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Actions of lattices in semisimple groups preserving a *G*-structure of finite type

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Abstract. In this paper we study actions of lattices in semisimple groups preserving a G-structure of finite type.

1. *Statement of results*

In this paper we continue the investigation begun in [**9**] concerning actions of arithmetic groups on compact manifolds. Here we shall focus attention on actions preserving a G-structure (possibly of higher order).

Let H be a connected semisimple Lie group with finite centre such that $\mathbb{R}$-rank $(H')\geq 2$ for every simple factor H' of H. Let $\Gamma\subset H$ be a lattice subgroup, i.e. Γ is discrete and H/Γ has a finite H-invariant measure. Let M be a compact manifold, $\dim(M)=n>0$. For any integer $k\geq 1$, we let $\mathrm{GL}(n,\mathbb{R})^{(k)}$ be the group of k-jets of diffeomorphisms of $\mathbb{R}^n$ leaving the origin fixed [**1**]. Suppose $G\subset \mathrm{GL}(n,\mathbb{R})^{(k)}$ is an algebraic subgroup, so that one may speak of a G-structure on M (which will be a structure of order k). If Q is a Lie group, we will denote its Lie algebra by $L(Q)$. The main point of this paper is to prove the following theorem.

THEOREM 1. *Let H, Γ, M be as above. Suppose that Γ acts smoothly on M and preserves a smooth volume density. Let $G\subset \mathrm{GL}(n,\mathbb{R})^{(k)}$ be an algebraic subgroup and suppose that the Γ-action on M preserves a G-structure. If*

(i) *Γ acts ergodically*;

(ii) *G is of finite type* (*in the sense of E. Cartan* [**3**]); *and*

(iii) *every homomorphism $L(H)\to L(G)$ is trivial*;

then there is a Γ-invariant C^0-Riemannian metric on M, and hence a Γ-invariant topological distance function. Thus there is a compact Lie group K, a closed subgroup $K_0\subset K$, and a homomorphism π: $\Gamma\to K$ with dense range such that the Γ-action on M is topologically conjugate to the Γ-action on K/K_0 defined by π.

We note that condition (iii) will be satisfied for example if $\mathbb{R}$-rank $(G/\mathrm{rad}(G))<\mathbb{R}$-rank (H') for every simple factor H' of H, where $\mathrm{rad}(G)$ is the radical of G. We recall that a Lorentzian metric on a manifold is an $\mathrm{O}(1,n-1)$ structure and an affine connection on M is a second order G-structure with $\mathbb{R}$-rank $(G/\mathrm{rad}(G))=n-1$. Both of these are of finite type [**1**]. Thus we obtain the following corollaries.

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Corollary 2. *Let* H, Γ *be as above. Then any smooth ergodic action of* Γ *preserving a Lorentzian metric on a compact manifold* M *actually leaves a* C^0*-Riemannian metric invariant. Thus the action is topologically conjugate to an action on* K/K_0 *as described in theorem* 1.

Corollary 3. *Let* H, Γ *be as above. Let* M *be a compact manifold,* $\dim M = n$. *Suppose there are an affine connection and a volume density on* M *such that* Γ *acts by volume preserving affine transformations of* M. *If*

(i) Γ *acts ergodically; and*

(ii) *every representation* $L(H) \to \mathfrak{sl}(n, \mathbb{R})$ *is trivial (e.g. if* $\mathbb{R}$*-rank* $(H') \geq n$ *for every simple factor* H' *of* H*);*

then there is a Γ*-invariant* C^0*-Riemannian metric, and the remaining conclusions of theorem* 1 *hold.*

For any Γ as above, it follows from the work of Margulis [**2**] that any compact Lie group K for which there is a dense range homomorphism $\Gamma \to K$ satisfies

$$\dim K \geq \min \{\dim H' | H' \text{ a simple factor of } H\}.$$

Since any compact group acting transitively on a topological n-manifold is a Lie group of dimension at most $n(n+1)/2$, Margulis' results combined with theorem 1 yield the non-existence of ergodic G-structure preserving actions in a suitable dimension range. One has for example, from corollary 3 and Margulis' result that the natural action of SL $(n, \mathbb{Z})$ on the torus T^n by group automorphisms is an ergodic affine volume preserving action of minimal dimension. More precisely:

Corollary 4. *Let* $n \geq 3$. *If* M *is a compact manifold with connection and volume density, and* $0 < \dim M < n$ *then there is no ergodic affine volume preserving action of* SL $(n, \mathbb{Z})$ *on* M.

Remarks. (i) Theorem 1 was conjectured in [**9**] without the assumption that the G-structure be of finite type. It remains open for G-structures of infinite type. Similarly the conjecture in [**9**] did not assume ergodicity and it remains open as to whether or not this hypothesis is necessary.

(ii) The notion of finite type in the sense of E. Cartan is based on the termination after finitely many steps of an inductively defined natural prolongation scheme. Tanaka [**4**], [**5**] has introduced another natural prolongation scheme for G-structures which will be finite for certain infinite Cartan type (in fact even non-elliptic) G-structures. Our proof of theorem 1 will be valid for G-structures of finite type in the sense of Tanaka as well.

(iii) Describing the ways in which Γ can act by automorphisms of a G-structure is a natural geometric generalization of the algebraic question of describing the homomorphisms of Γ into G. If G is an algebraic group, then for Γ as above this latter question has to a large extent been resolved by Margulis [**2**]. (See also [**8**].) Margulis' results imply for example that if there are only trivial homomorphisms $L(H) \to L(G/\text{rad}(G))$, then any homomorphism of Γ into G has a precompact image.

Theorem 1 can thus be considered as a geometric extension of this algebraic result of Margulis. We also remark that the group of automorphisms, A, of a G-structure of finite type is known to be a Lie group [**1**]. Furthermore, every simple subalgebra of $L(A/\text{rad } A)$ must then be a Lie subalgebra of $L(G/\text{rad } G)$ [**10**]. However, the group of connected components of A may well be infinite and hence one cannot simply apply [**10**] and Margulis' results.

One also has analogous (in fact stronger) results for actions of suitable lattices over p-adic fields. Namely:

THEOREM 5. *Let k be a totally disconnected local field of characteristic* 0, *H a connected k-group, almost k-simple, with k-rank*$(H)\geq 2$. *Let* $\Gamma\subset H_k$ *be a lattice. Then any volume preserving ergodic action of* Γ *on any compact manifold (of positive dimension) preserving a (possibly higher order) G-structure of finite type leaves a* C^0*-Riemannian metric invariant.*

In particular, this applies to ergodic actions by Lorentz transformations and to ergodic volume preserving actions by affine transformations. In [**9**] we conjectured that any volume preserving action of Γ on a compact manifold is isometric, and theorem 5 can be considered as a result in this direction.

2. *Preliminaries to the proofs*

We shall assume that the reader is familiar with G-structures, G-structures of higher order, and G-structures of finite type. (See [**1**], [**3**] for example.) We shall also assume familiarity with the discussion of measurable cocycles and measurable invariant metrics in [**9**, §§ 2, 3]. As in [**9**], our basic approach will be first to obtain measure theoretic information about the actions in question, and then to convert this to continuous information which will imply our results. Again as in [**9**] we will obtain the measure theoretic information by an application of the superrigidity theorem for cocycles which we proved in [**6**] (a generalization of Margulis' superrigidity theorem [**2**], [**8**]) and via Kazhdan's property; the conversion to continuous information will be achieved via the Sobolev embedding theorem.

For the following discussion, by an algebraic group we will mean the $\mathbb{R}$-points of an algebraic $\mathbb{R}$-group. If Q is a Lie group, $L(Q)$ will denote its Lie algebra. We let H and Γ be as in the second paragraph of § 1. Suppose G is an algebraic group. Let (S,μ) be a standard measure space and suppose Γ acts measurably on S so that μ is a finite Γ-invariant measure. Let $\alpha: S\times\Gamma\to G$ be a (measurable) cocycle. The proof of [**9**, theorem 2.8] (based on the superrigidity theorem and employing Kazhdan's property as well) shows the following.

THEOREM 6. *Let* H, Γ, G, α *be as above. Suppose every homomorphism* $L(H)\to L(G)$ *is trivial. Then* α *is (measurably) equivalent to a cocycle into a compact subgroup of G.*

We re-emphasize that the condition on the Lie algebras holds for example if $\mathbb{R}$-rank $(G/\text{rad }(G))<\mathbb{R}$-rank (H') for every simple factor H' of H. It holds in many other cases as well of course.

We shall need two other results proved in [**9**].

LEMMA 7. *Let M be a second countable topological space and μ a finite measure on M which is positive on open sets. Let G be a locally compact second countable group and $P \to M$ a (continuous) principal G-bundle. Suppose Γ is a countable group and that Γ acts by principal bundle automorphisms of P covering a μ-preserving ergodic action of Γ on M. Suppose there is a Γ-invariant function $f \in L^2(P) \cap C(P)$. (We remark that the left Haar measure on G defines a measure on each fibre, and hence together with μ defines a measure on P.) Then there is*

(i) *a compact subgroup $K \subset G$; and*

(ii) *an open Γ-invariant conull set $W \subset M$;*

such that there is a continuous Γ-invariant section $\phi: W \to P/K$ of the natural projection $P/K \to M$.

This result is not explicitly stated in [**9**] but it is easily extracted from [**9**, lemma 5.9] and the discussion preceding it. We remark that although the principal bundle in question in [**9**] is a reduction of the frame bundle of a manifold, this is irrelevant for deducing lemma 7.

The conclusion of the proof of [**9**, theorem 5.1] also shows the following.

LEMMA 8. *Let M be a compact manifold on which a group Γ acts ergodically by volume density preserving diffeomorphisms. Let $W \subset M$ be an open, conull Γ-invariant subset on which there is a Γ-invariant C^0-Riemannian metric. Then $W = M$ and the isometry group of M with respect to the induced topological distance function is transitive.*

3. *Proof of the theorems*

We prove theorem 1. Let $G^{(k)}$ be the kth prolongation of G. Then $G^{(k)}$ is an algebraic group which in fact is a semi-direct product $G \ltimes N_k$ where N_k is a connected unipotent group. We let $P^{(k)} \to M$ be the corresponding prolongation of the G-structure so that we have maps

$$\begin{array}{ccc} P^{(k)} & \xrightarrow{q} & P \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

Each map is a principal bundle projection, q defining an N_k-bundle, π the given G-bundle, and $\pi \circ q$ a $G^{(k)}$-bundle. Since G is of finite type, there is some k for which $G^{(l)} = G^{(k)}$ for all $l \geq k$, and we set $G^{(k)} = \tilde{G}$, $P^{(k)} = \tilde{P}$. The automorphism group of the G-structure, Aut (P), acts on $\tilde{P}$, and as a consequence of the choice of k, there is a complete parallelism on the manifold $\tilde{P}$ that is invariant under Aut (P). In particular, there is a Riemannian metric ξ on $\tilde{P}$ that is invariant under Aut (P). As described in [**9**, § 3], for any integer $r \geq 0$ the metric ξ on the vector bundle $T\tilde{P} \to \tilde{P}$ canonically determines a metric ξ_r on the vector bundle $J^r(\tilde{P}; \mathbb{R}) \to \tilde{P}$ of r-jets of smooth real valued functions on $\tilde{P}$. Since ξ is Aut (P)-invariant, it follows that ξ_r is as well. We form the space of L^2-sections of this bundle with respect to the metric ξ_r on the fibres, which we denote by $L^2(J^r(\tilde{P}; \mathbb{R}))_{\xi_r}$. The group Aut (P) acts naturally on the sections of this bundle and since ξ_r is Aut (P)-invariant, we obtain a unitary representation of Aut (P) on the Hilbert space $L^2(J^r(\tilde{P};\mathbb{R})_{\xi_r}$. The Sobolev space $L^{2,r}_{\xi_r}(\tilde{P})$ is, as usual, defined to be the completion of

$$\{f\in C^\infty(\tilde{P};\mathbb{R})|j^r(f)\in L^2(J^r(\tilde{P};\mathbb{R}))_{\xi_r}\}$$

with respect to the norm induced by the embedding in $L^2(J^r(\tilde{P};\mathbb{R}))_{\xi_r}$. Thus $L^{2,r}_{\xi_r}(\tilde{P})$ is a Hilbert space which we can identify with a subspace of $L^2(J^r(\tilde{P};\mathbb{R}))_{\xi_r}$, and this subspace is clearly Aut (P)-invariant. We thus obtain a unitary representation of Aut (P) on the Sobolev space $L^{2,r}_{\xi_r}(\tilde{P})$. We have a natural norm-decreasing linear map $i: L^{2,r}_{\xi_r}(\tilde{P})\to L^2(\tilde{P})$ with dense image which intertwines the unitary representations of Aut (P) on these spaces. We let $i^*: L^2(\tilde{P})\to L^{2,r}_{\xi_r}(\tilde{P})$ be the adjoint map. This is again a norm-decreasing linear map which is injective (owing to density of the image of i) and which intertwines the unitary representations of Aut (P). (This seemingly innocent last remark is in fact basic to the proof. If the representation on $L^{2,r}_{\xi_r}(\tilde{P})$ was not unitary, i^* would intertwine the adjoint actions on these Hilbert spaces, and on the Sobolev space this would not agree with the standard natural action. Thus Aut (P)-invariance of ξ_r, which we obtained from a complete parallelism, plays a crucial role.)

We have a homomorphism of Γ into Aut (P) and we now consider the above unitary representations restricted to Γ. As in [9, § 3], if we measurably trivialize the bundle $\tilde{P}\to M$ by writing $\tilde{P}\cong M\times\tilde{G}$, the action of Γ on $\tilde{P}$ will be given by

$$\gamma\cdot(m,g)=(\gamma m,\alpha(m,\gamma)^{-1}g)$$

where $\alpha: S\times\Gamma\to\tilde{G}$ is a cocycle. By theorem 6, α is (measurably) equivalent to a cocycle β such that $\beta(M\times\Gamma)\subset C$, where $C\subset\tilde{G}$ is a compact subgroup. Thus, if $f\in L^2(\tilde{G})$ is a non-0 function such that $f(ag)=f(g)$ for all $a\in C$, $g\in\tilde{G}$, the function $f'\in L^2(M\times\tilde{G})$ given by $f'(m,g)=f(g)$ will be Γ-invariant under the action on $M\times\tilde{G}$ defined by the cocycle β (i.e. $\gamma(m,g)=(\gamma m,\beta(m,\alpha)^{-1}g)$). Since $\alpha\sim\beta$, the Γ-actions these cocycles define are measurably conjugate and hence there is a non-0 Γ-invariant function $F\in L^2(\tilde{P})$. It follows that for any r, $i^*(F)\in L^{2,r}_{\xi_r}(\tilde{P})$ is also a non-0 Γ-invariant function. Since ξ_r is a smooth metric, if we choose r sufficiently large, the Sobolev embedding theorem implies that $i^*(F)\in L^2(\tilde{P})\cap C^0(\tilde{P})$.

We can now apply lemma 7. Thus, there is an open, conull, Γ-invariant set $W\subset M$, a compact subgroup $K\subset\tilde{G}$, and a continuous Γ-invariant section $\phi: W\to\tilde{P}/K$. However, $\tilde{G}=G\ltimes N$ where N is a connected unipotent group. Thus, we can assume K actually lies in G. Hence we have a natural Γ-map $\tilde{P}/K\to P/K$. Composing ϕ with this map we obtain a continuous Γ-invariant section $\psi: W\to P/K$. Since P is the frame bundle, K is contained in a conjugate of $O(n,\mathbb{R})$, and ψ thus defines a Γ-invariant C^0-Riemannian metric on W. We then apply lemma 8, from which all assertions in theorem 1 follow.

Theorem 5 follows in the same manner using the following version of theorem 6.

THEOREM 9. *Let Γ be as in theorem 5. Suppose S is a standard Borel Γ-space, that μ is a finite Γ-invariant measure on S, and that $\alpha: S\times\Gamma\to G$ is a cocycle where G is a real algebraic group. Then α is equivalent to a cocycle β with $\beta(S\times\Gamma)$ contained in a compact subgroup of G.*

As with theorem 6, theorem 9 follows from superrigidity for cocycles of Γ-actions (see [7], [**8**]) and Kazhdan's property via the arguments of [**9**, theorem 2.8]. (Cf. [**9**, proof of theorem 6.3].)

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