A NOTE ON NEAR-RINGS OF MAPPINGS

Dedicated to the memory of Hanna Neumann

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(Received 15 May 1972)

Communicated by M. F. Newman

Hanna Neumann was one of the pioneers of the algebraic theory of near-rings. In the functional analysis, the near-rings appear as the sets of non-linear mappings of a space into itself. The relations between the algebraic properties of these near-rings and the analytic structures of the spaces and mappings involved have not been fully investigated. The purpose of this note is to consider one of such problems.

Let E be a real Banach space and its dimension be not less than two. Let $\mathscr{A}$ be a near-ring whose elements are continuous mappings of E into itself. By $I(E)$ we denote the set of all constant mappings of E into itself, and we denote the set of all continuous linear mappings of E into itself by $L(E)$.

THEOREM. *If $\mathscr{A}$ contains $I(E)$ and $L(E)$, then every automorphism ϕ is inner.*

PROOF. By Yamamuro [5, Theorem 1], which is a modification of a method used by Magill [3], there exists a bijection h of E into itself such that

$$(*) \qquad \phi(f) = hfh^{-1} \qquad \text{for every } f \in \mathscr{A}.$$

Take any x such that $h^{-1}(x) \neq 0$ and take $\bar{a} \in \bar{E}$ (the conjugate space of E) such that $\langle h^{-1}(x), \bar{a} \rangle = 1$, where the left-hand side denotes the value of $\bar{a}$ at $h^{-1}(x)$. Then, for arbitrary $a \in E$ and $b \in E$, since the linear mappings $a \otimes \bar{a}$ and $b \otimes \bar{a}$, defined, for instance, by $(a \otimes \bar{a})(z) = \langle z, \bar{a} \rangle a$ for every $z \in E$, belong to $L(E) \subset \mathscr{A}$, we have

$$\phi(a \otimes \bar{a} + b \otimes \bar{a})(x) = \phi(a \otimes \bar{a})(x) + \phi(b \otimes \bar{a})(x),$$

which, together with (*), implies

$$h(a + b) = h(a) + h(b).$$

Hence, since we have the same property for h^{-1}, for any $u \in L(E)$, $\phi(u)$ is continuous and additive. Consequently, $\phi(u) \in L(E)$. In other words, ϕ maps $L(E)$ into $L(E)$. To show that ϕ is onto, take $u \in L(E)$. Since $\phi: \mathscr{A} \to \mathscr{A}$ is onto, there is $f \in \mathscr{A}$ such that $\phi(f) = u$. By (*), we have $f = h^{-1}uh$ and, hence, f is continuous and additive. Thus, $f \in L(E)$.

Therefore, ϕ is an automorphism of the ring $L(E)$ onto itself. Thus, a theorem of Eidelheit [2] implies that there is an invertible element h_1 of $L(E)$ such that

$$\phi(u) = h_1 u h_1^{-1} \qquad \text{for every } u \in L(E).$$

Then, by using $a \otimes \bar{a}$ defined above, we get $h_1 = h$, which means that $h \in L(E)$ and ϕ is inner.

We add a few remarks.

1. Let S be an s-category defined by Bonic and Frampton [1]. Then, for any Banach space E, *$S(E) = S(E,E)$ is a near-ring and it contains $I(E)$ and $L(E)$*. Therefore, every automorphism of the near-ring $S(E)$ is inner. The near-ring $C^0(E)$ of all continuous mappings of E into itself; the near-ring $D(E)$ of all Fréchet differentiable mappings of E into itself and the near-ring $C^n(E)$ of all n-times $(n = 1,2,\cdots)$ continuously differentiable mappings of E into itself are special cases of $S(E)$.

2. By the same method as the above proof, we can show that for Banach spaces E and F, if the near-rings $\mathscr{A}(E)$ and $\mathscr{A}(F)$ satisfy the conditions of the theorem and isomorphic algebraically, then E and F are topologically and algebraically isomorphic. Therefore, for instance, if $S(E)$ and $S(F)$ are isomorphic as near-rings, then E and F are isomorphic as Banach spaces.

3. As we have mentioned above, Eidelheit [2] has shown that every automorphism of the ring $L(E)$ is inner. Later, Rickart [4] has shown that every automorphism of the *semigroup* $L(E)$ is inner .The corresponding semigroup version of our theorem has not been obtained.

References

[1] R. Bonic and J. Frampton, 'Smooth functions on Banach manifolds', *J. Math. Mec.* 15 (1966), 877–898.

[2] M. Eidelheit, 'On isomorphisms of rings of linear operators', *Studia Math.* 9 (1940), 97–105.

[3] K. D. Magill, Jr., 'Automorphisms of the semigroup of all differentiable functions', *Glasgow Math. J.* 8 (1967), 63–66.

[4] C. E. Rickart, 'One-to-one mappings of rings and lattices', *Bull. Amer. Math. Soc.* 54 (1968), 758–764.

[5] S. Yamamuro, 'A note on semigroups of mappings on Banach spaces', *J. Austral. Math. Soc.* 9 (1969), 455–464.

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