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HYPERBOLIC DIFFERENTIAL EQUATIONS AND WAVE PROPAGATION

BY
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Mr. President, ladies and gentlemen, I have had great pleasure in accepting the invitation to deliver this tenth Jeffery–Williams lecture. As one of that generation who were strongly influenced by Ralph Jeffery and Lloyd Williams, I also appreciate the challenge of maintaining the high standard that these lectures named in their honour have established. May I also say that the lecture[23] earlier this morning by Professor Dieudonnè was certainly fortunate for us. Without prearrangement, and by an advantageous turn of events, there has been given that one lecture on partial differential equations, by that one lecturer, that could best prepare you to appreciate what the position of hyperbolic differential equations is within the domain of modern mathematics. Now I shall be more detailed and more specific and I trust that you will at least find the contrast refreshing.

1. **The hyperbolic type.** Hyperbolic differential equations were the last of the three major types to receive a thorough and satisfactory development as a part of the theory of linear partial differential equations. The beginnings of the topic do go back equally far, to D'Alembert, Poisson, Riemann, and Hadamard, so there is a chronological parallel in the developing theories of the different types of partial differential equations. But apart from that, there is quite a relative contrast rather than a parallel. Whereas the study of elliptic and parabolic equations led to the global study of smooth solutions, the study of hyperbolic equations showed just the opposite tendency, namely that of the propagation of singularities. And where the study of elliptic differential equations led, partly through analytic function theory, to many fine formulas, precise results, and particular developments, still the theory of hyperbolic differential equations was held back for several decades because no one could readily express what was going on. Only with the advent of the theory of distributions did it become possible to look at the subject as a single topic, as in Courant.[(5)]

There is a further parallel with the theory of non-existence. For the non-existence of solutions, which occurs on a dense set among linear partial differential equations with variable coefficients, was banished long ago from the

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subset of hyperbolic equations. For the solution of the Cauchy problem, or initial value problem, for hyperbolic differential equations, including the wave equation and higher equations, there was first developed a reasonably comprehensive C^∞ existence theorem (Courant,[5] p. 642, Gårding,[9] Leray.[15] Thus we have on the one hand, for hyperbolic equations, a very complete theory of existence, and on the other, for elliptic equations, a detailed expression of the form, or character, of the solution. (Hörmander[13]).

The formulas for the Fourier transform in n dimensions, namely

$$Ff(\xi)=\hat{f}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{ix\cdot\xi}f(x)\,dx$$

and its inverse

$$F^{-1}\hat{f}(x)=f(x)=\frac{1}{(2\pi)^{n/2}}\int_{R^n} e^{-ix\cdot\xi}\hat{f}(\xi)\,d\xi,$$

will be used to study one further contrast with the elliptic type, namely the finite domain of support, or finite speed of wave propagation. Whereas the value of a solution of an elliptic differential equation expresses everything that happens everywhere, the value of a solution of a hyperbolic equation may reflect only those events occurring on a particular set. This is embodied within the Fourier transform by the Paley–Wiener–Schwartz theorem, which states that an analytic function is the Fourier–Laplace transform of a distribution of compact support, if and only if the condition (Hörmander,[13] p. 21)

$$|u(\zeta)|\le C(1+|\zeta|)^N e^{A|\operatorname{Im}\zeta|}$$

holds for some constants C, N and A.

Given this condition, it is often possible to show that a solution of the wave equation is zero outside a certain domain. Let me give a particular calculation for the wave equation, to show something of the nature of the transformations involved in finding an elementary solution. I shall follow the Dirac notation or symbolism for the delta function or delta distribution. This is, after all, the same line of argument that Professor Dieudonnè referred to earlier this morning.

2. **Elementary solution of the wave equation.** Suppose we Fourier transform the wave equation

$$\frac{\partial^2 u}{\partial t^2}=\Delta u+\delta_n(x)\,\delta(t)$$

in which the right hand term is defined by an inner product over the space variables in n dimensions and the single time dimension. Then the problem of finding the elementary solution becomes the problem of inverting by the

Fourier transform a particular function

$$\hat{u}(\xi, t)=\frac{\sin |\xi| t}{|\xi|}$$

where the absolute value $|\xi|$ denotes the Euclidean norm in n dimensions. Here $\hat{u}(\xi, t)$ is the solution of the Fourier-transformed differential equation

$$\hat{u}_{tt}(\xi, t)=-|\xi|^2\hat{u}(\xi, t)+\delta(t)$$

with $\hat{u}(\xi, t)\equiv 0$ for $t<0$. Consequently the elementary solution is the inverse transform

$$\begin{aligned} u(x, t)&=\frac{1}{(2\pi)^n}\int_{R^n} e^{ix\cdot\xi}\frac{\sin|\xi|t}{|\xi|}\,d\xi \\ &=\frac{1}{(2\pi)^n}\int_0^\infty \frac{\sin|\xi|t}{|\xi|}|\xi|^{n-1}\,d|\xi|\int_\Omega e^{i|x||\xi|\mu}\,d\Omega_n \end{aligned}$$

where $\mu=\cos\theta$ with θ the angle between x and ξ, and $d\Omega_m$ denotes the solid angle element in R^m. Now

$$\begin{aligned} d\Omega_n&=\sin^{n-2}\theta\,d\theta\,d\Omega_{n-1} \\ &=-(1-\mu^2)^{(n-3)/2}\,d\mu\,d\Omega_{n-1} \end{aligned}$$

so that

$$u(x, t)=\omega_{n-1}\int_0^\infty M(|x|\,|\xi|)\sin(|\xi|t)|\xi|^{n-2}\,d|\xi|$$

where

$$M(x, t)=\frac{1}{(2\pi)^n}\int_{-1}^1 e^{is\mu}(1-\mu^2)^{(n-3)/2}\,d\mu$$

and ω_{n-1} denotes total solid angle in R^{n-1}.

Since the factor $|\xi|^{n-2}$ in the expression above may lead to divergence of the integral in the classical (though not the distribution) sense, we consider the following device that applies for n odd. We write

$$u(x, t)=\omega_{n-1}(-1)^{(n-1)/2}\left(\frac{\partial}{\partial t}\right)^{n-2}\int_0^\infty M(|x|\,|\xi|)\cos(|\xi|t)\,d|\xi|$$

and note that the integral can be written as a Fourier integral

$$\frac{1}{2|x|}\int_{-\infty}^\infty M(s)e^{ist/|x|}\,ds, \qquad s=|x|\,|\xi|.$$

From the definition of $M(s)$ it is now clear that $(2\pi)^{n/2}M(s)$ and the geometric factor $(1-\mu^2)^{(n-3)/2}H(1-\mu^2)$ are themselves a Fourier transform pair. Hence the Fourier integral above is equal to

$$\frac{1}{(2\pi)^n}\frac{1}{2|x|}\left[1-\frac{t^2}{|x|^2}\right]^{n-3/2} H\left[1-\frac{t^2}{|x|^2}\right].$$

Thus the elementary solution u now takes the following form for n odd;

$$u(x,t)=\frac{\omega_{n-1}}{(2\pi)^{n-1}}(-1)^{(n-1)/2}\left(\frac{\partial}{\partial t}\right)^{n-2}\left\{\frac{1}{2|x|}\left(1-\frac{t^2}{|x|^2}\right)^{(n-3)/2} H\left(1-\frac{t^2}{|x|^2}\right)\right\}.$$

As the polynomial being differentiated has degree $n-3$, the coefficient of the Heaviside function H in its Leibnitz expansion will be zero. Since also $H'=\delta$, the support of all other terms lies on the light cone $t=|x|$.

Thus the initial value problem with data $u(x,0)=0$, $u_t(x,0)=\psi(x)$ has solution (Courant,[5] vol. II, p. 686, Gelfand and Shilov,[11] pp. 154, 288)

$$u(x,t)=\frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^t (t^2-r^2)^{(n-3)/2} rQ(x,r)\,dr$$

where the spherical mean value

$$Q(x,r)=\frac{1}{\omega_n}\int_{\Omega_n}\psi(x+\alpha r)\,d\Omega_n,\qquad |\alpha|=1.$$

It can be shown by direct calculation or by the method of descent, that the same formula gives the solution for n even.

Returning to the distribution notation, the elementary solution itself for n even or odd can be written

$$u(x,t)=c\delta^{((n-3)/2)}(t^2-|x|^2)H(t).$$

Comparing these expressions we are reminded again that the solution distribution is a linear functional on the space of initial values or data functions $\psi(x)$. Its support is in odd space dimensions the surface, and in even space dimensions the interior of the light cone with vertex at the field point (Fig. 1). Thus the initial value problem, at least in the case of constant coefficients and including the wave equation, received very early an explicit treatment that was later brought within the distribution framework.

3. **The mixed or initial and boundary value problem.** A more difficult problem, and one that has not yet been fully worked out in every case, is the mixed initial and boundary value problem for the wave equation. Imagine a time axis, a domain R of space, and a space time cylinder $D=R\times[0,T]$. This domain for the solution corresponds to waves reverberating in a region R (Fig. 2).

Consider a wave equation

$$L(u)=u_{tt}-\Delta u+b\cdot\nabla u+cu=f$$

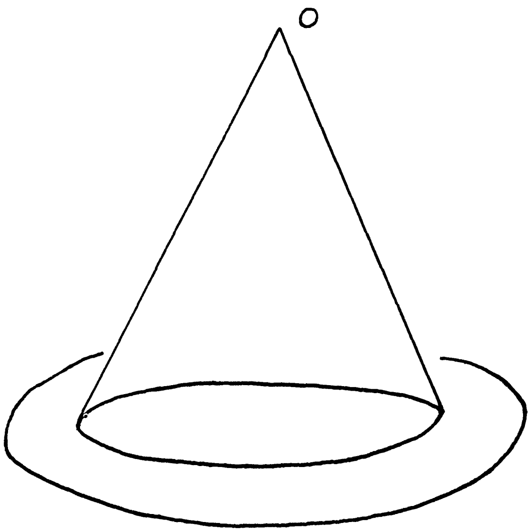

Fig. 1. Retrograde wave cone.

in D, with initial or Cauchy data

$$u(x, t) = u_0(x), \qquad u_t(x, t) = u_1(x), \qquad x \in R$$

and a boundary condition of the Dirichlet type

$$u(x, t) = g(x, t), \qquad (x, t) \in S \times [0, T]$$

where $S = \partial R$. Boundary conditions of the Neumann type $\partial u/\partial n = g$ or the more general Robin type $(\partial u/\partial n) + hu = g$ are also appropriate for this second order equation. This problem can be discussed by analogy with the elliptic

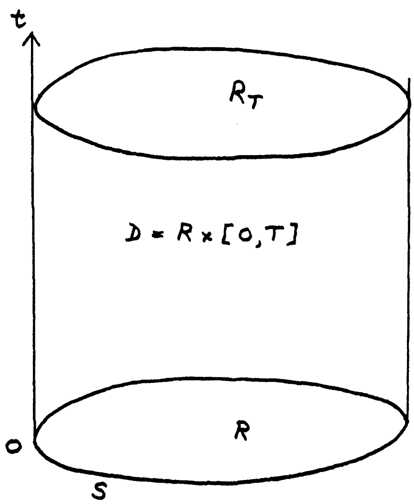

Fig. 2. Mixed problem on a space-time cylinder.

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differential equation, provided that the domain R does not vary with the time t. (Duff,[7] p. 45). Imagine that we can calculate the eigenvalues $\lambda_n = k_n^2$ and complete orthonormal eigenfunctions $u_n(P)$ of the space domain R and the chosen boundary condition $u = 0$ for the Helmholtz equation

$$\Delta u_n + k_n^2 u_n = 0.$$

We can then calculate as follows a Green's function for the wave equation $u_{tt} = c^2 \Delta u$, with Dirichlet boundary condition and initial conditions $u = 0$, $u_t = g$ on R. Let the solution be expanded as a Fourier series

$$u(x, t) = \sum_n c_n(t) u_n(P)$$

where the coefficients $c_n(t)$ are given by

$$c_n(t) = \int_R u(P) u_n(P)\, dV_p.$$

It is easily seen that $c_n(0) = 0$ and

$$c_n'(0) = \int_R g(P) u_n(P)\, dV_p = g_n$$

the Fourier coefficient of $g(P)$. The differential equation for $c_n(t)$ is

$$c_n''(t) + c^2 k_n^2 c_n(t) = 0$$

and thus it is easily shown that

$$c_n(t) = g_n \frac{\sin(k_n ct)}{k_n c}.$$

There is an evident analogy with the earlier Fourier transform of the elementary solution in R^n. Then the solution function is given by

$$u(R, t) = \sum_n c_n(t) u_n(P)$$
$$= \int_R K(P, Q, t) g(Q)\, dV_Q$$

where

$$K(P, Q, t) = \sum_n u_n(P) u_n(Q) \frac{\sin(k_n ct)}{k_n c}.$$

While this series is by no means convergent in the classical sense, it does always converge in the distribution sense. By summation of this series can be synthesized the wave fields, the wave fronts, the reflected waves and the support and character of the solution that will reverberate within that domain. In an exterior or infinite domain there will be a continuous spectrum leading to an integral rather than a sum, while for the full Euclidean space R^n we again encounter the expression derived earlier for the elementary solution. In fact only recently has the recognition of distribution convergence made possible the explicit representation and calculation of Green's functions on a scale comparable to that of elliptic theory. And yet the analogy is extremely close.

Here are some properties of the Green's "function" K: setting $Lu = (1/c^2)u_{tt} - \Delta u$ we have, applying the operator

$$L_P K(P, Q, t) = \delta(t)\,\delta(P, Q)$$

where the point-supported delta function has the dyadic expansion

$$\delta(P, Q) = \sum_n u_n(P)u_n(Q).$$

The initial values at $t=0$ of interest are

$$\lim_{t\to 0+} K(P, Q, t) = 0$$

$$\lim_{t\to 0+} K_t(P, Q, t) = \delta(P, Q)$$

while the boundary values are

$$K(P, Q, t) = 0, \qquad P \in S.$$

A general representation formula for the most general initial-boundary value problem can be found by applying Green's theorem to the product region $d = R \times [0, T]$ with functions $u(Q, T)$ and $K(P, Q, T+\varepsilon-T)$. Letting $\varepsilon \to 0$ we can deduce the general solution

$$\begin{aligned} u(P, t) = {} & -c^2 \int_D K(P, Q, t-\tau)Lu(Q, \tau)\,dV_Q\,d\tau \\ & + \int_R K(P, Q, t)\frac{\partial u}{\partial t}(Q, 0)\,dV_Q - \int_R K_t(P, Q, t)u(Q, 0)\,dV_Q \\ & + c^2 \int_0^t d\tau \int_S \frac{\partial K}{\partial n_Q}(P, Q, t-\tau)u(Q, \tau)\,dS_Q. \end{aligned}$$

4. **Mixed problems for equations with variable coefficients.** While the case of a fixed domain is essentially solved, the case where a transformation of coordinates involving time is needed to reduce the problem to a fixed domain is very much more difficult. For we then are confronted by an equation with variable coefficients and this brings forth a new problem.

A classical property of the hyperbolic equation is that of the finite speed of wave propagation. Given a point P of space-time, it can be shown that the value of a solution at P depends only on values of the data within and on the retrograde wave cone or characteristic cone with vertex at P. Thus any change of data outside that retrograde cone will not influence the values at P. The domain of dependence has compact support.

Let us now look at the mixed initial-boundary value problem in the case of variable coefficients, and where we try to construct a suitable integral estimate for the solution. While the method of integral estimates has close connections with elliptic theory as in the Dirichlet problem, it is also the means by which the Cauchy problem was first solved for equations with variable coefficients in a wide class. It can be shown that the energy integral, a quadratic integral in u and its derivatives up to a suitable order one less than the order of the equation, grows at a limited rate which is at most exponential in the time variable. This makes it possible to bound the energy integral at any particular time. It is then possible to apply the Sobolev inequalities and certain other instrumentalities of functional analysis to show that the solution exists. Convergence theorems, or certain existence theorems of operator theory, can be used to establish existence and to show that the solution also has a certain lesser degree of pointwise regularity in the smoothness of derivatives. For the C^∞ case the smoothness of solutions is also C^∞ (with certain characteristic exceptions) but in the C^r case the solution loses $\frac{1}{2}n - m + 1$ derivatives, where n is the space dimension and m the order of the equation.

Returning to the mixed problem for the wave equation, let us consider more general boundary conditions. For simplicity we can work locally, taking advantage of the finite speed of wave propagation, so we select an initial hyperplane $t=0$ and a boundary hyperplane $x=0$ for the domain $x>0$. Let y_i, $i=1,\ldots,n-1$ denote space variables parametrizing the boundary, and let $y_1=y$. Then the most general linear first order boundary condition is

$$Bu = pu_t + qu_x + ru_y + wu = g.$$

By formal reductions we can arrange that $g=0$ and $w=0$ without loss of generality. Now let us inquire what possible energy estimates can be established with quadratic integrals of first order derivatives, for the wave equation

$$Lu = u_{tt} - u_{xx} - u_{yy} - \sum_i u_{y_i y_i} = f.$$

Thus we shall need a *multiplier*

$$Mu = \alpha u_t + \beta u_x + \gamma u_y$$

in the integral identity. The idea of a multiplier expression of first order derivatives was suggested by Zaremba and has since been exploited by a number of workers. (Courant,[5] Gårding[9]).

After some calculation we obtain an integral identity

$$\iint_D MuLu\, dV\, dt = \int_{R_T} \left[\frac{\alpha}{2}\left(u_t^2 + u_x^2 + u_y^2 + \sum_j u_{y_j}^2\right) + \beta u_x u_t + \gamma u_y u_t\right] dV$$
$$+ \int_S \int_0^T \left[-\frac{\beta}{2}\left(u_t^2 + u_x^2 - u_y^2 - \sum_j u_{y_j}^2\right) - \alpha u_x u_t - \gamma u_x u_y\right] dS\, dt$$
$$+ I + Q$$

where the term I contains initial data and Q is a quadratic expression in first derivatives integrated over $D = R \times [0, T]$. Our energy integral is essentially the quadratic expression

$$E(t) = \int_{R_T} \left[\frac{\alpha}{2}\left(u_t^2 + u_x^2 + u_y^2 + \sum_j u_{y_j}^2\right) + \beta u_x u_t + \gamma u_y u_t\right] dV$$

and the form integrated is positive definite if and only if $\alpha > 0$ and $\alpha^2 > \beta^2 + \gamma^2$. We assume this condition which forces the multiplier coefficient vector (α, β, γ) to be *timelike*, that is, to fall within the forward nappe of the light-cone. Then the term Q can be bounded by an integral

$$\int_0^T E(t)c(t)\, dt,$$

and we must now assure ourselves only that the boundary term or "wall" term is suitably semi-definite. The integrated surface terms contain the quadratic form

$$Q(0) = -\frac{\beta}{2} u_t^2 - \frac{\beta}{2} u_x^2 + \frac{\beta}{2} u_y^2 + \frac{\beta}{2} \sum_j u_{y_j}^2 - \alpha u_x u_t - \gamma u_x u_y$$

which is scarcely a reassuring combination of positive and negative terms. Clearly therefore this problem does not always have solutions, that is, non-empty sets of boundary coefficients leading to quadratic estimates.

Nonetheless, we consider the boundary condition $Bu = pu_t + qu_x + ru_y = 0$ where $p > 0$ without loss of generality. In the estimates choose $\alpha = p$, $\beta = 0$, $\gamma = r$ which is compatible with $\alpha^2 > \beta^2 + \gamma^2$ provided $|r| < p$. Then

$$\begin{aligned} -Q(0) &= 2\alpha u_x u_t + 2\gamma u_x u_y \\ &= 2u_x(pu_t + ru_y) \\ &= -2u_x(qu_x) \\ &= -2qu_x^2 \geq 0 \end{aligned}$$

provided $q \leq 0$. We conclude that the estimate will hold if $p > 0$, $q \leq 0$ and $|r| < p$.

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Noting that the initial data terms I in the integral are bounded, say by K, we now obtain a Gronwall inequality

$$E(t) \leq K + \int_0^t c(t')E(t')\,dt',$$

from which the estimate

$$E(t) \leq K \exp\left[\int_0^t c(t')\,dt'\right]$$

is easily deduced. The construction and properties of the solution then follow as described earlier.

The mixed problem for the second order hyperbolic equation, with complex coefficients in the boundary condition, was recently systematically studied by Ikawa[14] and his results are the most complete. The integral identity has also been reviewed by Gårding[10] who has established an elegant invariant expression for the various boundary terms.

I would now like to give one particular example of the mixed initial boundary value problem, which will show how the singularities of the solution evolve. From the earliest work in this subject it has been clear that if a signal issued suddenly from a point source, then the forward characteristic cone with vertex at that space-time event would be the carrier of the singularities, or singular support, of the solution. A nonzero intensity of radiation would propagate along the bicharacteristic strips, forming in modern terminology the Lagrangian paths and the wave front set. As long as this signal propagates through empty space the characteristic cone is the only singular support. However when the wave is reflected at a wall, there may occur an additional singularity. I shall illustrate with the reflected part of the elementary solution when there is a first derivative linear boundary condition at the wall $x = 0$, a problem first considered by Bondi.[3]

If the boundary condition involves only even or only odd derivatives with respect to x, the solution can be constructed by the method of reflected images and its singular support is the reflected wave cone. But if the boundary condition involves both even and odd terms in $\partial/\partial x$, there may appear a further tangential singularity which is an oblique ultrasonic wave front. (Fig. 3). The intersection of this wave front with the wall will travel along the wall faster

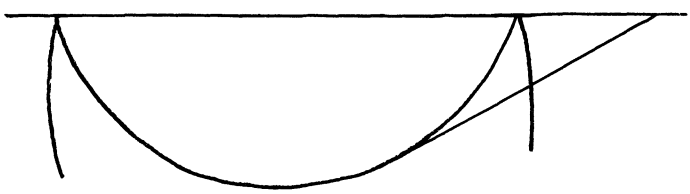

Fig. 3. Oblique ultrasonic wave front.

than the velocity of propagation: this is an apparent or virtual motion and the wave front itself propagates at the correct "speed of light", or of sound, in the direction normal to itself. For the calculation of this solution, which has a higher order singularity on the new oblique wave front, we refer to Bondi[(3)] and Duff.[(7)] The ultrasonic wave may in some cases be supported exactly on this wave front, in which case the enclosed domain behind it is known as a 'lacuna', just as in the case of the wave equation in odd space dimension where the solution is supported exactly on the wave cone and so is independent of data values within that cone as well as outside it.

5. **Higher order hyperbolic equations.** Let us now look at the higher order hyperbolic equations. Whereas for the wave equation we are dealing with a spherical wave front, for higher order hyperbolic equations the wave surfaces are more complicated. The condition that an equation $P_m(D)u=0$ of order m should be hyperbolic is essentially the condition that the wave surface, or more basically the normal cone $P_m(\xi)=0$ which is its tangential dual, should be totally real (Fig. 4). This is the exact opposite extreme to the elliptic case when zero is the only real point of this algebraic surface. Some sections of normal surfaces and their tangentially dual wave surfaces are shown in the figures, based on cases given by Atiyah, Bott and Gårding,[(2)] and Duff.[(7)] The second order case is a spherical or ellipsoidal surface. The third order case shows an unpaired or odd component dual to a cusped triangle (Fig. 5). The fourth order case is shown with a perturbation which is a strictly hyperbolic surface having only simple points corresponding to simple and distinct real zeros of the algebraic surface (Fig. 6). The limiting case of a non-strictly hyperbolic equation having multiple points corresponds on the wave surface to the formation

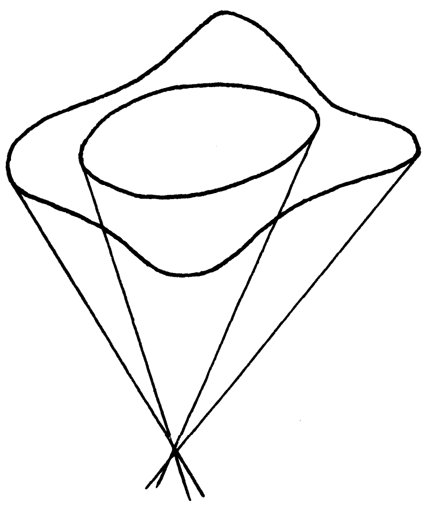

Fig. 4. Normal cone $P(\xi)=0$.

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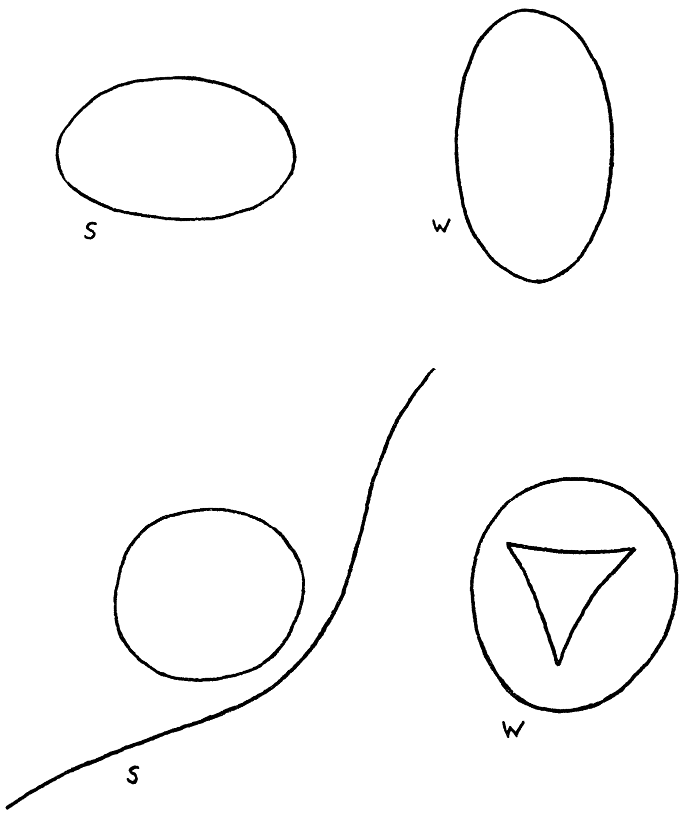

Fig. 5. Normal and wave surfaces of orders 2 and 3.

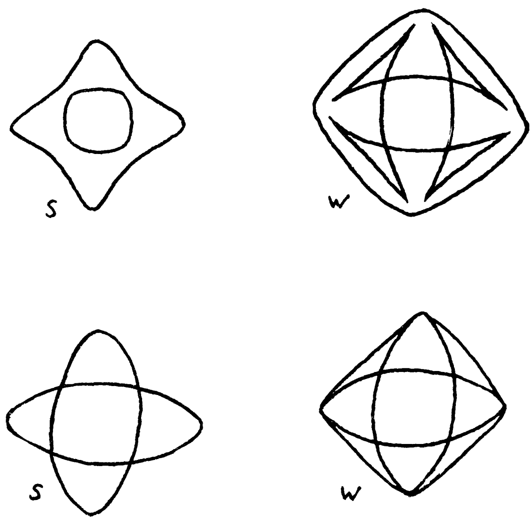

Fig. 6. Normal and wave surfaces of order 4. Above: strictly hyperbolic, Below: Non-strictly hyperbolic.

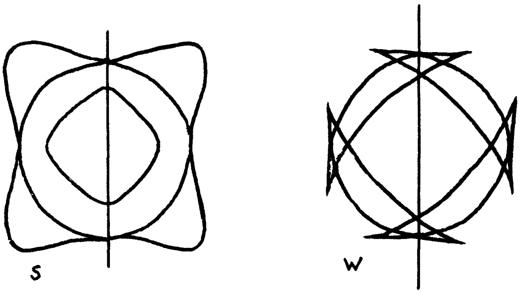

Fig. 7. Inverse and wave surfaces of nickel in cross-section. Cubic symmetry.

of two-sided limiting planar components. This occurs for example at the conical points of the Fresnel surface of crystal optics which play a role in the conical refraction of light in a crystalline medium. In the general strictly hyperbolic case the cusped wave front sheets correspond to the inflection loci of the normal surfaces which in the limiting cases tend to multiple points.

The next example of order 6 is typical of those in elastic wave propagation in an anisotropic medium (Fig. 7). Here the tangentially dual wave surface may unfold as many as nine fronts expanding past a given point. The following case in Fig. 8 is also of order 6, and suggests a product polynomial with a lower dimensional factor indicated by the parallels. This is essentially the case of magnetohydrodynamic wave propagation which does involve a preferred one-dimensional component. The second example in Fig. 8 shows three crossed ellipses and a dual wave surface with trebly unfolded cusps.

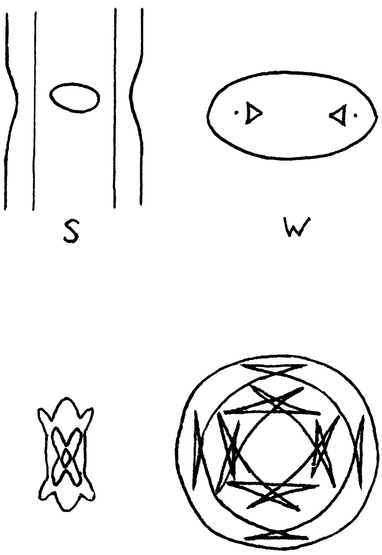

Fig. 8. Normal and wave surfaces of order 6.

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With these geometric complications in mind, let us look at the elementary solution. This can again be calculated for an equation with constant coefficients

$$P(D)u = P(D_t, D_x)u = \delta_n(x)\delta(t)$$

by means of Fourier transforms. Thus $P(D_t, \xi)\hat{u}(\xi, t) = \delta(t)$ and this ordinary differential equation in t has the Green's function

$$G(\xi, t) = i^{m+1} \sum_{k=1}^{m} \frac{e^{i\lambda_k(\xi)t}}{\dfrac{\partial P}{\partial \lambda}(\xi, \lambda_k(\xi))}$$

$$= \frac{1}{2\pi i^m} \oint \frac{e^{i\lambda t}\, d\lambda}{P(\lambda, \xi)}$$

where the contour encircles all (real and simple!) roots $\lambda = \lambda_k(\xi)$ of $P(\lambda, \xi) = 0$. Then the elementary solution is

$$K(x, t) = G(\ , t) = \frac{1}{(2\pi)^{n+1} i^m} \int_{R^n} e^{ix\cdot\xi}\, d\xi \oint \frac{e^{i\lambda t}\, d\lambda}{P(\xi, \lambda)}.$$

Such expressions were studied by Herglotz and Petrowsky during the 20's and 30's and Abelian integral expressions for even and odd space dimensions were constructed by them (Herglotz,(12) Petrowsky(18)). Detailed studies have been made by Petrowsky, and later Atiyah, Bott, Gårding(2) of the regions defined by the sheets of the wave cone and of the lacunas, those regions in which the elementary solution vanishes, or at least is of polynomial form so that certain of its derivatives will vanish.

6. **The Riemann matrix of a hyperbolic system.** However I should like to give here some details of the corresponding formulas for the case of a first order symmetric hyperbolic system which is the form that appears most frequently in physical applications. Consider a linear system of first order,

$$\frac{\partial \mathbf{u}}{\partial t} = \sum_{\nu=1}^{n} A_\nu \frac{\partial \mathbf{u}}{\partial x_\nu}$$

where $\mathbf{u}$ is an m-component vector and A_ν are $m \times m$ constant matrices. We assume the system is hyperbolic, that is the characteristic roots of the matrix pencil $A(\eta) = \sum_{\nu=1}^{n} A_\nu \eta_\nu$ are all real for real η_ν, a condition clearly satisfied for symmetric coefficient matrices A_ν. For a hyperbolic system, the initial problem $\mathbf{u}(x, 0) = \mathbf{g}(x)$ is correctly set and the solution can be expressed as a convolution

$$\mathbf{u}(x, t) = \int_{R^n} R(x - z, t)\mathbf{g}(z)\, dz$$

where $R(x, t)$ denotes the Riemann matrix or matrix elementary solution: thus

$\boldsymbol{R}$ satisfies the differential system in matrix form

$$\frac{\partial \mathbf{R}}{\partial t}=\sum_{\nu=1}^{n} A_\nu \frac{\partial R}{\partial x_\nu},$$

while initially, $R(x,0)=\delta(x)E$ with E the $m\times m$ unit matrix. For example, the one-dimensional wave system of rank two: $u_t=v_x$, $v_t=u_x$ has Riemann matrix

$$R(x,t)=\frac{1}{2}\begin{pmatrix}\delta(x+t)+\delta(x-t), & \delta(x+t)-\delta(x-t)\\ \delta(x+t)-\delta(x-t), & \delta(x+t)+\delta(x-t)\end{pmatrix}.$$

Calculation of the Fourier transform shows that (Duff[(6)])

$$\hat{R}(\xi,t)=\frac{1}{(2\pi)^{n/2}}\exp(-iA(\xi)t)$$

so that the inverse transform yields

$$R(x,t)=\frac{1}{(2\pi)^n}\int_{R^n}\exp(-i(x\cdot\xi E+A(\xi)t))\,d\xi.$$

To analyse this expression we need in effect just those techniques mentioned by Professor Dieudonné in his earlier lecture. First, the finite dimensional vector space (the infinite dimensional spaces having not received as yet such a detailed development). Second we require spectral theory, which expresses itself here in the construction of the real characteristic roots $\lambda_k(\xi)$ of $A(\xi)$ satisfying

$$\det(A(\xi)-\lambda E)=0.$$

Then the k^{th} sheet of the normal surface is given by the equation $\lambda_k(\xi)=1$. We shall also find it useful to construct the pseudo-differential operators related to $\lambda_k(\xi)$. Let T be the diagonalizing matrix of $A(\xi)$ so that $A(\xi)=T\operatorname{diag}(\lambda_k(\xi))T^{-1}$. Also we may observe that

$$T=(t_i(\lambda_j,\eta))$$

where the polarization vector t_i is a homogeneous function of λ_j, η of degree zero. Then it follows that

$$\exp(-iA(\xi)t)=T\exp(-i\operatorname{diag}(\lambda_k(\xi)))T^{-1},$$

and hence

$$R(x,t)=\frac{1}{(2\pi)^n}\int_{|\eta|=1}T\operatorname{diag}\int_0^\infty\exp(-i(x\cdot\eta+t\lambda_k(\eta))|\xi|)\cdot|\xi|^{n-1}\,d|\xi|\,T^{-1}\,d\Omega_\eta$$

where $\xi=|\xi|\eta$, $|\eta|=1$. Setting the phase $s=x\cdot\eta+t\lambda_k(\eta)$ we consider the inner integral

$$\int_0^\infty\exp(-is|\xi|)|\xi|^{n-1}\,d|\xi|=(-i)^n(n-1)!(s-i0)^{-n}.$$

This useful distribution formula is in effect the Fourier transform of a distribution as set forth in Gelfand–Shilov,[11] vol. 1, p. 172. Here

$$(s-i0)^{-n}=s^{-n}+\frac{i\pi(-1)^{n-1}}{(n-1)!}\,\delta^{(n-1)}(s)$$

where s^{-n} is a generalized function with exact definition given by Gelfand and Shilov. We now find

$$\begin{aligned}R(x,t)&=\frac{(-i)^n(n-1)!}{(2\pi i)^n}\int_{|\eta|=1} T(\eta)\mathrm{diag}(x\cdot\eta+t\lambda_k(\eta)-i0)^{-n}T^{-1}(\eta)\,d\Omega_\eta\\&=\frac{(n-1)!}{(2\pi i)^n}\int_{|\eta|=1}(x\cdot\eta E+tA(\eta)-i0)^{-n}\,d\Omega_\eta.\end{aligned}$$

The Riemann matrix is thus expressed as the spherical mean of the spectral resolution of the singular matrix distribution function $(s-i0)^{-n}$. Note that for $t=0$ the phases all coincide and $R(x,t)$ reduces to

$$\frac{(n-1)!}{(2\pi i)^n}\int_{|\eta|=1}(x\cdot\eta-i0)^{-n}\,d\Omega_\eta\cdot E=\delta_n(x)E$$

according to the plane wave representation of the delta function (see Gelfand–Shilov,[11] vol. 1, p. 77).

In component form we can write

$$R_{ik}(x,t)=\frac{(n-1)!}{(2\pi i)^n}\int_{|\eta|=1}\sum_j\frac{t_i(\lambda_j,\eta)\bar{t}_k(\lambda_j,\eta)}{(x\cdot\eta+t\lambda_j(\eta)-i0)^n}\,d\Omega_\eta$$

and to discuss the singularity on the j^{th} wave front sheet W_j we consider the phase $S_j=x\cdot\eta+t\lambda_j(\eta)$. Let s denote the normal distance of a field point from the wave front W_j which is an envelope of plane wave fronts with $S_j(\eta)=0$. By expanding the root $\lambda_j(\eta)$ about the stationary point $\eta^1=0$ of this phase, we find, asymptotically

$$S_j=s+\frac{t}{2}c_{kl}\eta_k^1\eta_l^1+\cdots$$

where the second order coefficients c_{kl} describe the curvature of the normal surface (Fig. 9). Then, applying a distributional form of the principle of stationary phase, we find the leading singular term of the elementary solution to be

$$\frac{(n-1)!}{2^{(3n-1)/2}\pi^{(n+1)/2}i^n t^{(n-1)/2}}\frac{t_i(\lambda_j,\eta)\bar{t}_k(\lambda_j,\eta)}{\sqrt{\det c_{kl}}}I_-^{n_+/2}I_+^{n_-/2}(s-i0)^{-n}$$

Fig. 9. Singularity at an ordinary point.

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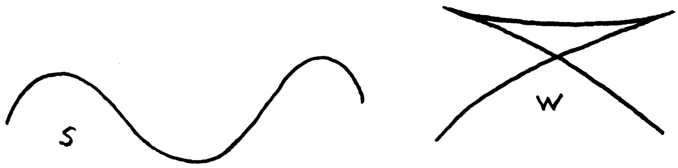

Fig. 10. Singularity at a cusp locus of W corresponding to an inflection point of S.

where (n_+, n_-) is the signature of the quadratic phase expansion above, and the fractional integrals are given by

$$I_{\pm}^{\alpha}f(x)=\frac{(\pm)^{\alpha}}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt=\frac{1}{\Gamma(\alpha)}\int_0^b u^{\alpha-1}f(x\mp u)\,du$$

(Duff,[6,7] Ludwig[16]). Here the det c_{kl} can be shown to be the Gaussian curvature of the normal surface at the value of η involved. In the strictly convex case either n_+ or n_- will be zero, and then wherever the delta-function part of $(s-i0)^{-n}$ appears, the support of the singular term will be a point, or will lie entirely on one side of the wave sheet.

The cusp formations and multiple sheets of the wave surface correspond to inflections or vanishing curvatures on the normal surface. In the calculation of singularities this leads to higher order cases of the stationary phase lemma, and to fractional powers in the expansions.

Geometrically we may imagine a wave front as an envelope of plane wave fronts, where the plane waves initially present at the source are the spherical average representing the delta function. They are separated by a spectral resolution into the variously polarized plane waves that propagate with each characteristic wave velocity along the corresponding wave sheet. The singular distribution $(s-i0)^{-n}$ is then smoothed by the spherical superposition of wave elements to a degree determined by the curvature on the normal surface. The leading singular term is

$$R_{ik}(x,t)\sim ct_i(1,\xi)\bar{t}_k(1,\xi)t^{-\Sigma(1/n_s)}I_{\pm}^{\Sigma(1/n_s)}(x\cdot y+t-i0)^{-n}$$

where n_s is the index or "order" of curvature in the s^{th} dimension. Thus n_s is equal to 2 in the case of nonzero curvature, and $n_s=3,4,\ldots$ in higher order cases of vanishing curvature. In these higher cases the spatial singularity is fractionally sharper, while the exponent of decay for long times is fractionally less (Fig. 10). At a double point of the normal surface the index n_s is in effect reduced to 1, while the two corresponding surfaces of the wave front have a common tangent (Fig. 11). On that tangent or ruled surface, the singularity is $\frac{1}{2}$

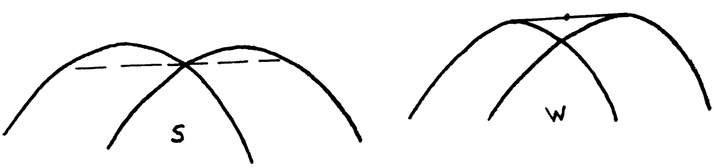

Fig. 11. Singularity on a ruled surface of W corresponding to a double point of S.

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degree lower for each such dimension, while the attenuation in t is as much higher.

7. **Wave reflections.** Let us now consider the general problem of wave reflection from a planar wall. Rather than give details of the construction of the solution, I will discuss the associated dual geometry. Consider the case of a fourth order equation that gives rise to two wave fronts, one fast front and one slow one. This is the situation for the equations of crystal optics, to take an example. Figure 12 shows the situation for the reflection of a slow incident wave front. Now the dual relation between the wave fronts and the algebraic normal surface inverts the order of the sheets, so that the slow wave front corresponds to the outer sheet of the normal surface. At the first moment of incidence on the wall, and shortly thereafter, the bundle of plane waves representing the reflected wave front is supported on a small region of the outer normal surface. As time goes on the angular extent of this reflected

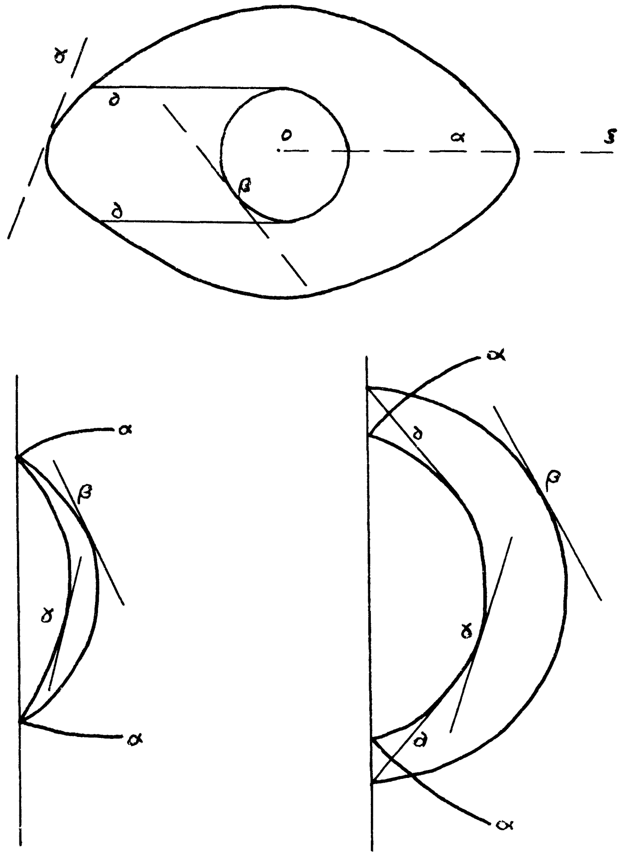

Fig. 12. Reflection of a slow incident wave front. (a) early stage, (b) late stage. Above: normal surface.

bundle will increase, and from an early moment it will also generate a component from the other sheet, according to the algebra of the spectral resolution and the boundary conditions. This correspondence is shown by the horizontals or parallels ∂ in the diagram; these one-dimensional generators are dual to wave elements parallel to the planar boundary. Then at a certain later instant, the supporting set of the fast wave surface will reach a branch point. This corresponds on the planar boundary to the instant of breakaway of the fast reflected wave front at the instant no elements of its enveloping set of plane waves can travel as slowly parallel to the boundary as the newly adjoined slow wave front elements. Thus the fast front breaks away and its trace on the boundary generates a planar wave singularity tangent to the slow wave front. This is known as a head wave.

In this general problem there are also ultrasonic waves, as shown in the diagram. Imagine that the homogeneous equation, or system, has a plane wave solution with wave front oblique to the boundary, such that ultrasonic speeds of the boundary intersection occur (Fig. 13). This will arise if the boundary

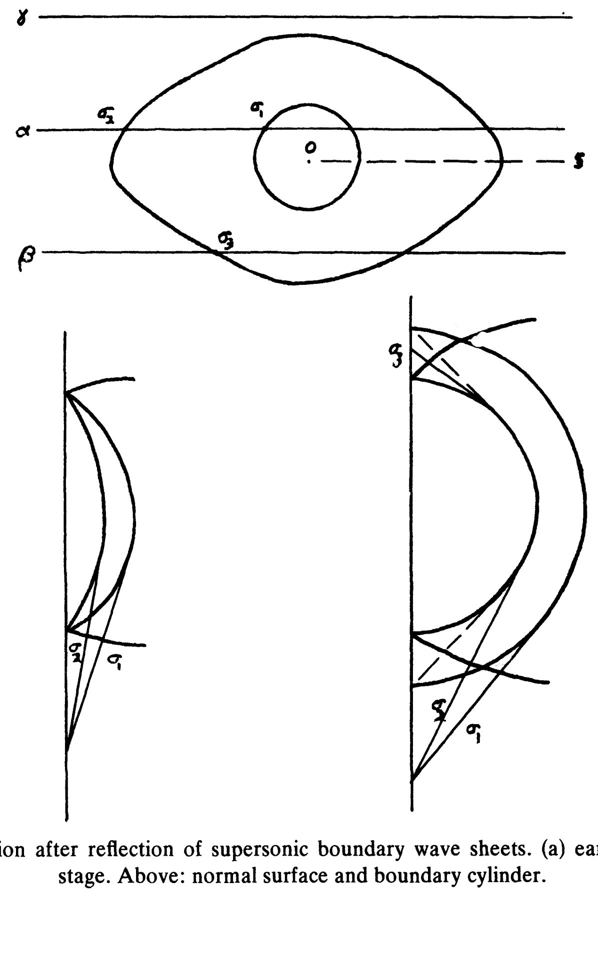

Fig. 13. Formation after reflection of supersonic boundary wave sheets. (a) early stage, (b) late stage. Above: normal surface and boundary cylinder.

conditions are redundant, or dependent, or non-coercive, for the particular propagation direction, and the boundary discriminant vanishes. Looking through the literature of applied mathematics, one can observe that the occurrence of such zeros forms a kind of history of the topics treated. For there are zeros named after Kelvin, Rayleigh, Love, Stoneley, and others, who discovered them in particular problems. The presence of a locus of zeros corresponds to a cylindrical boundary wave surface, the elements of which give rise by intersections with the algebraic normal surface followed by the usual envelope process to the planar boundary wave fronts of the supersonic type, and to the subsonic surface waves of the Rayleigh wave type in a semi-infinite elastic medium. As with case γ of Fig. 13, these last may be numerically large, will decrease exponentially with depth or distance from the boundary, and are generally smooth.

It is still not known whether other types of singularities can arise in a hyperbolic problem, due for instance to lower order terms in the differential equation or the boundary conditions. Very recently, Tsuji[20] and Wakabayashi[21] have studied the reflected wave front singularities in very general cases. However their concrete examples of lateral waves are of nearly the same geometrical character as those discussed here.

8. **Variable multiplicities.** It has been shown that the existence problem for hyperbolic equations has been solved in a comparatively satisfactory way. That is, the Cauchy problem is well-posed. The general question, just how large a class of equations have solutions is still subject to some uncertainty. The natural property of hyperbolic equations is, that all characteristic roots are real and distinct. Thus the difficulties of proving an existence theorem are centred about the occurrence of non-real or multiple roots. The most extensive recent result related to multiple roots is a theorem by Zeman[22] which may be described briefly, as follows.

Let $P(x, t, D_{x_i}, D_t)v = f$ be a hyperbolic differential equation with variable coefficients of order m, where

$$P(x, t, D_{x_i}, D_t)v = P_m v + P_{m-1} + \cdots$$

with principal part P_m and k^{th} order terms P_k for $k = 0, 1, \ldots, m$. Here $P_m(0, 0, 0, 1) \neq 0$ so the plane $t = 0$ is not characteristic. We use the multi index notations

$$D_{x_i} = \frac{1}{i}\frac{\partial}{\partial x_i}, \qquad D_t = \frac{1}{i}\frac{\partial}{\partial t}, \qquad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j$$

and assume that P_k are C^∞ in x and t. Let L_x^0 be the ring of pseudo differential operators of order zero in the x variables, and define the norm

$$\|u\|_s^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi.$$

Define the commutator

$$[A, B] = AB - BA$$

for differential or pseudo differential operators A, B.

The characteristic roots $\lambda_j(x, t, \xi)$ now depend on x and t and are still positively homogeneous of degree 1 in ξ; by definition we have

$$P_m(x, t, \xi, \lambda) = \prod_{j=1}^{m} (\lambda - \lambda_j(x, t, \xi)).$$

We assume the most general hyperbolic condition $\operatorname{Im} \lambda_j(x, t, \xi) \geq 0$, for $t \geq 0$ and $|\xi| = 1$, $\xi \in R^n$, thus excluding any exponential solutions that could increase arbitrarily rapidly for large times $t > 0$. We also assume that these roots are C^{m-1} in x, t, and ξ in this domain, as is necessary if we are to commute smoothly the factors $\partial_j = D_t - \lambda_j(x, t, \xi)$ in P_m as pseudo differential operators. Let the maximum multiplicity of the roots λ_j be denoted by $r(x, t, \xi)$ so that r is a positive integer, $1 \leq r \leq m$.

To complete these preparations, let W be the module over L_x^0 generated by all $W_{(m-i)}$, $i = 1, 2, \ldots, m$, where W_{m-i} in turn is generated by the product operators $\partial_1 \cdots \partial_{m-i}$ each containing exactly $m - i$ distinct factors. We can now state Zeman's form of the existence theorem, as follows:

Let $[\partial_i, \partial_j] = a\partial_i + b\partial_j + N$ where $a, b, N \in L_x^0$ for $1 \leq i \leq m$, $1 \leq j \leq m$. Also let $P_{m-j} \in W_{m-j}$ for $0 < j \leq r - 1$. Then the Cauchy problem for $Pu = f$ is well-posed.

The proof of existence and continuous dependence on data depends on an estimate

$$\sum_{|\alpha| \leq m-r} \left(\frac{1}{T}\right)^{m-|\alpha|} \|D_u^\alpha\|_s \leq C\|Pu\|_s$$

where $u \in C_0^\infty(\Omega)$ and $\Omega = R_x[0, T]$, while $D_t^j u(x, 0) = 0$ for $j = 1, \ldots, m-1$. For this estimate, the commutators $[\partial_i, \partial_j]$ must fall within the pseudodifferential module L_x^0 as described above.

This elegant theorem illustrates clearly a growing tendency in partial differential equations, which Professor Dieudonné described earlier, namely the use of pseudodifferential operators. They represent the properties of characteristic roots by constructed combinations, which are particularly simple, in fact linear, because the characteristic roots are homogeneous of degree 1 in ξ. The presence of the commutators represents the problem of smooth transition from one sheet of the normal surface to another, which arises in the case of multiple roots. How far this line of development can go in the study of more general types of hyperbolic and linear partial differential operators, only time will tell (see also Chazarain[4] and Mizohata-Ohya[17]).

9. **Control of solutions.** To conclude this lecture, I would like to mention a special topic – the theory of control of solutions of hyperbolic differential

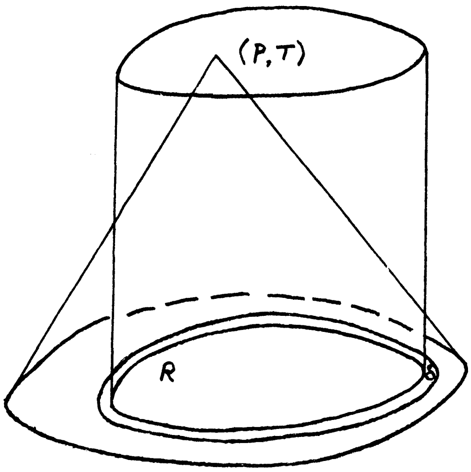

Fig. 14. Wave cone covering extended domain.

equations. As physical systems are described by such equations, and mankind wishes to control everything, everywhere, so he must learn the control of solutions of the partial differential equations. I shall now give one result of Russell[(19)].

Consider a region R with boundary B, and a boundary condition $Bu=f$, together with initial data $u=\phi$, $u_t=\psi$ at $t=0$, for the wave equation $Lu=0$. Choose the function f, defined on the boundary over a time interval $[0,T]$, such that u will vanish identically in R at time T, where T is minimal.

To see how this problem might be approached, let us take the case of odd space dimension, say three dimensions. First suppose that region R can be smoothly embedded in a larger neighbouring region R_δ (Fig. 14). Then consider a typical solution u and extend it to a larger space-time cylinder domain by smoothing its initial values to zero outside R but within R_δ. Let us then consider the initial value problem in the full three-dimensional space with the given initial values now having compact support. The value of this solution $u_1(P,t)$ at a point (P,T) where T is so large that the retrograde characteristic cone meets the initial plane $t=0$ entirely outside R_δ, must be a zero value: $u_1(P,T)=0$. Now let us choose $f=Bu$, so that the control function f has values obtained from the smoothed-off solution. This amounts to saying, in a sense, "let Nature take its course". It does effectively show that there is a time T, which is the maximal wave transit time of R, such that the problem has a solution with control interval time at least equal to T. It is evident that no solution is possible for shorter times since a wave propagating within R can endure that long without contact or control opportunity at the boundary of R.

For this problem there is a substantial difference with the even-dimensional case in which the solution does not vanish in the interior of the wave cone. However the elementary solution does become small for large times within the wave cone, and reduction to an integral equation leads to a solution for sufficiently large T. Whether the solution is possible for the same minimal T as in odd dimensions, is still an open problem.

The last problem I wish to mention is one involving periodic cycles and the convergence of a given solution to a time-periodic solution. This I first encountered in the problem of tidal energy. Another instance is in the control of the daily cycle of pressure variations in a natural gas pipeline system. In such problems there is a hyperbolic equation or system, such as the one-dimensional equations of gas dynamics, or the oceanographic equations for an estuary. Extensive work has been done on the Principle of Limiting Amplitude for linear or mildly nonlinear equations, which shows that convergence to a sinusoidal or periodic solution will occur in the steady-state (Amerio–Prouse[(1)]). In the control problem, however, the solution is influenced by values of certain parameters, such as value settings or pumping periods. Thus switching conditions, and influence variables which satisfy the adjoint differential equations, must be defined to control the controls. In numerical cases these combined systems can be very unstable, or temperamental, in the time dimension, so that the existence and still more the calculation of such solutions cannot always be demonstrated (Duff[(8)]).

The time variation of such a system does closely parallel the course of evolution with time, and hence the common phrase "evolution equation". But the limiting periodic or steady state solution, which can be likened to the cycle of successive generations in nature, shows in each of its values the influence of every other stage of the cycle. Thus in many instances is seen an "artificial intelligence" or apparent anticipation of cause and effect, which also occurs in Nature—the system prepares itself to do the right thing at the right time. In Nature, the control pattern for the cycle of successive generations is the survival of the fittest. Such a process actually works itself out in a numerical calculation by continual adjustment of the control parameters as the evolving solution converges to its steady state or periodic limit.

Perhaps we have now reached our limit, so this may be a fitting note on which to conclude this survey of hyperbolic equations and wave propagation.

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