ON SOME FAMILIES OF ANALYTIC FUNCTIONS ON RIEMANN SURFACES

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Dedicated to Professor K. Noshiro on the occasion of his 60th birthday

1. Throughout this paper all functions are single-valued. Let R be a Riemann surface. *We shall denote by* $\varphi^\wedge$ *the least harmonic majorant of a function* φ *defined in* R *if it has the meaning.* We define the families $H_p(R)$ (for $p>0$) and $S(R)(=D(R)$ in [17]) of analytic functions in R by the following:

> f is in $H_p(R)$ if and only if the subharmonic function $|f|^p$ has a harmonic majorant in R;
>
> f is in $S(R)$ if and only if the subharmonic function $\log^+(|f|/\mu)$ has a harmonic majorant in R for some positive constant μ (and consequently for all $\mu>0$) and $(\log^+(|f|/\mu))^\wedge(z_0)\to$ as $\mu\to+\infty$, where z_0 is a fixed point in R ([17]).

We shall call $H_p=H_p(R)$ (resp. $S=S(R)$) the Hardy class (resp. the Smirnov class) in R.

A harmonic function u in R is said to be *quasi-bounded* ([13]) if it can be represented as: $u=u_1-u_2$, where $u_j (j=1,2)$ is the limiting function of a monotone non-decreasing sequence of non-negative and bounded harmonic functions in R.

A *closed polar set* E in a Riemann surface R is a closed set in R such that for every open parameter disc V in R, there exists a superharmonic function $s_V>0$ defined in V with the property that $s_V=+\infty$ at every point in $V\cap E$, or equivalently, $V\cap E$ is a set of capacity zero in V ([1], [2]). It is known that $R-E$ is connected.

Tumarkin and Havinson [17] (resp. Parreau [13]) investigated the null set E in a plane domain (resp. in a Riemann surface) R for the class S (resp. H_p) under the condition that E is a compact set of logarithmic capacity zero (resp. a closed, not necessarily compact, polar set) and proved: if an analytic function f defined in $R-E$ belongs to the class $S(R-E)$

Received March 20, 1967

https://doi.org/10.1017/S0027763000012630 Published online by Cambridge University Press

(resp. $H_p(R-E)$), then there exists an analytic function $\tilde{f}$ defined in R belonging to the class $S(R)$ (resp. $H_p(R)$) such that the restriction of $\tilde{f}$ to $R-E$ coincides with f.

In this paper we shall show, using the notion of quasi-bounded harmonic functions, that in these theorems the well-known fact that the closed polar set E is removable for bounded and harmonic functions ([1], [2]) is essential.

As for S-part we shall prove the following:

THEOREM 1. *Any analytic function f in a Riemann surface R belongs to the Smirnov class $S(R)$ if and only if the subharmonic function $\log^+|f|$ has a quasi-bounded harmonic majorant in R.*

Using a version of Gårding and Hörmander's theorem [7] as a lemma, we shall prove:

THEOREM 2. *Any analytic function f in a Riemann surface R belongs to the Hardy class $H_p(R)$ (for $p>0$) if and only if the subharmonic function $|f|^p$ has a quasi-bounded harmonic majorant in R.*

Seeing the above characterizations for the two classes, we are tempted to say the following:

THEOREM 3. *Let $\Psi(r)$ be a continuous extended real-valued function defined for $r \geq 0$ satisfying the condition that for any finite positive real number c, the set of r such that the inequality $\Psi(r) \leq c$ holds is bounded (from above). Let R be a Riemann surface, E be a closed polar set lying in R and f be an anaylytic function defined in $R-E$ such that the composite function $\Psi(|f|)$ has a quasi-bounded harmonic majorant in $R-E$.*

Then there exists an analytic function $\tilde{f}$ defined in R such that the composite function $\Psi(|\tilde{f}|)$ has a quasi-bounded harmonic majorant in R and the restriction of $\tilde{f}$ to $R-E$ coincides with the function f.

As corollaries we have an extension of Tumarkin-Havinson's theorem and a new proof of Parreau's.

At the end, we shall give an example for the classification theory of open Riemann surfaces, which admits a non-constant analytic Lindelöfian function [9] and no non-constant analytic function in the Smirnov class.

2. Let R be a Riemann surface, $HP'(R)$ be the family of all the harmonic functions u in R such that the subharmonic function $|u|$ has a harmonic majorant in R. It is well-known (see for example, [3]) that $HP'(R)$ forms a vector lattice under the lattice operations:

$$u \vee v = (\text{the least harmonic majorant of } \max(u, v));$$
$$u \wedge v = -(-u) \vee (-v)$$

for u, v in $HP'(R)$. For u in $HP'(R)$ we define Mu as follows:

$$Mu = u \vee 0 - u \wedge 0 .$$

We know that $Mu = u \vee (-u)$ and $M(Mu) = Mu$. A function u in $HP'(R)$ is, by definition, *quasi-bounded* if

$$Mu = \lim_{n \to +\infty} (Mu) \wedge n ,$$

or equivalently,

$$\lim_{n \to +\infty} (Mu - n) \vee 0 = 0 ,$$

where n are positive numbers which can be considered as elements in $HP'(R)$ and the limit is taken in the sense of the lattice operation, namely, $(Mu) \wedge n$ (resp. $(Mu - n) \vee 0$) tends to Mu (resp. 0) non-decreasingly (resp. non-increasingly) in R. A function u in $HP'(R)$ is called *singular* if

$$\lim_{n \to +\infty} (Mu) \wedge n = 0 .$$

It is shown by Parreau [13] that any u in $HP'(R)$ can be decomposed uniquely as:

$$u = u_B + u_S ,$$

where u_B is quasi-bounded and u_S is singular. The operator $u \to u_B$ (resp. $u \to u_S$) from $HP'(R)$ into itself is linear, positive, i.e., $u \geq 0$ implies $u_B \geq 0$ (resp. $u_S \geq 0$) and idempotent, i.e., $(u_B)_B = u_B$ (resp. $(u_S)_S = u_S$). Of course, u is quasi-bounded (resp. singular) if and only if $u_S = 0$ (resp. $u_B = 0$).

In the remainder of this paper we shall assume that the Riemann surface R is hyperbolic since the situation is obvious in the parabolic case.

A subharmonic function v in R having a harmonic majorant in R can be decomposed uniquely as:

$$v = v^\wedge - p ,$$

https://doi.org/10.1017/S0027763000012630 Published online by Cambridge University Press

where $v^{\wedge}$ is the least harmonic majorant of v and $p \geq 0$ is a Green's potential in R (F. Riesz's decomposition).

We shall say that a subharmonic function v in R is *quasi-bounded* if $v^{\wedge}$ in the above decomposition is in $HP'(R)$ and quasi-bounded. A subharmonic function v having a quasi-bounded harmonic majorant u and a quasi-bounded harmonic minorant w simultaneously is quasi-bounded for $0 = w_S \leq (v^{\wedge})_S \leq u_S = 0$. Especially, a non-negative subharmonic function is quasi-bounded if and only if it has a quasi-bounded harmonic majorant.

Let $\{R_n\}_{n=1}^{\infty}$ be a normal exhaustion of R in Pfluger's sense, $\partial R_n = \Gamma_n$ be the boundary of R_n (consisting of a finite number of piecewise analytic closed Jordan curves), z_0 be a fixed point in R_1 and ω_{n,z_0} be the harmonic measure of Γ_n with respect to the domain R_n measured at the point z_0 (for $n = 1, 2, \ldots$). Then obviously we have:

$$v^{\wedge}(z_0) = \lim_{n \to +\infty} \int_{\Gamma_n} v(z) d\omega_{n,z_0}(z).$$

An extended real-valued function $f(z)$ defined for points z in R is said to be *uniformly absolutely integrable* with respect to the system $\{(\Gamma_n, \omega_{n,z_0})\}_{n=1}^{\infty}$ (we shall say simply "U.A.I. for z_0 and $\{R_n\}$") if the followings are satisfied:

(a) $$\sup_n \int_{\Gamma_n} |f(z)| d\omega_{n,z_0}(z) < \infty,$$

and

(b) for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \int_{A_n} f(z) d\omega_{n,z_0}(z) \right| < \varepsilon$$

uniformly for $n = 1, 2, \ldots$, if only $A_n \subset \Gamma_n$ and $\omega_{n,z_0}(A_n) < \delta$.

According to de la Vallée Poussin [18] and Doob [4], [6], a function $f(z)$ in R is U.A.I. for z_0 and $\{R_n\}$ if and only if there exists a non-negative monotone non-decreasing convex function $\Phi(r)$ defined for $r \geq 0$ satisfying the conditions:

(i) $$\lim_{r \to +\infty} \Phi(r) / r = +\infty$$

and

(ii) $$\sup_n \int_{\Gamma_n} \Phi(|f(z)|) d\omega_{n,z_0}(z) < \infty.$$

We shall call this *de la Vallée Poussin-Doob's lemma.*

In particular, if a subharmonic function $v(z) \geq 0$ in R is U.A.I. for z_0 and $\{R_n\}$, then the condition (ii) above can be read as:

(ii)′ The subharmonic function $\Phi(v)$ has a harmonic majorant in R.

We state some lemmas which will be used later.

LEMMA 1. *Let v be a quasi-bounded subharmonic function in a Riemann surface R. Then v is* U.A.I. *for arbitrary point z_0 in R and arbitrary exhaustion $\{R_n\}$, z_0 in R_1. Conversely assume that a subharmonic function v in R is* U.A.I. *for at least one point z_0 and at least one exhaustion $\{R_n\}$, z_0 in R_1. Then v is a quasi-bounded subharmonic function in R.*

Proof. We know that any harmonic function belongs to $HP'(R)$ and is quasi-bounded if and only if it is U.A.I. for one point z_0 and for one exhaustion $\{R_n\}$, z_0 in R_1 (and consequently for all) (see [4]). It is easy to check that Green's potential $p \geq 0$ is always U.A.I. for z_0 and $\{R_n\}$ since

$$\int_{\Gamma_n} p(z)d\omega_{n,z_0}(z) \to 0 \text{ as } n \to +\infty.$$

Using the above two facts, we have immediately the assertions.

LEMMA 2. *A subharmonic function v is quasi-bounded if and only if there exists a non-negative monotone non-decreasing convex function $\Phi(r)$ defined for $r \geq 0$ satisfying the conditions* (i) *and* (ii).

Proof. This is a consequence of de la Vallée Poussin-Doob's lemma and Lemma 1.

3. Here we remark the relations between some families of analytic functions defined in a Riemann surface R. We define the families $AB(R)$ and $AL(R)$ of analytic functions in R by the following:

f is in $AB(R)$ if and only if $|f|$ is bounded in R;

f is in $AL(R)$ if and only if the subharmonic function $\log^+|f|$ has a harmonic majorant in R.

Then the following inclusion relations:

$$AB(R) \subset H_p(R) \subset S(R) \subset AL(R) \qquad (\text{for } p > 0)$$

are proved by the inequalities:

$$\log^+(|f|/\mu) \leq |f|^p/(p\cdot\mu^p)$$

and

$$\log^+|f| \leq \log^+(|f|/\mu) + \log^+\mu .$$

REMARK. The functions f in the class $AL(R)$ are Lindelöfian analytic functions in the sense of Heins [9] and in the special case where R is the unit open disc, are analytic functions of bounded type in Nevanlinna's sense [12]. The Smirnov class $S(R)$ was first investigated by V.I. Smirnov [16].

Now we give

Proof of Theorem 1. Let $\mu \geq 1$. Then we obtain

$$\log^+(|f|/\mu) = \max(\log^+|f| - \log\mu, 0).$$

Consequently we have

$$\begin{aligned}(\log^+(|f|/\mu))^\wedge &= (\max(\log^+|f| - \log\mu, 0))^\wedge \\ &= (\max((\log^+|f|)^\wedge - \log\mu, 0))^\wedge \\ &= ((\log^+|f|)^\wedge - n) \vee 0,\end{aligned}$$

where $n = \log\mu$ and $\varphi^\wedge$ is the least harmonic majorant of φ (see §1). Hence the condition that

$$(\log^+(|f|/\mu))^\wedge\ (z_0) \to 0 \text{ as } \mu \to +\infty$$

is equivalent to the condition that

$$\lim_{n\to+\infty}((\log^+|f|)^\wedge - n) \vee 0 = 0$$

by Harnack's theorem, or $(\log^+|f|)^\wedge$, the least harmonic majorant of $\log^+|f|$, is quasi-bounded. Q.E.D.

REMARK. It is easy to show that $\log^+|f|$ has a quasi-bounded harmonic majorant in R if and only if $\log|f|$ has a quasi-bounded harmonic majorant in R.

By Lemma 1 with $v = \log^+|f|$ and by Theorem 1 we have

COROLLARY 1. (*An extended form of Theorem* 1 *in* [17]) *Any analytic function f is in the Smirnov class $S(R)$ if and only if the subharmonic function* $\log^+|f|$ *is* U.A.I. *for arbitrary fixed point z_0 in R and arbitrary exhaustion $\{R_n\}$, z_0 in R_1.*

COROLLARY 2. (*An extended form of Theorem* 2 *in* [17]) *Any analytic function* f *is in the Smirnov class* $S(R)$ *if and only if the subharmonic function* $\log^+|f|$ *has a harmonic majorant which is* U.A.I. *for arbitrary fixed point* z_0 *in* R *and arbitrary exhaustion* $\{R_n\}$, z_0 *in* R_1.

The following corollary shows that Gehring's class N^* in [8] is a special case of the Smirnov class $S(R)$ where R is the unit open disc.

COROLLARY 3. *Any analytic function* f *is in the class* $S(R)$ *if and only if there exists a non-negative monotone non-decreasing convex function* $\Phi(r)$ *satisfying the condition* (i) *in* §2 *and the subharmonic function* $\Phi(\log^+|f|)$ *has a harmonic majorant in* R.

Proof. This is a consequence of Theorem 1, Lemma 2 and (ii)′ in §2.

4. In this section we shall study the Hardy class $H_p(R)$.

Let Δ be Martin's boundary of a hyperbolic Riemann surface R and Δ_1 be the totality of minimal points on Δ. Let $K(z, \zeta)$ be Martin's kernel with respect to the fixed reference point z_0 in R, namely, $K(z_0, \zeta) = 1$ for any point ζ in $R \cup \Delta$. Then it is known that to any function u in the family $HP'(R)$, there corresponds a unique signed Baire measure $d\mu$ on Δ_1 of total mass finite such that

$$u(z) = \int_{\Delta_1} K(z, \zeta) d\mu(\zeta).$$

Let $d\omega$ be the measure on Δ_1 corresponding to the constant function 1, that is,

$$1 = \int_{\Delta_1} K(z, \zeta) d\omega(\zeta)$$

for any point z in R. Any function u in $HP'(R)$ has the fine limit $u^*(\zeta)$[1)] at $d\omega$-almost every point ζ in Δ_1 and the quasi-bounded part u_B of u is given by

$$u_B(z) = \int_{\Delta_1} K(z, \zeta) u^*(\zeta) d\omega(\zeta).$$

On the contrary, the singular part u_S of u in $HP'(R)$ is represented as

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1) In this section we shall denote by u^* the fine limit of any function u if it has the meaning.

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$$u_S(z) = \int_{\Delta_1} K(z,\zeta)\,d\mu_S(\zeta)\,,$$

where $d\mu_S$ is a singular measure on Δ_1 with respect to $d\omega$ and u_S has the fine limit zero at $d\omega$-almost every point in Δ_1. In conclusion:

$$d\mu(\zeta) = u^*(\zeta)d\omega(\zeta) + d\mu_S(\zeta)\,,$$

u^* is integrable with respect to $d\omega$.

Let v be a subharmonic function in R and have a harmonic function in $HP'(R)$ as a majorant. Then F. Riesz's decomposition of v becomes:

$$v = v^\wedge - p\,,$$

where, in this case, $v^\wedge$ is in $HP'(R)$. Green's potential p has the fine limit zero at $d\omega$-almost every point in Δ_1. Consequently we may write in this case

$$v^* = (v^\wedge)^* = ((v^\wedge)_B)^*\,.$$

As to the notion of the fine limit at Martin's compactification, see Naïm [11] and Doob [5].

Now we are ready to state a generalization of Gårding and Hörmander's theorem ([7]).[2]

Lemma 3. *Let v be a subharmonic function defined in R. Let $\varphi(r)$ be a non-negative monotone non-decreasing convex function defined for $-\infty < r < +\infty$ satisfying the condition*

(A) $\lim_{r\to+\infty} \varphi(r)/r = +\infty$

and assume that

(B) *the subharmonic function $\varphi(v)$ has a harmonic majorant in R, where we set $\varphi(-\infty) = \lim_{r\to-\infty}\varphi(r)$.*

Then

(C) *the least harmonic majorant $v^\wedge$ of v exists and is in $HP'(R)$,*

(D) *the singular measure $d\mu_S$ on Δ_1 corresponding to the singular part $(v^\wedge)_S$ of $v^\wedge$ is non-positive,*

[2] E.D. Solomentsev proved partly the same results as Gårding and Hörmander's in his paper: Izv. Akad. Nauk SSSR (1938), pp. 571–582.

https://doi.org/10.1017/S0027763000012630 Published online by Cambridge University Press

(E) *the least harmonic majorant* $(\varphi(v))^\wedge$ *of the subharmonic function* $\varphi(v)$ *exists and is quasi-bounded,*

and

(F) $$(\varphi(v))^\wedge(z) = \int_{\Delta_1} K(z,\zeta)\varphi(v^*(\zeta))d\omega(\zeta).$$

Proof. There exists a finite number $c>0$ such that $\varphi(r)$ is strictly increasing for $r>c-1$. Set $v_c = \max(v,c)$. Then v_c and consequently $\varphi(v_c)$ are subharmonic. Let $\Gamma_{n,c}$ be the set of points z on $\Gamma_n = \partial R_n$ such that $v(z)\geq c$ holds $(n=1,2,\ldots)$. Then we have

$$\begin{aligned}\varphi(v_c(z_0)) &\leq \int_{\Gamma_n} \varphi(v_c(z))d\omega_{n,z_0}(z)\\ &= \int_{\Gamma_{n,c}} \varphi(v)d\omega_{n,z_0} + \varphi(c)\omega_{n,z_0}(\Gamma_n - \Gamma_{n,c})\\ &\leq \int_{\Gamma_n} \varphi(v)d\omega_{n,z_0} + \varphi(c)\\ &\leq h(z_0) + \varphi(c)\end{aligned}$$

for arbitrary point z_0 in R, where h is a harmonic majorant of $\varphi(v)$ in R. Hence $\varphi(v_c)\leq h+\varphi(c)$ in R and we have $v_c\leq\varphi^{-1}(h+\varphi(c))$, the right hand side being superharmonic, so that $(v_c)^\wedge\leq\varphi^{-1}(h+\varphi(c))$, or $\varphi((v_c)^\wedge)\leq h+\varphi(c)$. The assertion (C) is immediate since $v\leq v_c\leq(v_c)^\wedge$.

Let $\Phi(r)$ be the restriction of $\varphi(r)$ to $r\geq 0$ and set $u=(v_c)^\wedge$. Then from above

$$\Phi(u) = \varphi((v_c)^\wedge)\leq h+\varphi(c).$$

By de la Vallée Poussin-Doob's lemma, u is U.A.I. for z_0 and $\{R_n\}$ so that u is a non-negative quasi-bounded harmonic function in R. This shows the assertion (D) for $v^\wedge\leq u$ implies $(v^\wedge)_S\leq u_S=0$.

Set $u_n = u\wedge n$ for positive integer $n\geq c$ so that $u_n\nearrow u$ by the definition. Then we have

(*) $$\lim_{n\to+\infty}(\varphi(u_n))^\wedge = (\varphi(u))^\wedge.$$

In fact, on the one hand, $(\varphi(u_n))^\wedge\leq(\varphi(u))^\wedge$ and on the other hand, $\lim_{n\to+\infty}(\varphi(u_n))^\wedge\geq\varphi(u)$, this can be shown as follows. From $\varphi(u_n)\leq(\varphi(u_n))^\wedge$

we have $u_n \leq \varphi^{-1}((\varphi(u_n))^\wedge)$ for $u_n \geq c$. Consequently $u_n \leq \varphi^{-1}(\lim_{n\to+\infty}(\varphi(u_n))^\wedge)$ and so $u \leq \varphi^{-1}(\lim_{n\to+\infty}(\varphi(u_n))^\wedge)$ or $\varphi(u) \leq \lim_{n\to+\infty}(\varphi(u_n))^\wedge$.

Now (*) means that $(\varphi(u))^\wedge$ is quasi-bounded. Therefore $0 \leq ((\varphi(v))^\wedge)_S \leq ((\varphi(u))^\wedge)_S = 0$ which proves our assertion (E).

The last assertion (F) follows from (E) and the continuity of the function $\varphi(r)$.

Using Lemma 3, we can prove our Theorem 2 which is an extension of F. and M. Riesz's theorem ([14], R is the unit open disc and $p = 1$).

Proof of Theorem 2. "if"-part is obvious. Let f be in the Hardy class $H_p(R)$ and set $v = p(\log|f|)$, $\varphi(r) = e^r$. Apply Lemma 3 to v and $\varphi(r)$. Obviously the conditions (A) and (B) are satisfied because $\varphi(v) = |f|^p$. The conclusion (E) proves our Theorem 2.

5. Let E be a closed polar set in a Riemann surface R. It is known that for any bounded and harmonic function u defined in $R-E$ there exists a bounded and harmonic function $\tilde{u}$ defined in R such that the restriction of $\tilde{u}$ to $R-E$ coincides with u ([1], [2]). For clarity, we shall show the following

LEMMA 4. *Let E be a closed polar set in a Riemann surface R and assume that u is a quasi-bounded harmonic function defined in $R-E$. Then there exists a quasi-bounded harmonic function $\tilde{u}$ defined in R such that the restriction of $\tilde{u}$ to $R-E$ coincides with u.*

Proof. We can consider only the case $u \geq 0$ (Jordan decomposition in the lattice $HP'(R)$). By the definition, u is the limiting function of a monotone non-decreasing sequence of bounded and harmonic functions and vice versa and hence our assertion is immediate.

Proof of Theorem 3. Let u be a quasi-bounded harmonic majorant of $\Psi(|f|)$ in $R-E$. By Lemma 4, u can be continued to R so that the resulting function $\tilde{u}$ is quasi-bounded harmonic in R. Consequently $\tilde{u}$ is bounded in any relatively compact open set G in R and hence f is bounded and analytic in $G-E$ because of the property of the function $\Psi(r)$. Hence f can be continued analytically to R and we have the assertions.

REMARK. We can take as $\Psi(r)$, for example, r^p (for $p>0$), $\log^+ r$, $\log r$, $\log(\log^+ r)$, $(\log^+\log^+ r)^p$ (for $p>0$), . . . , etc.

COROLLARY 1. (*An extension of Tumarkin-Havinson's theorem* [17]) *Let E be a closed polar set lying in a Riemann surface R. If a function f is in the Smirnov class $S(R-E)$, then there exists an analytic function $\tilde{f}$ in the Smirnov class $S(R)$ such that the restriction of $\tilde{f}$ to $R-E$ coincides with f.*

Proof. This is a consequence of Theorem 1 and Theorem 3 with $\Psi(r)=\log^+ r$.

COROLLARY 2. (*Parreau* [13], *Theorem* 20) *Let E be a closed polar set lying in a Riemann surface R. If a function f is in the class $H_p(R-E)$ for $p>0$, then there exists $\tilde{f}$ in the class $H_p(R)$ such that the restriction of $\tilde{f}$ to $R-E$ coincides with f.*

Proof. This is a consequence of Theorem 2 and Theorem 3 with $\Psi(r)=r^p$.

REMARK. Parreau's theorem can be proved, using Corollary 1 above, if we assume the fact that the polar set E is removable for non-negative superharmonic functions ([1], [2]).

W. Rudin ([15], at p. 49) pointed out that *the analogous assertion for the class AL is false.*

6. As usual we shall denote by O_X the totality of open Riemann surfaces R (including parabolic types) on which the given family $X(R)$ of functions consists only of constants. Then we have

$$O_{AL}\subset O_S\subset O_{H_p}\subset O_{AB} \qquad (\text{for } p>0).$$

Parreau ([13], p. 192) proved that the inclusion relation $O_{AL}\subset O_{H_p}$ (for $p>0$) is proper, using P.J. Myrberg's example in [10]. Using the fact that one point is removable for the Smirnov class S and the inequality: $\log^+|\alpha-\beta|^2\leq 2(\log^+|\alpha|+\log^+|\beta|+\log 2)$, for complex numbers α and β, we can prove that the inclusion relation $O_{AL}\subset O_S$ is proper by the same method as in [10].

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