A CONTRIBUTION TO THE SOLUTION OF THE COMPACT CORRECTION PROBLEM FOR OPERATORS ON A BANACH SPACE

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1. Introduction. We consider the hypothesis that an operator T on a given Banach space can always be perturbed by a compact operator K in such a way that, whenever a complex number λ is in the semi-Fredholm region of $T+K$, then $T+K-\lambda$ is either bounded below or surjective. The hypothesis has its origin in the work of West **[11]**, who proved it for Riesz operators on Hilbert space. In this paper, we reduce the general Banach space problem to one of considering only operators of a special type, operators which are, in a spectral sense, natural generalizations of the Riesz operators studied by West.

We shall adopt the following notation and terminology, where X denotes a Banach space.

$B(X)$ will denote the algebra of bounded linear operators on X, and, for $T \in B(X)$
ker(T) denotes the null space of T;
nul(T), the *nullity* of T, is the dimension of ker(T);
def(T), the *defect* of T, is the dimension of $X/\overline{T(X)}$;
ind(T), the *index* of T, is nul(T)—def(T), provided not both those quantities are infinite;
min ind(T), the *minimum index* of T, is the smaller of nul(T) and def(T).

If Y is a linear subspace of X, then $T|_Y: Y \to X$ is the restriction of T to Y, and $T_Y: Y \to Y$ will denote a compression of T to Y. If Y is invariant for T, then T_Y is bounded and is independent of the projection used for the compression. Otherwise, T_Y need not, of course, be bounded. Where it is significant, the exact compression intended will be obvious from the text.

We shall be concerned with the following subsets of $B(X)$:
$\Phi_+(X) = \{T \in B(X)\colon T(X)$ closed and nul$(T) < \infty\}$ is the set of upper semi-Fredholm operators on X;
$\Phi_-(X) = \{T \in B(X)\colon T(X)$ closed and def$(T) < \infty\}$ is the set of lower semi-Fredholm operators on X.

For $T \in B(X)$,
$\sigma(T) = \{\lambda \in \mathbb{C}\colon \lambda - T$ is not invertible in $B(X)\}$ is the *spectrum* of T;
$\sigma_{s-F}(T) = \{\lambda \in \mathbb{C}\colon \lambda - T \notin \Phi_+(X) \cup \Phi_-(X)\}$ is the *semi-Fredholm spectrum* of T;
$\rho_{s-F}(T) = \mathbb{C} \backslash \sigma_{s-F}(T)$ is the *semi-Fredholm region* of T;
$\sigma_W(T) = \sigma(T) \backslash \{\lambda \in \rho_{s-F}(T)\colon \text{ind}(\lambda - T) = 0\}$ is the *Weyl spectrum* of T;
$\sigma_s(T) = \sigma_{s-F}(T) \cup \{\lambda \in \rho_{s-F}(T)\colon \text{min ind}(\lambda - T) \neq 0\}$ will be called the *subspectrum* of T;
$\sigma_p^0(T) = \{\lambda \in \sigma(T) \backslash \sigma_W(T)\colon \mu \notin \sigma(T)$ for $|\mu - \lambda|$ sufficiently small and non-zero$\}$ is the set of *Riesz points* of the spectrum of T.

An operator $T \in B(X)$ for which

$$\sigma_s(T) \backslash \sigma_{s-F}(T) = \sigma_p^0(T)$$

will be called a *West operator*. An operator $T \in B(X)$ for which there exists a compact operator $K \in B(X)$ such that

$$\sigma(T+K) = \sigma(T) \backslash \sigma_p^0(T)$$

will be said to have a *West decomposition*. If $T \in B(X)$ and there exists a compact operator $K \in B(X)$ such that

$$\sigma_s(T+K) = \sigma_{s-F}(T)$$

then K will be called a *compact correction* for T.

West [**11**] showed that every Riesz operator on a Hilbert space has a compact correction. His result was extended by Stampfli [**10**]. The conjecture we are considering is that every operator on a given Banach space X has a compact correction. This was shown to be true for Hilbert spaces by Apostol [**2**], and for a certain special class of Banach spaces, including c_0 and l^p $(1 \leq p < \infty)$, by Davidson and Herrero [**3**]. The general problem remains unsolved. Our principal task here is to show that the above conjecture is equivalent to the conjecture that every West operator on X has a West decomposition.

2. Reducing the problem to countable proportions and discovering the compliant nature of the remaining points. In this section, we shall show that the nub of the compact correction problem is the removal from an operator's subspectrum of a countable set of isolated points, each of which is a point of finite ascent or descent for the operator. We begin with two lemmas.

LEMMA 2.1. *Let X be a Banach space and let T be a semi-Fredholm operator on X with* $\operatorname{ind}(T) \leq 0$ (*resp.* $\operatorname{ind}(T) \geq 0$). *Then there exists a finite rank operator $F \in B(X)$ such that $T + \lambda F$ is bounded below* (*resp. surjective*) *for all $\lambda \in \mathbb{C} \backslash \{0\}$.*

Proof. We shall assume, without loss of generality, that $\operatorname{ind}(T) \leq 0$. Let $P \in B(X)$ be a projection onto $\ker(T)$, and let $Q \in B(X)$ be a projection of the same finite rank as P which satisfies $QT = 0$.

Choose A, $B \in B(X)$ such that $AQB = P$, and set $F = QB$. Then F is finite rank, so we certainly have

$$T + \lambda F \in \Phi_+(X) \qquad (\lambda \in \mathbb{C} \backslash \{0\}),$$

and it suffices to show that the $T + \lambda F$ are injective. But

$$\begin{aligned}(T+\lambda F)x = 0 &\Rightarrow Tx = -\lambda Fx \\ &\Rightarrow Tx = 0 = Fx \qquad (\text{since } QT = 0 \text{ and } QF = F) \\ &\Rightarrow x = Px = AFx = 0.\end{aligned}$$

The second lemma is a splitting of the well known Riesz–Schauder theorem for poles

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of finite multiplicity of the resolvent set. This splitting was originally due to Laffey and West [**6**], who used holomorphic left inverses of Allan [**1**] to achieve the result for Fredholm operators. That their results belong properly to ring theory and Banach algebra theory was shown in [**8**]. In the lemma below, we do the splitting for all semi-Fredholm operators on Banach space. The proof is adapted from one given by West [unpublished] for Atkinson operators.

LEMMA 2.2. *Let X be a Banach space and let T be an upper (resp. lower) semi-Fredholm operator on X. Suppose there exists a positive real number ε such that $T-\lambda$ is bounded below (resp. surjective) for all $\lambda \in \mathbb{C}$ with $0<|\lambda|<\varepsilon$. Then T has finite ascent (resp. descent).*

Proof. We suppose that $T \in \Phi_+(X)$. The result for $T \in \Phi_-(X)$ will follows by duality.

Let

$$W=\bigcap_{n=1}^{\infty} T^n(X).$$

Then W is a closed invariant subspace for T, and it is easy to show that T_W is surjective (see, for example, Kato [**5**, p. 241]). Hence $(T-\lambda)_W$ is surjective for sufficiently small λ.

Now it is clear that $(T-\lambda)_W$ is bounded below whenever $T-\lambda$ is; so it follows immediately that T_W is invertible in $B(W)$, for, otherwise, T_W would be in the boundary of the invertible elements of $B(W)$, and, being therefore a two-sided topological divisor of zero, could not be surjective. In particular,

$$W \cap \ker(T)=\{0\}.$$

Since $\mathrm{nul}(T)<\infty$, we have, for sufficiently large $n \in \mathbb{N}$.

$$\ker(T) \cap T^n(X)=\ker(T) \cap W=\{0\},$$

and hence

$$\ker(T^{n+1})=\ker(T^n).$$

We now utilize Lemma 2.1 in a method of proof not unlike that used by Stampfli [**10**] when he removed connected index-zero components from the spectrum of an operator on Hilbert space.

THEOREM 2.3. *Let X be a Banach space and let ε be a positive real number. Let $T \in B(X)$. Then there exists an operator $K \in B(X)$ in the closure of the finite rank operators, and of norm not exceeding ε, such that the set $\sigma_s(T+K)\backslash\sigma_{s-F}(T)$ has accumulation points only in $\sigma_{s-F}(T)$, and is therefore countable. Moreover, for each $\mu \in \sigma_s(T+K)\backslash\sigma_{s-F}(T)$, the operator $T+K-\mu$ has either finite ascent or finite descent, depending on whether its index is non-positive or non-negative.*

Proof. Let $\Lambda=\{\lambda_n : n \in \mathbb{N}\}$ be a dense subset of $\rho_{s-F}(T)$.

We choose a sequence $(\varepsilon_n)_{n=0}^{\infty}$ of positive real numbers and a sequence $(F_n)_{n=1}^{\infty}$ of finite rank operators inductively as follows. Let $\varepsilon_0=\frac{1}{2}\varepsilon$. For each successive $n \in \mathbb{N}$, we

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choose ε_n and F_n according to the following rule: use Lemma 2.1 to choose F_n to have norm less than $\frac{1}{2}\varepsilon_{n-1}$ and be such that $T-\lambda_n+\sum_{i=1}^{n} F_i$ is bounded below or surjective; and use the openness of the set of operators which are either bounded below or surjective to choose $\varepsilon_n<\frac{1}{2}\varepsilon_{n-1}$ such that, for each $U\in B(X)$ with $\|U\|<\varepsilon_n$, the operator $T-\lambda_n+U+\sum_{i=1}^{n} F_i$ is bounded below or surjective.

Now $\sum_{n=1}^{\infty} F_n$ converges to an operator $K\in B(X)$ with $\|K\|<\varepsilon$. Furthermore, for each $n\in\mathbb{N}$, we have

$$\left\|\sum_{i=n+1}^{\infty} F_i\right\|\leq\varepsilon_n$$

so that $T+K-\lambda_n$ is bounded below or surjective.

The semi-Fredholm punctured neighbourhood theorem states that, for each $\alpha\in\rho_{s-F}(T)$, we can find a complex neighbourhood G_α of α such that $\mathrm{nul}(T+K-\mu)$ and $\mathrm{def}(T+K-\mu)$ are constants for $\mu\in G_\alpha\backslash\{\alpha\}$, and do not exceed $\mathrm{nul}(T+K-\alpha)$ and $\mathrm{def}(T+K-\alpha)$ respectively. Each G_α has non-empty intersection with Λ, so that one of those constant is zero. It follows that the set $\sigma_s(T+K)\backslash\sigma_{s-F}(T)$ has no accumulation point in $\rho_{s-F}(T)$ and is therefore countable. The rest of the result is a consequence of Lemma 2.2.

3. The efficacy of this approach. Before proceeding to our main result, we shall step aside to demonstrate the efficacy of our approach by using Theorem 2.3 to give a direct proof of Apostol's Theorem [**2**] on Hilbert space. Our proof is somewhat easier than that of Apostol. One interest here is to see where the proof fails in the general Banach space context, and, towards that end, we shall couch much of the proof in the language of Banach spaces.

We begin with two easy lemmas:

LEMMA 3.1. *Let X be a Banach space and let $T\in\Phi_+(X)$ have non-positive index. Let Y be a closed linear subspace of X which is invariant for T. Then $Ty\in\Phi_+(Y)$.*

Proof. By Lemma 2.1 we can find a finite rank operator $F\in B(X)$ such that $T+F$ is bounded below. Then $(T+F)|_Y:Y\to X$ is bounded below, so that $T|_Y:Y\to X$ is upper semi-Fredholm, and hence $T_Y\in\Phi_+(Y)$.

LEMMA 3.2. *Let X be a Banach space and let $T\in B(X)$. Let Y be a closed linear subspace of X which is invariant for T. Suppose T has finite ascent, that $\ker(T^n)\subseteq Y$ ($n\in\mathbb{N}$) and T_Y is Fredholm of index zero. Let Z be any algebraic complement for Y in X. Then T_Z is an injective linear transformation on Z.*

Proof. We may assume, without loss of generality, that the ascent of T is one.

Suppose there exists $z\in Z\backslash\{0\}$ such that $T_Zz=0$. Then $Tz\notin T(Y)$, since $\ker(T)\subseteq Y$. So the codimension in Y of $T(Y)\oplus\mathbb{C}Tz$ is less than the finite quantity $\mathrm{def}(T_Y)$, which in

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turn is equal to $\text{nul}(T_Y)$. Since $\ker(T_Y)=\ker(T)$, it follows that $T(X)\cap\ker(T)\neq\{0\}$, contradicting the assumption that T has ascent one.

THEOREM 3.3. *Let H be a Hilbert space and let $V\in B(H)$. Let ε be a positive real number. Then V has a compact correction $K\in B(H)$ with*

$$\|K\|<\varepsilon+\sup\{\text{dist}(\lambda,\sigma_{s-F}(V))\colon\lambda\in\sigma_s(V)\}.$$

Proof. Firstly, we make a correction K' to V as described in Theorem 2.3, being careful to ensure that K' satisfies both

$$\|K'\|<\tfrac{1}{2}\varepsilon$$

and

$$\sup\{\text{dist}(\lambda,\sigma(V))\colon\lambda\in\sigma(V+K')\}<\tfrac{1}{2}\varepsilon.$$

Put $T=V+K'$.

We shall assume, without loss of generality, that the set

$$\Lambda=\{\lambda\in\sigma_s(T)\backslash\sigma_{s-F}(T)\colon \text{ind}(\lambda-T)\leq 0\}$$

is infinite, and that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is an enumeration of its elements.

Let k_n be the ascent of $T-\lambda_n$ $(n\in\mathbb{N})$ and let P_n be the orthogonal projection onto Y_n, where

$$Y_n=\bigoplus_{i=1}^{n}\ker(T-\lambda_i)^{k_i}\qquad(n\in\mathbb{N}).$$

Set $Y=\overline{\bigcup_{n\in\mathbb{N}}Y_n}$. Now choose $(\alpha_n)_{n\in\mathbb{N}}$ on the boundary of $\sigma_{s-F}(T)$ such that

$$|\lambda_n-\alpha_n|=\text{dist}(\lambda_n,\sigma_{s-F}(T))\qquad(n\in\mathbb{N}).$$

Then it is clear that, taking $P_0=0$, $\sum_{n=1}^{\infty}(\alpha_n-\lambda_n)(P_n-P_{n-1})$ converges to a compact operator $K''\in B(H)$ with $\|K''\|=\sup_{n\in\mathbb{N}}|\alpha_n-\lambda_n|$. Now Y_n $(n\in\mathbb{N})$ and Y are closed invariant subspaces for both T and K'', and it is easy to check that $\sigma(T_{Y_n})=\{\lambda_1,\ldots,\lambda_n\}$ and $\sigma((T+K'')_{Y_n})=\{\alpha_1,\ldots,\alpha_n\}$. In particular,

$$(T-\mu)_Y \textit{ has dense range for } \mu\in\mathbb{C}\backslash\Lambda \tag{$*$}$$

and

$$(T+K''-\mu)_Y \text{ has dense range for } \mu\in\mathbb{C}\backslash(\alpha_n)_{n\in\mathbb{N}} \tag{$**$}.$$

Let Z be any algebraic complement of Y in H. We note that $K''_Z=0$ and we make the following three checks.

(i) If $T-\mu$ is surjective, then so is $T+K''-\mu$.

Proof. If $T-\mu$ is onto H, then $\mu\notin(\alpha_n)_{n\in\mathbb{N}}$, so $(T+K''-\mu)_Y$ has dense range, by (**), and $(T-\mu)_Z$ is onto Z. So $T+K''-\mu$ has dense range, and, being lower semi-Fredholm, is onto H.

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(ii) If $T-\mu$ is bounded below, then so is $T+K''-\mu$.

Proof. Suppose $T-\mu$ is bounded below. Then $\mu \notin \Lambda \bigcup (\alpha_n)_{n\in\mathbb{N}}$, so $(T-\mu)_Y$ has dense range, by (*); it is clearly bounded below, so is invertible in $B(Y)$. Therefore $(T+K''-\mu)_Y$ is Fredholm of index zero; also, it has dense range, by (**), so it is invertible in $B(Y)$. Furthermore, since $(T-\mu)_Y$ is invertible and $T-\mu$ is injective, we have $(T-\mu)_Z$ injective. So $T+K''-\mu$ is injective, and, being upper semi-Fredholm, is bounded below.

(iii) If $\lambda\in\Lambda$ then $T+K''-\lambda$ is bounded below.

Proof. If $\lambda\in\Lambda$, then $\lambda\notin(\alpha_n)_{n\in\mathbb{N}}$, so $(T+K''-\lambda)_Y$ has dense range, by (**), and is upper semi-Fredholm by Lemma 3.1, so is onto Y. Now, for $\mu\in\mathbb{C}\backslash\{\lambda\}$ and $|\mu-\lambda|$ sufficiently small, we have both $(T-\mu)_Y$ and $(T+K''-\mu)_Y$ invertible in $B(Y)$ from (ii) above; so $(T+K''-\lambda)_Y$ is invertible in $B(Y)$ and $(T-\lambda)_Y$ is Fredholm of index zero. So $T-\lambda$ satisfies the hypotheses of Lemma 3.2. Therefore $(T-\lambda)_Z$ is injective, and so is $T+K''-\lambda$. Being upper semi-Fredholm, $T+K''-\lambda$ is therefore bounded below.

We can now perturb $(T+K'')^*$ in a similar fashion with a compact operator $L\in B(H)$. We put $K=K'+K''+L^*$, and it is clear that K is a compact correction for V.

As a finishing touch, it is easy to see that the spaces on which K'' and L^* act are orthogonal, so that $\|K\|$ satisfies the required inequality.

The norm estimate achieved in Theorem 3.3 is not the best possible by any means. Herrero [**4**] has made an analysis of that problem and has achieved sharp estimates. Apostol's estimate was better than that given above, but is achievable by incorporating the work of the next section.

Before leaving this section, we note that the statements (*) and (**) in Theorem 3.3 are true for any choice of projections P_n onto the spaces Y_n. Provided P_n and α_n can be chosen in such a way as to ensure the convergence of

$$\sum_{n=1}^{\infty}(\alpha_n-\lambda_n)(P_n-P_{n-1})$$

then the erasure of Λ from $\sigma_s(T)$ can be effected. Pursuing this line of reasoning would give us an analogue of the theorems proved by Laurie and Radjavi [**7**] for Riesz operators.

4. Reduction to West operators. In this section, we show that a trick used by Apostol [**2**, Lemma 4.2] for operators on Hilbert space is applicable in the general context. We use it to complete our reduction.

LEMMA 4.1. *Let Y and Z be Banach spaces, let $C\in B(Z,Y)$ be of finite rank; suppose $A\in B(Y)$ satisfies $C(Z)\subseteq A(Y)$ and suppose $B\in B(Z)$ has closed range and satisfies* $\ker(B)\subseteq\ker(C)$. *Then there exist solutions $T_1\in B(Z,Y)$ and $T_2\in B(Z,Y)$ of the equation*

$$T_1B=C=AT_2$$

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with

$$T_1(Z) = C(Z) \quad \textit{and} \quad \ker(T_2) = \ker(C).$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of $C(Z)$ and let $\phi_1, \ldots, \phi_n \in Z^*$ be such that

$$C = \sum_{i=1}^{n} e_i \otimes \phi_i.$$

The condition on B implies that the following set of equations defines bounded functionals $\psi_i \in B(Z)^*$:

$$\psi_i(Bz) = \phi_i(z) \qquad (z \in Z, i = 1, \ldots, n).$$

We extend the ψ_i to Z by the Hahn–Banach theorem and set

$$T_1 = \sum_{i=1}^{n} e_i \otimes \psi_i.$$

Then $T_1 B = C$ and $T_1(Z) = C(Z)$.

Also, since $C(Z) \subseteq A(Y)$, we can choose $f_1, \ldots, f_n \in Y$ such that

$$Af_i = e_i \qquad (1 \le i \le n).$$

We set

$$T_2 = \sum_{i=1}^{n} f_i \otimes \phi_i.$$

Then

$$AT_2 = C \quad \text{and} \quad \ker(T_2) = \ker(C).$$

LEMMA 4.2. *Let X be a Banach space and let $X = Y \oplus Z$ be a decomposition of X into closed subspaces. Suppose $T \in B(X)$ has representation $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ on $Y \oplus Z = X$, where $C \in B(Z, Y)$ is of finite rank, and either*

(i) *$B \in B(Z)$ is bounded below and $A \in B(Y)$ is nilpotent,*

or

(ii) *$A \in B(Y)$ is surjective and $B \in B(Z)$ is nilpotent.*

Then there exists a bounded projection of X onto Y which commutes with T.

Proof. We prove the Lemma in case (i). The other case can be handled similarly.

We suppose, then, that B is bounded below and that $n \in \mathbb{N}$ satisfies $A^n = 0$. We use Lemma 4.1 to solve successively the n pairs of equations:

$$F_1 B = C \quad \text{and} \quad F_1(Z) = C(Z),$$
$$F_i B = AF_{i-1} \quad \text{and} \quad F_i(Z) = A^{i-1}C(Z) \qquad (2 \le i \le n).$$

We put $F = -(F_1 + \ldots + F_n)$ and note that $AF_n = 0$. Then $AF - FB = C$, so that the projection

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

commutes with T.

Now we are ready to apply Apostol's argument to this context.

LEMMA 4.3. *Let X be a Banach space; let T be an upper (resp. lower) semi-Fredholm operator on X with* $\operatorname{ind}(T) \neq 0$. *Let ε be a positive real number. Suppose that $T-\lambda$ is bounded below (resp. surjective) for each non-zero λ in some complex neighbourhood of zero. Then there exists a finite rank operator $F \in B(X)$ with $\|F\| < \varepsilon$ which satisfies both*

(i) *$T+F$ bounded below (resp. surjective)*

and

(ii) $\min \operatorname{ind}(T-\lambda) = 0 \Rightarrow \min \operatorname{ind}(T+F-\lambda) = 0 \ (\lambda \in \mathbb{C})$.

Proof. We suppose that $T-\lambda$ is bounded below for each non-zero λ in some complex neighbourhood of zero. The other case can be treated similarly. By Lemma 2.2, T has finite ascent, say k. Suppose $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ on $Y \oplus Z' = X$ where $Y = \ker T^k$, and Z' is a closed complement for Y in X. We assume $Y \neq \{0\}$, since that case is trivial.

Since $\begin{bmatrix} A & C \\ 0 & 0 \end{bmatrix}$ is finite rank it follows that $\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$ is semi-Fredholm, so that $B \in B(Z')$ has closed range.

Also $T^{2k} = \begin{bmatrix} 0 & DB^k \\ 0 & B^{2k} \end{bmatrix}$ for some $D \in B(Z', Y)$, so that, for $z \in Z'$,

$$B^k z = 0 \Rightarrow T^{2k} z = 0 \Rightarrow T^k z = 0 \Rightarrow z = 0.$$

It follows that B is injective and therefore bounded below. Consequently, we can apply Lemma 4.2 to represent

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \qquad \text{on } Y \oplus Z = X$$

where Z is an appropriate closed complement of Y in X. T_1 is nilpotent and T_2 is bounded below. Furthermore, T_2 is not onto Z since that would clearly force $\operatorname{ind}(T) = 0$.

Now, it is easy to see, by writing T_1 in Jordan form, that we can choose a finite rank operator $F_{11} \in B(Y)$ of arbitrarily small norm such that $T_1 + F_{11}$ is nilpotent and $\operatorname{nul}(T_1 + F_{11}) = 1$.

We choose our F_{11} accordingly such that $\|F_1\| < \frac{1}{2}\varepsilon$ where

$$F_1 = \begin{bmatrix} F_{11} & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{on } Y \oplus Z = X.$$

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Then, we let $e \in Z \backslash T_2(Z)$ and we choose $\phi \in Y^*$ such that

$$\phi(\ker(T_1 + F_{11})) \neq \{0\} \quad \text{and} \quad \|F_2\| < \tfrac{1}{2}\varepsilon$$

where

$$F_2 = \begin{bmatrix} 0 & 0 \\ e \otimes \phi & 0 \end{bmatrix} \qquad \text{on } Y \oplus Z = X.$$

We now put $F = F_1 + F_2$ and it is elementary to check that F satisfies the requirements of the lemma.

THEOREM 4.4. *Let X be a Banach space and let $T \in B(X)$. Let ε be a positive real number. Then there exists an operator $K \in B(X)$ in the closure of the finite rank operators, and with norm less than ε, such that $T + K$ is a West operator.*

Proof. Firstly we apply Theorem 2.3 to T, constructing an operator $K_1 \in B(X)$ with $\|K_1\| < \frac{1}{2}\varepsilon$ such that $\sigma_s(T + K_1) \backslash \sigma_{s-F}(T)$ accumulates only on the boundary of $\sigma_{s-F}(T)$.

Let $T^* = T + K_1$, and let

$$\Lambda = \{\lambda \in \sigma_s(T^*) \backslash \sigma_{s-F}(T) \colon \operatorname{ind}(T^* - \lambda) \neq 0\}.$$

Suppose, without loss of generality, that Λ is an infinite set and let $\{\lambda_n : n \in \mathbb{N}\}$ be an enumeration of the elements of Λ.

We construct a sequence $(\varepsilon_n)_{n=0}^{\infty}$ of positive real numbers, and a sequence $(F_n)_{n=0}^{\infty}$ of finite rank operators inductively as follows. Let $\varepsilon_0 < \min(1, \frac{1}{2}\varepsilon)$ and let $F_0 = 0$. For each successive $n \in \mathbb{N}$, we choose ε_n and F_n according to the following rules: we use Lemma 4.3 to find F_n with norm less than $\frac{1}{2}\varepsilon_{n-1}$ which satisfies both

(i) $T^* - \lambda_n + \sum_{i=0}^{n} F_i$ is bounded below or surjective and, for $\lambda \in \mathbb{C}$,

(ii) $\min \operatorname{ind}\left(T^* - \lambda + \sum_{i=0}^{n-1} F_i\right) = 0 \Rightarrow \min \operatorname{ind}\left(T^* - \lambda + \sum_{i=0}^{n} F_i\right) = 0$.

Now we use the openness of the sets of surjective and bounded below operators to choose $\varepsilon_n < \frac{1}{2}\varepsilon_{n-1}$ such that

$$\lambda \in S_n \quad \text{and} \quad \|B\| < \varepsilon_n \Rightarrow \min \operatorname{ind}\left(T^* - \lambda + B + \sum_{i=0}^{n} F_i\right) = 0,$$

where

$$S_n = \left\{\lambda \in \mathbb{C} \colon \varepsilon_{n-1} \leq \operatorname{dist}\left(\lambda, \sigma_s\left(T^* + \sum_{i=0}^{n} F_i\right)\right) \leq 1/\varepsilon_{n-1}\right\}.$$

This is possible since S_n is a compact subset of $\mathbb{C} \backslash \sigma_s\left(T^* + \sum_{i=0}^{n} F_i\right)$ and the function d_n defined on $\mathbb{C}$ by

$$d_n(\lambda) = \operatorname{dist}\left(T^* - \lambda + \sum_{i=0}^{n} F_i,\ B(X) \backslash \{S \in \Phi_+(X) \cup \Phi_-(X) \colon \min \operatorname{ind}(S) = 0\}\right)$$

is continuous.

Now $\sum_{n=0}^{\infty} F_n$ converges to an operator $K_2 \in B(X)$ with $\|K_2\| < \frac{1}{2}\varepsilon$. Furthermore, we have

$$\left\| \sum_{i=n+1}^{\infty} F_i \right\| < \varepsilon_n \qquad \text{for each } n \in \mathbb{N}$$

and

$$\bigcup_{n \in \mathbb{N}} S_n \supseteq \rho_{s-F}(T) \backslash \sigma_p^0(T^*)$$

so that

$$\sigma_s(T^* + K_2) \subseteq \sigma_{s-F}(T) \cup \sigma_p^0(T^*).$$

We put $K = K_1 + K_2$. Then $\|K\| < \varepsilon$ and $T + K$ is a West operator.

We note that, even if we are interested in achieving a compact correction of minimal norm for an operator, then the upper semi-continuity of the spectrum ensures that we may take Theorem 4.4 as our starting point.

The following three results are easy consequences of the foregoing, and we state them without proof.

Corollary 4.5. *Let X be a Banach space and suppose that every West operator on X has a West decomposition. Then every operator on X has a compact correction.*

Corollary 4.6. *Let X be a Banach space and let $T \in B(X)$. Suppose that $\sigma_W(T) = \sigma(T)$. Then T has a compact correction of arbitrarily small norm.*

Corollary 4.7. *Let X be a Banach space and let U be the unilateral shift on l^2. Let α be any complex number with modulus greater than the essential spectral radius of T. Then*

$$T \oplus \alpha U \in B(X \oplus l^2)$$

has a compact correction.

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