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FROM PATH LIFTING AND UNIQUE ARC LIFTING TO UNIQUE PATH LIFTING

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1. Introduction. In [4] it was conjectured that a light map $p : E \to B$ for which paths can be lifted and lifting of arcs is unique is a Serre fibration. As is well-known this implies that paths have unique liftings. In this paper we shall prove several special cases of this conjecture.

The two main theorems are: (3.5) Let p be a light compact map of a metric space E onto a connected semi-locally contractible along arcs metric space B. If arcs can be lifted uniquely then p is locally trivial. (Hence paths can be lifted uniquely.); (3.9) Let p be a light compact map of a metric space E onto a semi-locally arc simply connected, connected, arcwise connectable metric space B. If arcs have unique liftings then p is locally trivial.

We shall then show that the above results are about as good as to be expected by presenting three examples of maps where paths can be lifted, arcs have unique liftings and paths do not have unique liftings. In the first example the base space is the unit interval I, which satisfies all the above requirements for base spaces and the domain is a locally contractible non-separable complete metric space. In the second example the base space is again the unit interval and the domain is a non-locally connected subset of the unit square. In view of the first two examples one might expect that compactness and local connectedness are the keys, however in the third example both the base space and the domain are Peano continua.

2. Definitions and notation. The following definitions and notation will be needed.

(2.1) *Definition.* A map (continuous function) $p : E \to B$ is *compact* if the inverse of every compact set is compact.

(2.2) *Definition.* A map $p : E \to B$ has the property that *paths can be lifted* if given any path $\sigma : I \to B$ and any point e in $p^{-1}\sigma(0)$ there exists a path $\tau : I \to E$ such that $\tau(0) = e$ and $p\tau = \sigma$. The map τ is called a *lifting* of σ. If all such liftings are unique then we say *paths can be lifted uniquely.*

(2.3) *Definition.* A map $p : E \to B$ is locally trivial if there exists an open cover $\{U_\alpha\}$ of B, a collection $\{\phi_\alpha\}$ of homeomorphisms and a space F such that F is homeomorphic to $p^{-1}(b)$ for all b in B and $\phi_\alpha : U_\alpha \times F \to p^{-1}(U_\alpha)$ satisfies $p\phi_\alpha(u, f) = u$.

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(2.4) *Definition.* A space B is *semi-locally contractible along arcs* if there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of B and a collection of maps $H_\alpha : U_\alpha \times I \to B$ such that $H_\alpha(u, 0) = b_\alpha$, $H_\alpha(u, 1) = u$ and $H_{\alpha u} : I \to B$ (defined by $H_{\alpha u}(t) = H_\alpha(u, t)$) is either an arc or a constant path.

(2.5) *Definition.* If σ and τ are two arcs in a space B and σ and τ have the same endpoints, then σ and τ are *homotopic along arcs* if there exists a map $H : I \times I \to B$ such that $H(s, 0) = \sigma(s)$, $H(s, 1) = \tau(s)$ and H_s is either an arc or a constant path.

(2.6) *Definition.* A space B is *semi-locally arc simply connected* if there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of B such that if σ and τ are arcs in U_α and σ and τ have the same endpoints then σ and τ are homotopic along arcs.

(2.7) *Definition.* A space B is *arcwise connectable* if given any convergent sequence $\{v_i\}$ in B there is an arc whose image contains v_j for all j in an infinite set M.

(2.8) *Definition.* A map $p : E \to B$ has the *arc lifting property* (ALP) if the map $q : E^I \to Z = \{(e, w) \in E \times B^I | p(e) = w(0)\}$ defined by $q(\tau) = (\tau(0), p\tau)$ admits a section over $A_p = \{(e, w) \in Z | w$ is either an arc or a constant map.$\}$

3. Sufficient conditions for paths to have unique liftings.

(3.1) THEOREM. *If p is a light compact mapping from a metric space E onto a metric space B and arcs can be lifted uniquely, then p has the* ALP.

Proof. Define $\lambda : A_p \to E^I$ by $\lambda(e, w)$ is the unique lifting of w with initial point e. The proof that λ is continuous follows almost verbatim from [**4**, (5.4)].

(3.2) LEMMA. *If p is a light map with the* ALP *then arcs can be lifted uniquely.*

Proof. Let α be an arbitrary arc in B and let β and γ be liftings of α with the same initial point. It shall be shown that $\beta(1) = \gamma(1)$. Since α is arbitrary it follows that arcs have unique liftings.

Define a map $G : I \times I \to B$ by

$$G(s, t) = \begin{cases} \lambda(2(t-1)s + 1) & \text{if } 0 \leqq s \leqq 1/2, \\ \lambda(2(1-t)s + 2t - 1) & \text{if } 1/2 \leqq s \leqq 1. \end{cases}$$

Note that $G(0, t) = G(s, 1) = G(1, t) = \alpha(1)$, and for each s the map $G_s : I \to B$ defined by $G_s(t) = G(s, t)$ is an arc or a constant path.

Let $\lambda : A_p \to E^I$ be a section for q guaranteed by the ALP. Define a map $H : I \to E$ by

$$H(s) = \begin{cases} \lambda(\beta(1-2s), G_s)(1) & \text{if } 0 \leqq s \leqq 1/2, \\ \lambda(\gamma(2s-1), G_s)(1) & \text{if } 1/2 \leqq s \leqq 1. \end{cases}$$

Since $pH(s) = \alpha(1)$ for each s in I the lightness of p guarantees that H is constant. Since G_0 and G_1 are both constant paths it follows that $\beta(1) = H(0) = H(1) = \gamma(1)$.

(3.3) THEOREM. *Let p be a light map from a space E onto a connected, semi-locally contractible along arcs space B. If p has the* ALP *then p is locally trivial.*

Proof. Let b be an element of B. Since B is semi-locally contractible along arcs there exists an open set U containing b, a point b_0 in B and a map $H : U \times I \to B$ such that $H(u, 0) = b_0$, $H(u, 1) = u$, and H_u is constant or an arc for $u \in U$. There also exists $\lambda : A_p \to E^I$ as in the definition of ALP.

Define $\psi : p^{-1}(b_0) \times U \to p^{-1}(U)$ by

$$\psi(e, u) = \lambda(e, H_u)(1).$$

Define $\Phi : p^{-1}(U) \to p^{-1}(b_0) \times U$ by $\Phi(e) = (\lambda(e, {}^*H_{p(e)})(1), p(e))$. (If σ is a path ${}^*\sigma$ is the path defined by ${}^*\sigma(t) = \sigma(1 - t)$.)

It is easily seen that ψ and Φ are continuous, and that $p\psi(e, u) = u$. Since arcs have unique liftings (by (3.2)) it follows that ψ and Φ are inverses of each other. Hence ψ is a homeomorphism. Since B is connected all fibers are homeomorphic.

(3.4) *Note.* If U in the above proof can be chosen to be B then p is trivial.

(3.5) COROLLARY. *Let p be a light compact map from a metric space E onto a connected semi-locally contractible along arcs metric space B. If arcs can be lifted uniquely then p is locally trivial.*

Proof. This follows from (3.1) and (3.3).

(3.6) COROLLARY. *If p is a light compact map from a metric space to a metric manifold* (*with or without boundary*) *for which arcs have unique liftings then paths have unique liftings.*

(3.7) COROLLARY. *If p is a light compact mapping from a metric space E onto a Peano continuum B and p has the property that arcs can be lifted uniquely, then there exists a dense arcwise connected subset C of B such that $p|p^{-1}(C)$ is trivial.*

Proof. This follows from (3.1), (3.3), (3.4) and [**1**, Theorem 2], and [**3**, Proposition 3].

(3.8) THEOREM. *Let p be a light map from a first countable space E onto a semi-locally arc simply connected, locally arcwise connectable, first countable, connected space B. If p has the* ALP *then p is locally trivial.*

Proof. Let b be a point of B. Let U be a neighbourhood of b given by the semi-locally arc simple connectedness. Let V be the arc component of U that contains the point b. Since B is locally arcwise connectable and first countable

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it follows that V is open. Let $\lambda : A_p \to E^I$ be given by the ALP. Define $\psi : p^{-1}(b) \times V \to p^{-1}(V)$ by $\psi(e, v) = \lambda(e, \omega)(1)$ where ω is any arc in V from b to v. (If $b = v$ let ω be the constant path.)

To see that ψ does not depend on the choice of ω let β and γ be two paths in V from b to v. By the hypothesis there exists a map $H : I \times I \to B$ such that $H(0, t) = b$, $H(1, t) = v$, $H(s, 0) = \beta(s)$, $H(s, 1) = \gamma(s)$ and H_s is an arc or a constant path. The map $\delta : I \to E$ defined by

$$\delta(s) = \lambda(\lambda(e, \beta)(s), H_s)(1)$$

is a lifting of γ with initial point e. Since by (3.2) arcs have unique liftings it follows that $\delta = \lambda(e, \gamma)$. Finally since H_1 is constant $\lambda(e, \gamma)(1) = \delta(1) = \lambda(\lambda(e, \beta)(1), H_1)(1) = \lambda(e, \beta)(1)$.

To complete the proof of this theorem it will be shown that ψ is a homeomorphism. To see that ψ is continuous let $\{(e_i, v_i)\}$ be a sequence in $p^{-1}(b) \times V$ that converges to a point (e_0, v_0). It suffices to show that some subsequence of $\{\psi(e_i, v_i)\}$ converges to $\psi(e_0, v_0)$. Since B is arcwise connectable there is an arc σ whose image contains v_i for all i in an infinite set M. Since V is arcwise connected σ may be chosen so that $\sigma(0) = b$. For each i in M or $i = 0$ let $t_i = \sigma^{-1}(v_i)$ and let σ_i be the path from b to v_i defined by $\sigma_i(s) = \sigma(t_i s)$. Since $\lim_{i \in M} \sigma_i = \sigma_0$ it follows that

$$\lim_{i \in M} \psi(e_i, v_i) = \lim_{i \in M} \lambda(e_i, \sigma_i)(1) = \lambda(e_0, \sigma_0)(1) = \psi(e_0, v_0).$$

To see that ψ has a continuous inverse define $\Gamma : p^{-1}(V) \to p^{-1}(b) \times V$ by $\Gamma(e) = (\lambda(e, \omega)(1), p(e))$ where ω is any arc in V from $p(e)$ to b. As with ψ Γ does not depend on the choice of ω. Since arcs have unique liftings it follows that ψ and Γ are inverses of each other. As with the map ψ the map $\pi_1 \Gamma : p^{-1}(V) \to p^{-1}(b)$ defined by $\pi_1 \Gamma(e) = \lambda(e, \omega)(1)$ is continuous from which it readily follows that Γ is continuous.

(3.9) COROLLARY. *Let p be a light compact map from a metric space E onto a semi-locally arc simply connected, arcwise connectable connected metric space B. If p has the property that arcs can be lifted uniquely then P is locally trivial.*

Proof. This follows from (3.1) and (3.8).

4. Examples. In this section we present three examples of light maps $p : E \to B$ for which paths can be lifted, arcs have unique liftings, and paths do not have unique liftings.

(4.1) *Example.* Let $f : I \to I$ be a map that is not one-to-one or constant on any subinterval of I, such as a continuous nowhere differentiable function. Let d be the metric defined on $I \times I$ by $d((a, b), (c, d)) = |b - d|$ if $a = c$, and $d((a, b), (c, d)) = |b - f(a)| + |a - c| + |f(c) - d|$ if $a \neq c$. Let $B = I$ and let $E = I \times I$ with the topology induced by d. Let $p : E \to B$ be defined by $p((a, b)) = b$. Notice that E is the union of a family of intervals, each of which is mapped homeomorphically onto B by the map p, so paths can be lifted. Notice also that if g is a path in E whose image contains two points (a, b) and (c, d) where $a < c$ then $g(I)$ must contain all points of the form $(x, f(x))$ where $a \leqq x \leqq c$, so that pg is not one-to-one. Consequently, arcs have unique liftings. Now, the path f has two distinct liftings from the initial point $(0, f(0))$, namely the map h defined by $h(t) = (0, f(t))$ and the map g defined by $g(t) = (t, f(t))$. It is easy to see that E is a locally contractible non-separable complete metric space.

(4.2) *Example.* Let $B = I$. The domain E shall be a subset of $I \times I$ with the relative topology and $p : E \to B$ is defined by $p((a, b)) = b$. To describe the set E we utilize a map $f : I \to I$ that is strictly increasing or strictly decreasing on each component of the complement of the Cantor set C and is not one-to-one on any open set that intersects C. If c is in C and is not an endpoint of some component of the complement of C let

$$I_c = \{(c, t) : 0 \leqq t \leqq 1\}.$$

If c is the left-hand endpoint of some component of the complement of C, (c, d), then let $I_c = \{(x, y) : x = c$ and $y \leqq f(c)$, or $c \leqq x \leqq d$ and $y = f(x)$, or $x = d$ and $y \geqq f(d)\}$ if f is increasing on (c, d), and let $I_c = \{(x, y) : x = c$ and $y \geqq f(c)$, or $c \leqq x \leqq d$ and $y = f(x)$, or $x = d$ and $y \leqq f(d)\}$ if f is decreasing on (c, d). Let E be the union of the I_c where c is any point of the Cantor set that is not an upper endpoint of some component of the complement of C. Clearly p maps each I_c homeomorphically onto I. We use the same techniques as were used in (4.1) above to show that paths can be lifted, arcs have unique liftings, and the path f has two distinct liftings with initial point $(0, f(0))$.

(4.3) *Example.* We now construct an example $p : E \to B$ where E and B are Peano continua. The construction is done in two stages. First we construct an example $p' : E' \to B'$ where E' and B' are compact. We then embed E' and B' into Peano continua E and B in a way that allows us to extend p' to the desired map p.

Let S^1 denote the set of points in the plane (a, b) for which $a^2 + b^2 = 1$. For each positive integer n let S_n denote the circle in the plane of radius $r_n = 1/2n(n + 1)$ with center at $(0, r_n)$. For each positive integer n let $h_n : S_n \to S^1$ be the homeomorphism defined by $h_n((a, b)) = (r_n - b, a)/r_n$. Notice that $h_n((0, 0)) = (1, 0)$. Let $s : S^1 \to S^1$ be the squaring map $s(z) = z^2$. For each positive integer n let $k_n : S_n \to \mathbf{R}^3$ be the isometry defined by $k_n((a, b)) = (a, 0, 1/n - b)$. Notice that $k_n((0, 2r_n)) = k_{n+1}((0, 0))$, so that $\bigcup\{k_n(S_n) : n = 1, 2, 3, \ldots\}$ is connected. Let $B' = \bigcup\{S_n : n = 1, 2, 3, \ldots\}$. Let $E' \subset \mathbf{R}^3$ be the set of points (a, b, c) where $c = 0$ and $(a, b) \in B'$, or $c = 1/n$ and $(a, b) \in S_j$ for some positive integer j different than both n

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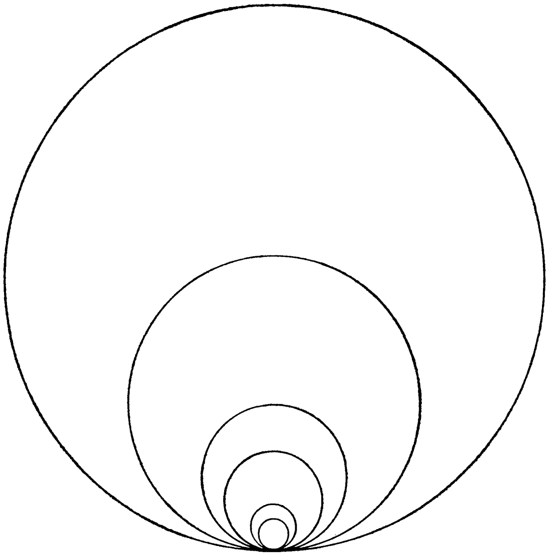

FIGURE 1. B'

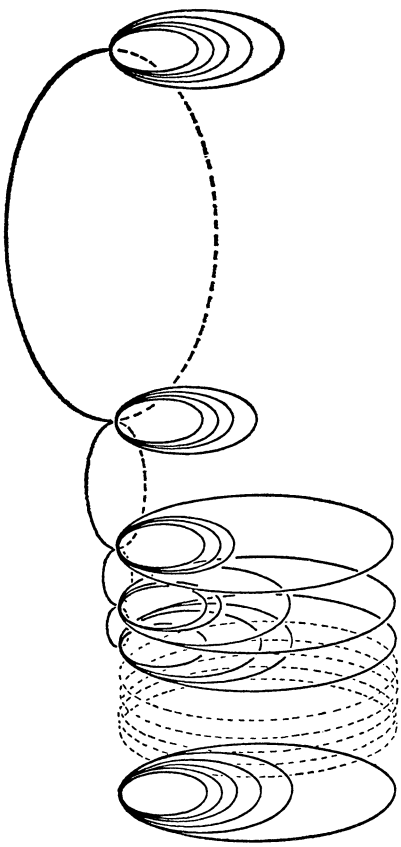

FIGURE 2. E'

and $n - 1$, or $(a, b, c) \in \bigcup\{k_n(S_n) : n = 1, 2, 3, \ldots\}$. Let $p' : E' \to B'$ be defined by $p'((a, b, c)) = (b, c)$ if $(a, b, c) \notin \bigcup\{k_n(S_n) : n = 1, 2, 3, \ldots\}$, and $p'((a, b, c)) = h_n^{-1} s h_n k_n^{-1}((a, b, c))$ if $(a, b, c) \in k_n(S_n)$. (See figures 1 and 2.)

(4.3.1) LEMMA. *Paths can be lifted.*

Proof. Since p' is an open map [**2**, Theorem 3] assures us that paths can be lifted.

(4.3.2) LEMMA. *Arcs have unique liftings.*

Proof. Notice that the restriction of p' to each component of $E' - p'^{-1}(0, 0)$ is one-to-one. Also if α and β are distinct arcs in E' with the same initial point $e \in p'^{-1}(0, 0)$ then $p'\alpha \neq p'\beta$.

(4.3.3) *A path with two distinct liftings.* Clearly the subset of E' consisting of the point $(0, 0, 0)$ and $\bigcup\{k_n(S_n) : n = 1, 2, \ldots\}$ is the image of a path g for which $g(0) = (0, 0, 0)$. The map $f = p'g$ has g as one lifting with initial point $(0, 0, 0)$. To obtain a second lifting let $h : B' \to E'$ be defined by $h(a, b) = (a, b, 0)$, then $g' = hf$ is another lifting of f with initial point $(0, 0, 0)$.

The Peano continua E and B and the map p are now constructed by utilizing the following induction. For each positive integer n define a map $p_n : E_n \to B_n$ and choose a point x_n in B_n and a positive integer m_n as follows: Let $B_1 = B'$, $E_1 = E'$, $p_1 = p'$, $x_1 = (0, 0)$, and $m_1 = 1$. Now suppose that B_n, E_n, p_n, x_n, and m_n have been chosen. Choose $x_{n+1} \in B_n$ so that the minimal distance of x_{n+1} to the $\{x_i : i = 1, 2, 3, \ldots, n\}$ is maximal. Let h be an isometry of $\mathbf{R}^2$ onto itself for which $h(0, 0) = x_{n+1}$ and the circle in B_n that contains x_{n+1} is $h(S_i)$ for some positive integer i. For each z in $p_n^{-1}(x_{n+1})$ let h_z be an isometry of the plane into $\mathbf{R}^3$ for which $h_z(0, 0) = z$ and the circle in E_n that contains the point z is $h_z(S_j)$ for some positive integer j. Next choose $m_{n+1} > m_n$ so that

$$h(\bigcup\{S_i : i \geqq m_{n+1}\}) \cap B_n = \{x_{n+1}\}, \; h_z(\bigcup\{S_i : i \geqq m_{n+1}\}) \cap E_n = \{z\}$$

for z in $p_n^{-1}(x_{n+1})$, and if $z = (a, b, c) \in p_n^{-1}(x_{n+1})$ with $c \leqq 1/m_{n+1}$ then the restriction of p_n to the circle in E_n that contains z is a monotone map. Denote $\bigcup\{S_i : i \geqq m_{n+1}\}$ by K. Let $B_{n+1} = B_n \cup h(K)$. Let $E_{n+1} = E_n \cup G \cup H$ where $G = \bigcup\{h_z(K) : z = (a, b, c) \in p_n^{-1}(x_{n+1})$ and $c > 1/m_{n+1}\}$, and $H = \{(a', b', c) : (a, b, c) \in p'^{-1}(K),\ c \leqq 1/m_{n+1}, \text{ and } (a', b') = h(a, b)\}$. Define p_{n+1} so that if $w \in E_n$ then $p_{n+1}(w) = p_n(w)$, if $w \in h_z(K) \subset G$ let $p_{n+1}(w) = h(h_z^{-1}(w))$, and if $w = (a', b', c) \in H$ then let $p_{n+1}(w) = h(p'(a, b, c))$ where $(a', b') = h(a, b)$. Finally let

$$B = (\bigcup\{B_n : n = 1, 2, 3, \ldots\})^-, \qquad E = (\bigcup\{E_n : n = 1, 2, 3, \ldots\})^-,$$

and let p be the map that extends each p_n to E. (See figures 3 and 4.)

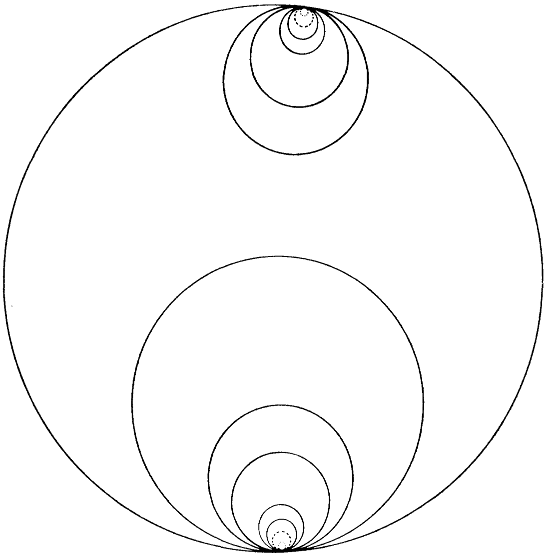

FIGURE 3. B_2

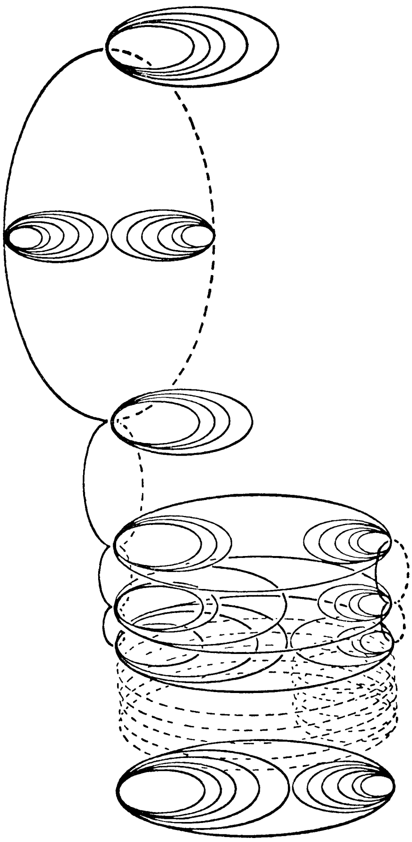

FIGURE 4. E_2

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