the interaction of plasma with the ambient electromagnetic field, similar to the phenomenon of stimulated absorption/emission. However, any linear perturbation or source can cause the energy in one plasma mode to be re-channelled into other modes, thus mixing the plasma and leading to dissipation. All one needs is a linear interaction between the waves and the plasma field, and a resonant amplifier, which tends to exaggerate a specific frequency.

In simple terms, a wave acts like a sinusoidal potential which scatters and drags the particle field. If the phase of the field is such that it strikes the upward slope of the wave, it is damped or reflected, losing energy. If the phase is such that the field 'rolls down' the downward slope of the wave, it is enhanced and gains energy. In a random system, the average effect is to dissipate or to dampen the field so that all particles or field modes tend to become uniform. In short, Landau damping is the re-organization of energy with the modes of a field due to scattering off wavelets of another field.

Let us consider an unbound particle displacement field with action

$$S = \frac{1}{\sigma_x}\int (\mathrm{d}x)\left\{-\frac{1}{2}m\dot{s}^2 - J^i s_i\right\}, \tag{7.105}$$

coupled through the current $J_i$ to the electromagnetic field. The position of a particle is

$$x^i = \overline{x}^i + \delta x^i = \overline{x}^i + s^i, \tag{7.106}$$

and its velocity is

$$\dot{x}^i = \overline{v}^i + \delta v. \tag{7.107}$$

The velocity of a free particle is constant until the instant of its infinitesimal perturbation by a wave, so we write

$$\overline{x}^i = \overline{v}t, \tag{7.108}$$

so that

$$k_\mu x^\mu = k_i x^i - \omega t = k_i s^i + (k_i \overline{v}^i - \omega)t. \tag{7.109}$$

The perturbation is found from the solution to the equation of motion:

$$s^i = \int (\mathrm{d}x) G^{ij}(x, x') E_j(x'), \tag{7.110}$$

or

$$\delta v^i = \frac{q}{m}\,\mathrm{Re}\,\frac{E_0^i \exp\mathrm{i}\left(k_i s^i + (k_i\overline{v}^i - \omega)t \pm \gamma t\right)}{\mathrm{i}(k_i\overline{v}^i - \omega) \pm \gamma}$$

$$s^i = \frac{q}{m}\,\mathrm{Re}\,\frac{E_0^i \exp\mathrm{i}\left(k_i s^i + (k_i\overline{v}^i - \omega)t \pm \gamma t\right)}{\left[\mathrm{i}(k_i\overline{v}^i - \omega) \pm \gamma\right]^2}. \tag{7.111}$$

An infinitesimal regulating parameter, $\gamma$, is introduced here in order to define a limit in what follows. This has causal implications for the system. It means that the field either grows from nothing in the infinite past or dissipates to nothing in the infinite future. This is reflected by the fact that its sign determines the sign of the work done. Eventually, we shall set $\gamma$ to zero. The work done by this interaction between the charged particle $q$ and the electric field $E^i$ is $qE_i x^i$. The rate of work is

$$q\frac{\mathrm{d}}{\mathrm{d}t}\left[E_i x^i\right] = q\partial_t E_i\, x^i + E_i v^i. \tag{7.112}$$

The two terms signify action and reaction, so that the total rate of work is zero, expressed by the total derivative. The second term is the rate of work done by the charge on the field. It is this which is non-zero and which leads to the dampening effect and apparent dissipation. Following Lifshitz and Pitaevskii [90], we calculate the rate of work per particle as follows,

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= \mathrm{Re}\; q v_i E^i \\ &= \mathrm{Re}\; q(\overline{v}^i + \delta v^i)E(t, \mathbf{x} + \mathbf{s}) \\ &= \mathrm{Re}\; q(\overline{v}^i + \delta v^i)(E_i(t, \mathbf{x}) + \partial_j E_i(t, \mathbf{x})s^j + \cdots).\end{aligned} \tag{7.113}$$

To first order, the average rate of work is thus

$$\begin{aligned}\left\langle\frac{\mathrm{d}w}{\mathrm{d}t}\right\rangle &= \mathrm{Re}\; q\overline{v}_i\langle(\partial_j E_i)s^i\rangle + \mathrm{Re}\; q\langle\delta v^i E_i\rangle \\ &= \frac{1}{2}q\overline{v}_i(\partial_j E_i^*)s^i + \frac{1}{2}q\delta v^i E_i^*.\end{aligned} \tag{7.114}$$

Here we have used the fact that

$$\mathrm{Re}\, A = \frac{1}{2}(A + A*) \tag{7.115}$$

and

$$\langle \mathrm{Re}\, A \cdot \mathrm{Re}\, B\rangle = \frac{1}{4}(AB^* + A^*B) = \frac{1}{2}\mathrm{Re}\,(AB^*), \tag{7.116}$$

since terms involving $A^2$ and $B^2$ contain $\mathrm{e}^{2\mathrm{i}\omega t}$ average to zero over time (after setting $\gamma \to 0$). Substituting for $s^i$ and $\delta v^i$, we obtain

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}t} \right\rangle = \frac{q^2}{2m}\, E^i E^j \left[ \frac{\mathrm{i}\overline{v}_i k_j}{[\mathrm{i}(k_i v^i - \omega) \pm \gamma]^2} + \frac{\delta_{ij}}{[\mathrm{i}(k_i v^i - \omega) \pm \gamma]} \right]. \tag{7.117}$$

The tensor structure makes it clear that the alignment, $k_i$, and polarization, $E_0^i$, of the electric wave to the direction of particle motion, $v_i$, is important in deciding the value of this expression. Physically, one imagines a wave (but not a simple transverse wave) moving in direction $k_i$ and particles surfing over the wave in a direction given by $v_i$. The extent to which the wave offers them resistance, or powers them along, decides what work is done on them. For transverse wave components, $k^i E_i = 0$, the first term vanishes. From the form of eqn. (7.117) we observe that it is possible to write

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}t} \right\rangle = \frac{q^2}{2m}\, E_i E_j\, \frac{\mathrm{d}}{\mathrm{d}(k_i v_j)} \left[ \frac{\pm\gamma\, (k_m v^m)}{[\mathrm{i}(k_i v^i - \omega) \pm \gamma]} \right], \tag{7.118}$$

and, using

$$\lim_{\gamma \to 0} \frac{\gamma}{z^2 + \gamma^2} = \pi\, \delta(z) \tag{7.119}$$

we have

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}t} \right\rangle = \pm \frac{q^2 \pi}{m}\, E_i E_j\, \frac{\mathrm{d}}{\mathrm{d}(k_i v_j)} (k_i v^i) \delta(k_j v^j - \omega). \tag{7.120}$$

To avoid unnecessary complication, let us consider the contribution to this which is most important in the dampening process, namely a one-dimensional alignment of $k_i$ and $v_i$:

$$\left\langle \frac{\mathrm{d}w}{\mathrm{d}t} \right\rangle = \pm \frac{q^2 \pi}{2m}\, |E_\parallel|^2\, \frac{\mathrm{d}}{\mathrm{d}(kv)} (kv)\; \delta(kv - \omega). \tag{7.121}$$

This expression is for one particle. For the whole particle field we must perform the weighted sum over the whole distribution, $f(\omega)$, giving the total rate of work:

$$\begin{aligned} \left\langle \frac{\mathrm{d}W}{\mathrm{d}t} \right\rangle &= \pm \frac{q^2 \pi}{2m}\, |E_\parallel|^2 \int \mathrm{d}\omega\; f(\omega)\, \frac{\mathrm{d}}{\mathrm{d}(kv)} (kv) \delta(kv - \omega) \\ &= \mp \frac{q^2 \pi}{m}\, |E_\parallel|^2 \int \mathrm{d}\omega \frac{\mathrm{d}f(\omega)}{\mathrm{d}\omega}\, (kv) \delta(kv - \omega) \\ &= \mp \frac{q^2 \mathbf{pi}\omega}{2m}\, |E_\parallel|^2\, \left. \frac{\mathrm{d}f(\omega)}{\mathrm{d}\omega} \right|_{v = \omega/k}. \end{aligned} \tag{7.122}$$

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The integral over the delta function picks out contributions when the velocity of particles, $v_i$, matches the phase velocity of the electromagnetic wave, $\omega/k_i$. This result can now be understood either in real space from eqn. (7.114) or in momentum space from eqn. (7.122). The appearance of the gradient of the electric field in eqn. (7.114) makes it clear that the dissipation is caused as a result of motion in the potential of the electric field. Eqn. (7.122) contains $\mathrm{d}f/\mathrm{d}\omega$, for frequencies where the phase velocity is in resonance with the velocity of the particles within the particle field; this tells us that particles with $v < \omega/k$ gain energy from the wave, whereas $v > \omega/k$ lose energy to it ($\gamma > 0$). The electric field will be dampened if the shape of the distribution is such that there are more particles with $v < \omega/k$ than with $v > \omega/k$. This is typical for long-tailed distributions like thermal distributions.

This can be compared with the discussion in section 6.1.4.

### *7.4.4 Laser cooling*

Another example of resonant scattering with many experimental applications is the phenomenon of laser cooling. This can be thought of as Landau damping for neutral atoms, using the dipole force as the breaking agent. We shall consider only how the phenomenon comes about in terms of classical fields, and sketch the differences in the quantum mechanical formulation. By now, this connection should be fairly familiar. The shift in energy of an electromagnetic field by virtue of its interaction with a field of dipoles moving at fractional speed $\beta^i$ is the work done in the rest frame of the atom,

$$\begin{aligned}\Delta W &= -\frac{1}{2}\int \mathrm{d}\sigma_x \; \mathbf{P}(x)\cdot\mathbf{E}(x) \\ &= \frac{q^2}{2m}\int (\mathrm{d}x')\mathrm{d}\sigma_x \; E^i(x) G^{\beta}_{ij}(x,x') E^j(x'), \end{aligned} \tag{7.123}$$

where

$$((1-\beta^i)^2\partial_t^2 - \gamma\partial_t + \kappa)G^{\beta}_{ij}(x,x') = \delta_{ij}\; c\delta(x,x') \tag{7.124}$$

(see eqn. (2.88)), and therefore the dipole force $\mathbf{F}$ on each atom may be deduced from $\mathrm{d}W = \mathbf{F}\cdot\mathrm{d}\mathbf{r}$. The imaginary part of the energy is the power exchanged by the electromagnetic field, which is related to the damping rate or here the cooling rate of the atoms. The force on an atom is the gradient of the real part of the work:

$$F_i^{\beta} = -\frac{q^2}{2m}\int \mathrm{d}\sigma_x \; \overset{x}{\partial_i}\left[E^j(x)\int (\mathrm{d}x')\; G^{\beta}_{jk}(x,x')E^k(x')\right]. \tag{7.125}$$

If we consider a source of monochromatic radiation interacting with the particle field (refractive index $n^i$),

$$E^i(x) = E_0^i\,\mathrm{e}^{\mathrm{i}kx} = E_0^i\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{7.126}$$

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Table 7.1. Doppler effect on momentum.

| Resonance-enhanced | Parallel | Anti-parallel |
|---|---|---|
| (diagonal) | $\hat{k}^i\beta_i > 0$ | $\hat{k}^i\beta_i < 0$ |
| $\omega_\beta > \omega_0$ | $F_i\hat{\beta}^i < 0$ | $F_i\hat{\beta}^i > 0$ |
| $\omega_\beta < \omega_0$ | $F_i\hat{\beta}^i > 0$ | $F_i\hat{\beta}^i < 0$ |

where the frequency $\omega$ is unspecified but satisfies $\mathbf{k}^2c^2 = \mathbf{n}\omega^2$, then we have

$$\begin{aligned} F_i^\beta &= -\frac{q^2}{2m}\,E_0^2\rho_N \int \mathrm{d}\sigma_x\,\partial_i\,\frac{\mathrm{e}^{\mathrm{i}x(k+k')+\mathrm{i}x'(k'-k)}}{-\omega_\beta^2+\omega_0^2+\mathrm{i}\gamma\omega} \\ &= -\frac{q^2}{2m}\,E_0^2\rho_N \int \mathrm{d}\sigma_x\,\partial_i\,\frac{\mathrm{e}^{\mathrm{i}2kx}}{-\omega_\beta^2+\omega_0^2+\mathrm{i}\gamma\omega}. \end{aligned} \tag{7.127}$$

This expression contains forward and backward moving photons of fixed frequency, $\omega$, and wavenumber, $k_i$. The sign of the force acting on the atoms depends on the frequency relative to the resonant frequency, $\omega_0$, and we are specifically interested in whether the force acts to accelerate the atoms or decelerate them relative to their initial velocity. The fact that atoms in the particle field move in all directions on average means that some will experience Doppler blue-shifted radiation frequencies and others will experience red-shifted frequencies, relative to the direction of photon wavevector, $k^i$. In effect, the Doppler effect shifts the resonant peak above and below its stationary value making two resonant 'side bands'. These side bands can lead to energy absorption. This is best summarized in a table (see table 7.1).

As the velocity component, $v^i = \beta^i c$, of a particle field increases, the value of $1 - \beta^i\hat{k}_i$ either increases (when $\hat{k}$ and $\beta^i$ point in opposing directions) or decreases (when $\hat{k}$ and $\beta^i$ point in the same direction). The component of velocity in the direction of the photons, $\mathbf{E}^i$, is given by $\hat{k}^i\beta_i$, and its sign has two effects. It can bring $\omega_\beta$ closer to or further from the resonant frequency, $\omega_0$, thus amplifying or attenuating the force on the particles. The force is greater for those values which are closest to resonance. It also decides whether the sign of the force is such that it tends to increase the magnitude of $\beta^i$ or decrease the magnitude of $\beta^i$. It may be seen from table 7.1 that the force is always such as to make the velocity tend to a value which makes $\omega_\beta = \omega_0$. Thus by sweeping the value of $\omega$ from a value just above resonance to resonance, it should be possible to achieve $\beta^i \to 0$. The lowest attainable temperature according to this simple model is limited by the value of $\omega_0$.

In order to reduce all of the components of the velocity to minimal values, it is desirable to bathe a system in crossed laser beams in three orthogonal directions.

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Such laser beams are called *optical molasses*, a kind of quagmire for resonant particle fields. Clearly, systems with a low-frequency resonance are desirable in order to push the magnitude of $\beta^i$ down to a minimum. The rate of energy loss is simply the damping constant, $\gamma$.

## 7.5 Hydrodynamics

The study of the way in which bulk matter fields spread through a system is called hydrodynamics. Because it deals with bulk matter, hydrodynamics is a macroscopic, statistical discussion. It involves such ideas as flow and diffusion, and is described by a number of essentially classical phenomenological equations.

### *7.5.1 Navier–Stokes equations*

The Navier–Stokes equations are the central equations of fluid dynamics. They are an interesting example of a vector field theory because they can be derived from an action principle in two different ways. Fluid dynamics describes a stationary system with a fluid flowing through it. The velocity is a function of position and time, since the flow might be irregular; moreover, because the fluid flows relative to a fixed pipe or container, the action is not invariant under boosts.

*Formulation as a particle field* Using a 'microscopic' formulation, we can treat a fluid as a particle displacement field without a restoring force (spring tension zero). We begin by considering such a field at rest:

$$S = \int (\mathrm{d}x) \left\{ \frac{1}{2}\rho \dot{s}^2 - \frac{1}{2}\eta(\partial^i s^j) \overset{\leftrightarrow}{\partial_t} (\partial_i s_,) + s^i (F_i - \partial_i P) \right\}. \quad (7.128)$$

Notice the term linear in the derivative which is dissipative and represents the effect of a viscous frictional force (see section 7.2). $\eta$ is the coefficient of viscosity. In this form, the equations have made an assumption which relates bulk and shear viscosity, leaving only a single effective viscosity. This is the form often used experimentally. Varying the action with respect to $s^i$ leads to the field equation

$$-\rho\, \ddot{s}_i + \eta\, \nabla^2\, \dot{s}_i + F_i - \partial_i P = 0. \quad (7.129)$$

Or, setting $v_i \equiv \dot{s}_i$,

$$\rho \dot{v}_i - \eta\, \nabla^2\, v_i + \partial_i P = F_i. \quad (7.130)$$

This is the equation of a velocity field at rest. In order to boost it into a moving frame, we could re-define positions by $x_i \rightarrow x_i - v_i t$, but it is more convenient

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to re-define the time coordinate to so-called retarded time (see section 9.5.2). With this transformation, we simple replace the time derivative for $v^i$ by

$$\frac{\mathrm{d}}{\mathrm{d}t_{\mathrm{ret}}} = \partial_t + v^i\,\partial_j. \tag{7.131}$$

This gives

$$\rho\frac{\mathrm{d}}{\mathrm{d}t_{\mathrm{ret}}}v_i - \eta\,\nabla^2\,v_i + \partial_i P = F_i. \tag{7.132}$$

In fluid dynamics, this derivative is sometimes called the *substantive derivative*; it is just the total derivative relative to a moving frame. This transformation of perspective introduces a non-linearity into the equation which was not originally present. It arises physically from a non-locality in the system; i.e. the fact that the velocity-dependent forces at a remote point lead to a delayed effect on the velocity at local point. Put another way, the velocity at one point interacts with the velocity at another point because of the flow, just as in a particle scattering problem. In particle theory parlance, we say that the velocity field scatters off itself, or is self-interacting. It would have been incorrect to apply this transformation to the action before variation since the action is a scalar and was not invariant under this transformation, thus it would amount to a change of the physics. Since the action is a generator of constraints, it would have additional consequences for the system, as we shall see below.

*Formulation as an effective velocity field* The description above is based upon a microscopic picture of a fluid as a collection of particles. We need not think like this, however. If we had never built a large enough microscope to be able to see atoms, then we might still believe that a fluid were a continuous substance. Let us then formulate the problem directly in terms of a velocity field. We may write the action

$$S = \tau\int(\mathrm{d}x)\left\{-\frac{1}{2}\rho\,v^i\,\overset{\leftrightarrow}{\partial_t}\,v_i + \frac{1}{2}\eta(\partial^i v^j)(\partial_i v_j) - v^i(F_i - \partial_i P)\right\}. \tag{7.133}$$

The constant scale $\tau$ has the dimensions of time and is necessary on purely dimensional grounds. The fact that we need such an arbitrary scale is an indication that this is just an average, smeared out field theory rather than a microscopic description. It has no physical effect on the equations of motion unless we later attempt to couple this action to another system where the same scale is absent or different. Such is the nature of dimensional analysis. The linear derivatives in the action are symmetrized for the reasons discussed in section 4.4.2. Varying this action with respect to the velocity $v^i$, and treating $\rho$ as a constant for the time being, leads to

$$\rho\,\partial_t v_i - \eta\,\nabla^2\,v_i + \partial_i P = F_i. \tag{7.134}$$

Changing to retarded time, as before, we have the Navier–Stokes equation,

$$\rho\, \partial_t v_i + \rho v^j (\partial_j v_i) - \eta\, \nabla^2\, v_i + \partial_i P = F_i. \tag{7.135}$$

Again, it would be incorrect to transform the action before deriving the field equations, since the action is a scalar and it is not invariant under this transformation.

Consider what would have happened if we had tried to account for the retardation terms in the action from the beginning. Consider the action

$$S = \tau \int (\mathrm{d}x) \bigg\{ -\frac{1}{2} \rho\, v^i \overset{\leftrightarrow}{\partial_t}\, v_i + \frac{1}{2} \rho\, v^i\, (\partial_i v_j) v^j - \frac{1}{2} \rho\, \partial_i (v^i v^j) v_j \\ + \frac{1}{2} \eta (\partial^i v^j)(\partial_i v_j) - v^i (F_i - \partial_i P) \bigg\}. \tag{7.136}$$

The action is now non-linear from the beginning since it contains the same retardation information as the transformed eqn. (7.132). The derivatives are symmetrized also in spatial directions. The variation of the action is also more complicated. We shall now let $\rho$ depend on $x$. After some calculation, variation with respect to $v^i$ leads to an equation which can be separated into parts:

$$(\partial_t \rho) v_i + \rho v_i (\partial_j v^j) + \frac{1}{2} (\partial_i \rho) v^2 = 0 \\ \rho\, (\partial_t v_i) + \rho v^j \partial_i v_j - \eta\, \nabla^2\, v_i + \partial_i P = F_i. \tag{7.137}$$

The first of these occurs because the density is no longer constant; it is tantalizingly close to the conservation equation for current

$$-\partial_t \rho = \partial_i (\rho v^i), \tag{7.138}$$

but alas is not quite correct. The equations of motion (7.137) are almost the same as before, but now the derivative terms are not quite correct. Instead of

$$v^j \partial_j v_i \tag{7.139}$$

we have the symmetrical

$$v^j \partial_i v_j. \tag{7.140}$$

This result is significant. The terms are not unrelated. In fact, since we can always add and subtract a term, it is possible to relate them by

$$v^j \partial_j v_i = v^j (\partial_i v_j) + v^j (\partial_j v_i - \partial_i v_j). \tag{7.141}$$

The latter term is the curl of the velocity. What this means is that the two terms are equivalent provided that the curl of the velocity vanishes. It vanishes in

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the absence of eddies or other phenomena which select a preferred direction in space or time. This is indicative of the symmetry of the action. Since the action was invariant under space and time reversal, it can only lead to equations of motion with the same properties. Physically, this restriction corresponds to purely *irrotational flow*. Notice how the symmetry which is implicit in the action leads directly to a symmetry in the field equations. The situation was different in our first formulation, where we chose to transform the action to retarded time (an intrinsically asymmetrical operation).

The problem of an $x$-dependent density $\rho$ is not resolvable here. The fundamental problem is that the flow equation is not reversible, whereas the action would like to be. If we omit the non-linear terms, the problem of finding an action which places no restriction on $\rho$ is straightforward, though not particularly well motivated. We shall not pursue this here. The lesson to be learned from this exercise is that, because the action is a scalar, the action principle will always tend to generate field equations consistent with the symmetries of the fields it is constructed from. Here we have tried to generate a term $v^j \partial_j v_i$ from an action principle, but the infinitesimal variation of this term led to new constraints since action is spacetime-reflection-invariant. The problem of accommodating an $x$-dependent density is confounded by these other problems. In short, non-covariant analyses do not lend themselves to a covariant formulation, but should be obtained as a special case of a more well defined problem as in the first method.

### *7.5.2 Diffusion*

Let us consider the rate at which conserved matter diffuses throughout a system when unencumbered by collisions. Consider a matter current, $J_\mu$, whose average, under the fluctuations of the system, is conserved:

$$\partial_\mu \langle J^\mu \rangle = 0. \tag{7.142}$$

We need not specify the nature of the averaging procedure, nor the origin of the fluctuations here. Phenomenologically one has a so-called constitutive relation [53], which expresses a phenomenological rate of flow in terms of local density gradients:

$$\langle J_i \rangle = -D\partial_i \langle \rho \rangle. \tag{7.143}$$

Substituting this into the conservation equation gives

$$(\partial_t - D\nabla^2)\langle \rho \rangle = 0. \tag{7.144}$$

This is a diffusion equation, with diffusion coefficient $D$. If we multiply this equation by the positions squared, $x^2$, and integrate over the entire system,

$$\int d\sigma \; x^2 (\partial_t - D\nabla^2)\langle \rho \rangle = 0, \tag{7.145}$$

we can interpret the diffusion constant in terms of the mean square displacement of the field. Integrating by parts, and assuming that there is no diffusion at the limits of the system, one obtains

$$\partial_t \langle x^2 \rangle - 2D \sim 0, \tag{7.146}$$

or

$$\langle x^2 \rangle \sim 2Dt, \tag{7.147}$$

which indicates that particles diffuse at a rate of $\sqrt{2D}$ metres per unit time. Notice that, since $D$ characterizes the diffusion of averaged quantities, it need not be a constant. We shall think of it as a slowly varying function of space and time. The variation, however, must be so slow that it is effectively constant over the dominant scales of the system. We shall derive a Kubo-type relation for this quantity [53].

From eqn. (7.144), we may solve

$$\langle \rho \rangle (x) = \int \frac{\mathrm{d}^n \mathbf{k} \mathrm{d}\omega}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \rho(k)\, \delta(-\mathrm{i}\omega - D\mathbf{k}^2), \tag{7.148}$$

or

$$G^{(\pm)}(k) = \frac{1}{\mp \mathrm{i}\omega - D\mathbf{k}^2}. \tag{7.149}$$

Thus

$$\langle \rho \rangle (x) = \int \frac{\mathrm{d}^n \mathbf{k} \mathrm{d}\omega}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - D\mathbf{k}^2 t} \rho(k). \tag{7.150}$$

To determine the effect of fluctuations in this system, consider adding an infinitesimal source,

$$(\partial_t - D\nabla^2)\langle \rho \rangle = F. \tag{7.151}$$

The purely mechanical retarded response to $F$ gives us the following relation:

$$\langle \rho \rangle (x) = \int (\mathrm{d}x')\; G_{\mathrm{r}}(x, x') F(x'), \tag{7.152}$$

where the retarded Green function may be evaluated by analogy with eqn. (5.77)

$$\begin{aligned} G_{\mathrm{r}}(x, x') &= \int \frac{\mathrm{d}^n \mathbf{k} \mathrm{d}\omega}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \left[ \frac{1}{\omega + \mathrm{i}D\mathbf{k}^2 - \mathrm{i}\epsilon} - \frac{1}{\omega - \mathrm{i}D\mathbf{k}^2 + \mathrm{i}\epsilon} \right] \\ &= \int \frac{\mathrm{d}^n \mathbf{k} \mathrm{d}\omega}{(2\pi)^{n+1}} \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)} \frac{-2\mathrm{i}D\mathbf{k}^2}{(\omega - \mathrm{i}\epsilon)^2 + D^2\mathbf{k}^4}. \end{aligned} \tag{7.153}$$

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From eqn. (6.67) we have

$$\langle\rho\rangle = \frac{\mathrm{i}}{\hbar}\int (\mathrm{d}x')\langle\rho(x)\rho(x')\rangle F(x'), \tag{7.154}$$

where

$$\frac{\mathrm{i}}{\hbar}\langle\rho(x)\rho(x')\rangle = \frac{\delta^2 W}{\delta F^2} = -\mathrm{i}\mathrm{Im}G_{\mathrm{F}}(x,x'). \tag{7.155}$$

The Feynman Green function may be evaluated using the phase, or weight $\exp(\mathrm{i}S/\hbar)$, by analogy with eqn. (5.95):

$$\begin{aligned} G_{\mathrm{F}}(x,x') &= \int \frac{\mathrm{d}^n\mathbf{k}\mathrm{d}\omega}{(2\pi)^{n+1}}\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\left[\frac{1}{\omega+\mathrm{i}D\mathbf{k}^2-\mathrm{i}\epsilon} - \frac{1}{\omega-\mathrm{i}D\mathbf{k}^2+\mathrm{i}\epsilon}\right] \\ &= \int \frac{\mathrm{d}^n\mathbf{k}\mathrm{d}\omega}{(2\pi)^{n+1}}\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\frac{-2\mathrm{i}D\mathbf{k}^2}{\omega^2+D^2\mathbf{k}^4-\mathrm{i}\epsilon}. \end{aligned} \tag{7.156}$$

For thermal or other distributions it will be somewhat different. We may now compare this (in momentum space) with the linear response equation:

$$\langle\rho\rangle(k) = \mathrm{Im}G_{\mathrm{F}}(k)F = \frac{2D\mathbf{k}^2}{\omega^2+D^2\mathbf{k}^4}\,F. \tag{7.157}$$

Thus, eliminating the source from both sides of this equation, we may define the instantaneous 'D.C.' ($\omega \to 0$) diffusion constant, given by the Kubo-type relation,

$$\langle D(\omega\to 0)\rangle = \lim_{\omega\to 0}\left(\lim_{\mathbf{k}\to 0}\frac{\omega^2}{\mathbf{k}^2}\,G_{\mathrm{F}}(k)\right). \tag{7.158}$$

If we take $G_{\mathrm{F}}$ from eqn. (7.156), we see the triviality of this relation for purely collisionless quantum fluctuations of the field, $\langle\rho\rangle$. By taking the fluctuation average to be $\exp(\mathrm{i}S/\hbar)$, we simply derive a tautology. However, once we switch on thermal fluctuations or quantum interactions (for which we need to know about quantum field theory), the Feynman Green function picks up a temperature dependence and a more complicated analytical structure, and this becomes non-trivial; see eqn. (6.61). Then it becomes possible to express $D$ in terms of independent parameters, rather than as the phenomenological constant in eqn. (7.143).

### 7.5.3 *Forced Brownian motion*

A phenomenological description of Brownian motion for particles in a field is given by the Langevin model. Newton's second law for a particle perturbed by random forces may be written in the form

$$m\frac{\mathrm{d}v^i}{\mathrm{d}t} = F^i - \alpha v^i, \tag{7.159}$$

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where $v^i$ is the velocity of a particle in a field and $\alpha$ is a coefficient of friction, by analogy with Stokes' law. This equation clearly expresses only a statistical phenomenology, and it cannot be derived from an action principle, since it contains explicitly velocity-dependent terms, which can only arise from statistical effects in a real dynamical system. The forcing term, $F^i$, is a random force. By this, we mean that the time average of this force is zero, i.e. it fluctuates in magnitude and direction in such a way that its time average vanishes:

$$\langle F(t)\rangle = \frac{1}{T}\int_{\bar{t}-T/2}^{\bar{t}+T/2} F(t)\,\mathrm{d}t = 0. \tag{7.160}$$

We may solve this equation simply, in the following ways.

*Green function approach* Consider the general solution of

$$a\frac{\mathrm{d}u}{\mathrm{d}t} + bu = f(t), \tag{7.161}$$

where $a$ and $b$ are positive constants. Using the method of Green functions, we solve this in the usual way. Writing this in operator/source form,

$$\left(a\frac{\mathrm{d}}{\mathrm{d}t} + b\right) u = f(t), \tag{7.162}$$

we have the formal solution in terms of the retarded Green function

$$u(t) = \int \mathrm{d}t' G_\mathrm{r}(t,t') f(t'), \tag{7.163}$$

where

$$\left(a\frac{\mathrm{d}}{\mathrm{d}t} + b\right) G_\mathrm{r}(t,t') = \delta(t,t'). \tag{7.164}$$

Taking the Fourier transform, we have

$$G_\mathrm{r}(t-t') = \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\omega(t-t')}}{(-\mathrm{i}a\omega + b)}. \tag{7.165}$$

This Green function has a simple pole for $t - t' > 0$ at $\omega = -\mathrm{i}b/a$, and the contour is completed in the lower half-plane for $\omega$, making the semi-circle at infinity vanish. The solution for the field $u(t)$ is thus

$$\begin{aligned} u(t) &= \int \mathrm{d}\tau \int \frac{\mathrm{d}\omega}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}\omega(t-\tau)}}{(-\mathrm{i}a\omega + b)} f(\tau) \\ &= \int_{-\infty}^{t} \mathrm{d}\tau \frac{1}{2\pi} - 2\pi\mathrm{i}\left(\frac{1}{-\mathrm{i}a}\mathrm{e}^{\frac{b}{a}(\tau-t)} f(\tau)\right) \\ &= \frac{1}{a}\int_{-\infty}^{t} \mathrm{d}\tau\ f(\tau)\,\mathrm{e}^{\frac{b}{a}(\tau-t)}. \end{aligned} \tag{7.166}$$

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The lower limit of the integral is written as minus infinity since we have not specified the time at which the force was switched on, but we could replace this by some finite time in the past by specifying boundary conditions more fully.

*Differential equation approach* Although the Green function method is straightforward and quite simple, this eqn. (7.161) can also be solved by an alternative method. When $f(t) = 0$ it is solved by separation of variables, giving

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{b}{a}\,u$$
$$u(t) = u_0\,\mathrm{e}^{-\frac{b}{a}t}, \tag{7.167}$$

for some constant $u_0$. This is therefore the complementary function for the differential equation. If the forcing term $f(t)$ is non-zero, this hints that we can make the equation integrable by multiplying through by the integrating factor $\exp(-bt/a)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{b}{a}t}u(t)\right) = \frac{1}{a}\left(a\frac{\mathrm{d}u}{\mathrm{d}t} + b\,u(t)\right)\mathrm{e}^{\frac{b}{a}t}$$
$$\mathrm{e}^{\frac{b}{a}t}u(t) = \frac{1}{a}\int_0^t \mathrm{d}\tau\; f(\tau)\mathrm{e}^{\frac{b}{a}\tau}$$
$$u(t) = \frac{1}{a}\int_0^t \mathrm{d}\tau\; f(\tau)\mathrm{e}^{\frac{b}{a}(\tau-t)}. \tag{7.168}$$

This is exactly analogous to making a gauge transformation in electrodynamics. Note that, since the integral limits are from 0 to $t$, $u(t)$ cannot diverge unless $f(t)$ diverges. The lower limit is by assumption. The general solution to eqn. (7.161) is therefore given by the particular integral in eqn. (7.168) plus an arbitrary constant times the function in eqn. (7.167). The solutions are typically characterized by exponential damping. This reproduces the answer in eqn. (7.166) marginally more quickly than the tried and trusted method of Green functions. This just goes to show that it never does any harm to consider alternative methods, even when in possession of powerful methods of general applicability.

*Diffusion and mobility* Langevin's equation plays a central role in the kinetic theory of diffusion and conduction. Let $\dot{x}^i = v^i$, then, multiplying through by $x$, we have

$$mx\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = m\left[\frac{\mathrm{d}}{\mathrm{d}t}(x\dot{x}) - \dot{x}^2\right] = -\alpha x\dot{x} + xF(t). \tag{7.169}$$

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Taking the kinetic (ensemble) average of both sides, and recalling that the fluctuating force has zero average, we have that

$$m\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(x\dot{x}) \right\rangle = m\frac{\mathrm{d}}{\mathrm{d}t}\langle x\dot{x}\rangle = kT - \alpha\langle x\dot{x}\rangle, \tag{7.170}$$

where we have used the result from kinetic theory (the equi-partition theorem) that $\frac{1}{2}m\langle \dot{x}^2\rangle = \frac{1}{2}kT$. We can solve this to give

$$\langle x\dot{x}\rangle = C\,\mathrm{e}^{-\alpha t/m} + \frac{kT}{\alpha}. \tag{7.171}$$

At large times, the first of these terms decays and the system reaches a steady state. We may integrate this to give

$$\langle x^2\rangle = \frac{2kT}{\alpha}t. \tag{7.172}$$

This tells us the mean square position. By comparing this to the diffusion equation in eqn. (7.146) we find the effective diffusion coefficient

$$D = \frac{kT}{\alpha}. \tag{7.173}$$

A related application is that of electrical conduction. Consider the same diffusion process for charges $e$ in a uniform electric field $E$. The average of the Langevin equation is now

$$m\frac{\mathrm{d}\langle v^i\rangle}{\mathrm{d}t} = eE^i - \alpha\langle v^i\rangle, \tag{7.174}$$

since $\langle F\rangle = 0$. In a steady state, the average acceleration is also zero, even though microscopically there might be collisions which cause fluctuations in the velocity. Thus we have, at steady state,

$$eE^i = \alpha\langle v^i\rangle. \tag{7.175}$$

We define the *mobility*, $\mu$, of the charges, for an isotropic system, as

$$\mu = \frac{\langle v^i\rangle}{E^i} = \frac{e}{\alpha}. \tag{7.176}$$

The mobility is related to the diffusion constant by the Einstein relation

$$\frac{\mu}{D} = \frac{e}{kT}. \tag{7.177}$$

In an anisotropic system, there might be different coefficients for diffusion and mobility in different directions. Then, eqn. (7.176) would become a tensor relation, $\mu_{ij}Ev^i/E_j$.

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## 7.6 Vortex fields in 2 + 1 dimensions

Although one generally avoids speaking of particulate matter in field theory, since classically it is used to describe mainly smooth, continuous fields, there are occasions on which the solutions to the equations of motion lead unambiguously to pointlike objects. One such situation is the case of vortices.

Vortices are charged, singular objects which arise in some physical systems such as the non-linear Schrödinger equation. Vortices have the property that they acquire a phase factor, by an Aharonov–Bohm-like effect, when they wind around one another. They can usually be thought of as pointlike objects which are penetrated by an infinitely thin line of magnetic flux. In $2 + 1$ dimensions, vortices are also referred to as *anyons*, and have a special relationship with Chern–Simons field theories. It might seem strange that a field variable $\phi(x)$, which covers all of space and time, could be made to represent such singular objects as vortices. As we shall see in the following example, this is made possible precisely by the singular nature of Green functions.

Consider a field, $\phi(x)$, representing pointlike objects in two spatial dimensions with coordinates denoted for simplicity by $r = (x, y)$. We define the winding angle, $\theta$, between any two pointlike objects in the field by

$$\theta(r - r') = \tan^{-1}\frac{\Delta y}{\Delta x} = \tan^{-1}\frac{y - y'}{x - x'}. \tag{7.178}$$

Notice that $\theta(r - r')$ is a function of coordinate differences between pairs of points. We shall, in fact, relate this winding angle to the Green function $g(x, x')$, for the Laplacian in two dimensions, which was calculated in section 5.4.4.

### 7.6.1 A vortex model

The study of Chern–Simons theories is motivated principally by two observations: namely that important aspects of the quantum Hall effect are described by a Chern–Simons theory, and that a viable theory of high-temperature superconductivity should be characterized by a parity-violating, anti-ferromagnetic state. Symmetry considerations lead to an action which does not possess space-reflection symmetry. The Chern–Simons action fits this prescription. These two physical systems are also believed to be essentially two-dimensional, planar systems.

In its most primitive form, the action for the Chern–Simons model may be written in $(2 + 1)$ dimensional flat spacetime as

$$S = \int \mathrm{d}t\mathrm{d}^2x \left( (D^\mu\Phi)^\dagger(D_\mu\Phi) + m^2\Phi^2 + \frac{\lambda}{6}\Phi^4 + \frac{1}{2}\mu\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \right). \tag{7.179}$$

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The equation of motion is thus

$$\frac{1}{2}\mu\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = J^{\mu}. \tag{7.180}$$

The gauge-invariant current, $J^{\mu}$, is introduced for convenience and represents the interaction with the matter fields arising from the gauge-covariant derivatives in eqn. (7.179). We shall not consider the full dynamics of this theory here; rather, it is interesting to see how the singular vortex phenomenon is reflected in the field variables.

### 7.6.2 Green functions

The basic Green function we shall use in the description of two-dimensional vortices is the inverse Laplacian which was derived in section 5.4.4, but it is also useful to define and elaborate on some additional symbols which are encountered in the literature. We shall use the symbol $r^i$ as an abbreviation for the coordinate difference $\Delta r^i = \Delta x^i = x^i - x^{i\prime}$, and the symbol $\Delta r$ for the scalar length of this vector. Some authors define a Green function vector by

$$\begin{aligned} G^i(r-r') &= \epsilon^{ij}\partial_j\, g(r-r') \\ &= -\frac{1}{2\pi}\epsilon^{ij}\frac{\hat{r}^j}{r-r'}, \end{aligned} \tag{7.181}$$

where $\hat{r}$ is a unit vector along $r - r'$. The two-dimensional curl of this function is thus

$$\begin{aligned} \nabla\times\mathbf{G}(r) &= \epsilon^{ij}\partial_i G_j(r-r') \\ &= \epsilon^{ij}\epsilon_{jk}\partial_{\mathrm{i}}\partial_k g(r-r') \\ &= -\nabla^2 g(r-r') \\ &= \delta(r-r'). \end{aligned} \tag{7.182}$$

In other words, $G^i(r - r')$ is the inverse of the curl operator.

### 7.6.3 Relationship between $\theta(r - r')$ and $g(r - r')$

To obtain a relationship between the coordinates and the winding function $\theta(r)$, we note that

$$\begin{aligned} \partial_i \tan\theta(r-r') &= \partial_i\left(\frac{\sin\theta(r-r')}{\cos\theta(r-r')}\right) \\ &= \partial_i\theta(r-r')\ \sec^2\theta(r-r') \\ &= \partial_i\theta(r-r')(1+\tan^2\theta(r-r')). \end{aligned} \tag{7.183}$$

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From eqn. (7.178), this translates into

$$\begin{aligned} \partial_i\theta &= \frac{\partial_i\left(\frac{\Delta y}{\Delta x}\right)}{1+\left(\frac{\Delta y}{\Delta x}\right)^2} \\ &= \frac{\Delta x(\partial_i\Delta y)-\Delta y(\partial_i\Delta x)}{r^2} \\ &= -\epsilon_{ij}\,\frac{\hat{r}^j}{r}. \end{aligned} \tag{7.184}$$

This last form is significant since the logarithm has a similar property, namely

$$\epsilon^{ij}\partial_j \ln|r-r'| = \epsilon^{ij}\frac{\hat{r}_j}{r-r'}, \tag{7.185}$$

and thus we immediately have the relationship:

$$-\frac{1}{2\pi}(\partial_i\theta(r-r')) = G(r) = -\epsilon_{ij}\partial_j g(r-r'). \tag{7.186}$$

It is understood that partial derivatives acting on $r-r'$ act on the first argument $r$.

### 7.6.4 Singular nature of $\theta(r-r')$

The consistency of the above relations supplies us with an unusual, and perhaps somewhat surprising relation, namely

$$\epsilon^{ij}\partial_{\mathrm{i}}\partial_j\theta(r-r') = 2\pi\,\delta(r-r') \tag{7.187}$$

or

$$[\partial_1,\partial_2]\theta(r-r') = 2\pi\delta(r-r'). \tag{7.188}$$

This relation tells us that the partial derivatives do not commute when acting on the function $\theta(r)$. This is the manifestation of a logarithmic singularity in the field, or, physically, the non-triviality of the phase accrued by winding vortices around one another. Although the field is formally continuous, it has this non-analytical property at every point.

Using complex coordinates $z = x^1 + \mathrm{i}x^2$ and conjugate variables $\overline{z}$, the above discussion leads to the relations in complex form:

$$\begin{aligned} \partial_z(\overline{z})^{-1} &= \partial_z\partial_{\overline{z}}\ln|z|^2 \\ &= \pi\delta(|z|). \end{aligned} \tag{7.189}$$

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