Yoshiaki Hashimoto
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HYPOELLIPTIC OVERDETERMINED SYSTEMS WITH VARIABLE COEFFICIENTS

YOSHIAKI HASHIMOTO

1. Introduction

Let

$$p(x, D)u = f \tag{1}$$

be a system of partial difierential equations. We shall say that $p(x, D)$ is hypoelliptic if the distribution solution u is in C^∞ wherever $f \in C^\infty$ (cf. section 2, Definition 5.)

Here we shall give a sufficient condition for the hypoellipticity of over-determined systems with variable coefficients.

For determined and overdetermined systems with constant coefficients a necessary and sufficient condition was obtained by Lech [5], Hörmander [1], Malgrange [6] and Matsuura [7]. On the other hand, Volevič [8] gave a sufficient condition for determined systems with variable coefficients. His condition corresponds to the formally hypoelliptocity in the scalar case. Furthermore, a more general sufficient condition was obtained by Hörmander [2]. For overdetermined systems with variable coefficients, Kato [3] gave a sufficient condition as an extension of the Volevič condition.

As in [2] our method is to construct a left parametrix by pseudo-differential operators. § 2 is devoted to some properties of matrices of pseudo-differential operators. In § 3 we shall state a main theorem on a sufficient condition for the hypoellipticity of overdetermined systems with variable coefficients. We shall prove this theorem in § 4.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space with coordinates $(x_1, \cdots, x_n)$ and Ω an open subset in R^n. We denote the set of all ν-dimensional vectors whose components are C^∞-functions in Ω and those with compact sup-

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ports in Ω by $C^\infty(\Omega; C^\nu)$ and by $C_0^\infty(\Omega; C^\nu)$, respectively.

Let $p(x,D)$ be a $\mu\times\nu$ matrix

$$p(x,D) = \sum_{|\alpha|\leqq m} a_\alpha(x)D^\alpha, \tag{2}$$

of partial differential operators, where $\alpha=(\alpha_1,\cdots,\alpha_n)$ ($\alpha_i\geqq 0$, integer) is a multi-index, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $\alpha!=\alpha_1!,\cdots,\alpha_n!$, $D_j=-i\dfrac{\partial}{\partial x_j}$ ($i=\sqrt{-1}$, $j=1,\cdots,n$), $D=(D_1,\cdots,D_n)$ and $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$. If we denote the Fourier transform of $u\in C_0^\infty(\Omega; C^\nu)$ by $\hat{u}(\xi)=\int u(x)e^{-i\langle x,\xi\rangle}dx$ where $\langle x,\xi\rangle=x_1\xi_1+\cdots+x_n\xi_n$, then $p(x,D)\,u$ is given by the μ-dimensional vector

$$p(x,D)u(x) = (2\pi)^{-n}\int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi. \tag{3}$$

Here we set $p(x,\xi)=\sum_\alpha a_\alpha(x)\xi^\alpha$.

The class of partial differential operators can be extended as follows.

Definition 1. If m, ρ and δ are real numbers with $1\geqq\rho>0$ and $\delta\geqq 0$, we denote by $S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ the set of all $p(x,\xi)\in C^\infty(\Omega\times R^n; C^\nu, C^\mu)$ such that for every compact subset K in Ω and all multi-indices α, β we have with a consatnt $C_{\alpha,\beta,K}$

$$|D_\xi^\alpha D_x^\beta p(x,\xi)| \leqq C_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},\ x\in K \text{ and } \xi\in R^n,$$

where $|\ \ |$ denotes an operator norm of matrices from C^ν into C^μ. Set $\bigcup_m S^m_{\rho,\delta}=S^\infty_{\rho,\delta}$ and $\bigcap_m S^m_{\rho,\delta}=S^{-\infty}_{\rho,\delta}$.

It is easy to see that matrices of partial differential operators are contained in $S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$. For $p\in S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ we define an operator $p(x,D)$ as follows:

Definition 2. (See [2], for example.) For $p\in S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ we can define

$$p(x,D)u(x) = (2\pi)^{-n}\int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi \tag{4}$$

where $u\in C_0^\infty(\Omega; C^\nu)$ and $x\in\Omega$.

Then we shall have the following properties for these operators, which are extensions of the results obtained by Hörmander [2] for the scalar case.

Proposition 1. *Let $p(x,\xi)\in S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ and assume that $\delta<1$. The operator $p(x,D)$ defined by (4) is then a continuous linear mapping of $C_0^\infty(\Omega; C^\nu)$ into*

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$C^\infty(\Omega; C^\mu)$, and it can be extended to a continuous linear mapping of $\mathscr{E}'(\Omega; C^\nu)$ into $\mathscr{D}'(\Omega; C^\mu)$. The distribution kernel of $p(x, D)$ is a C^∞-function outside the diagonal in $\Omega\times\Omega$; it is in $C^j(\Omega\times\Omega; C^\nu, C^\mu)$ if $j+m+n<0$. For every $u\in\mathscr{E}'(\Omega; C^\nu)$,

$$\text{(5)} \qquad \textit{sing supp } p(x, D)u \subset \textit{sing supp } u,$$

where sing supp u means the singular support of u.

Proof. The proof of this theorem is reduced to the scalar case in Hörmander's paper [2].

Since $p(x, D) = (p_{ij}(x, D))_{i=1,\cdots,\mu, j=1,\cdots,\nu}$, it follows, by Hörmander's result [2], that $p_{ij}(x, D)$ is a continuous linear mapping of $C_0^\infty(\Omega)$ into $C^\infty(\Omega)$ and it can be extended to a continuous linear mapping of $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$. Hence $p(x, D)$ becomes a contiunous linear mapping of $C_0^\infty(\Omega; C^\nu)$ into $C^\infty(\Omega; C^\mu)$, and it can be extended to a continuous linear mapping of $\mathscr{E}'(\Omega; C^\nu)$ into $\mathscr{D}'(\Omega; C^\mu)$. Furthermore considering the results for the scalar pseudo-differential operator $p_{ij}(x, D)$, we obtain that the distribution kernel of $p(x, D)$ is a C^∞-function outside the diagonal in $\Omega\times\Omega$ and that it is in $C^j(\Omega\times\Omega; C^\nu, C^\mu)$ if $j+m+n<0$. Finally from the scalar pseudo-local property, we have

$$\text{sing supp } p(x, D)\, u \subset \text{sing supp } u$$

for $u\in\mathscr{E}'(\Omega; C^\nu)$. Thus we have the proposition 1.

The following propositions are reduced to the scalar case in the same way as in Proposition 1. Hence we shall not describe the details.

PROPOSITION 2. *The space $S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ is a linear subspace. If $p\in S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ and $q\in S^{m'}_{\rho,\delta}(\Omega; C^\lambda, C^\nu)$, then $p^{(\alpha)}_{(\beta)}\in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$, and $pq\in S^{m+m'}_{\rho,\delta}(\Omega; C^\lambda, C^\mu)$. If $p_j\in S^{m_j}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$, $j=0, 1, 2, \cdots$ and $m_j\to -\infty$, one can find $p\in S^{m_0}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ such that for every k*

$$\text{(6)} \qquad p-\sum_{j<k} p_j \in S^{m_k}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$$

where $m_k = \max_{j\geq k} m_j$. The function p is uniquely determined modulo $S^{-\infty}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$.

We shall say that p has an asymptotic expansion $\sum_{j=0}^\infty p_j$ and we express $p\sim\sum_{j=0}^\infty p_j$.

PROPOSIITON 3. *If $p\in S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$, the kernel of $p(x, D)$ is in C^∞ if and only if $p\in S^{-\infty}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$.*

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PROPOSITION 4. *Let* $p_j \in S^{m_j}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$, $j = 0, 1, 2, \cdots$, *and assume that* $m_j \to -\infty$ *when* $j \to \infty$. *Let* $p \in C^\infty(\Omega \times R^n)$ *and assume that for all multi-indices* α, β *and compact sets* K *there exist some* C *and* μ *depending on* α, β *and* K *such that*

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C(1 + |\xi|)^\mu, \quad x \in K.$$

If there exist numbers $\mu_k \to -\infty$ *such that*

$$|p(x, \xi) - \sum_{j<k} p_j(x, \xi)| \leqq C_{K,k}(1 + |\xi|)^{\mu_k}, \quad x \in K, \tag{7}$$

it follows that $p \in S^{m_0}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$, *where* $m_0 = \sup m_j$ *and that* $p \sim \sum_{j=0}^{\infty} p_j$.

PROPOSITION 5. (*Leibniz' formula*) *Let* $p \in S^{m'}_{\rho',\delta'}(\Omega; C^\nu, C^\mu)$ *and* $q \in S^{m''}_{\rho'',\delta''}(\Omega; C^\lambda, C^\nu)$ *where* $\delta' < \rho'' \leqq 1$. *Set* $\delta = \max(\delta', \delta'')$, $\rho = \min(\rho', \rho'')$ *and choose* $f \in C^\infty_0(\Omega)$. *Then there is an element* $r \in S^{m'+m''}_{\rho,\delta}(\Omega; C^\lambda, C^\mu)$ *such that* $r(x, D)\, u = q(x, D)\, f p(x, D) u$ *for* $u \in C^\infty_0(\Omega; C^\nu)$, *and*

$$r(x, \xi) \sim \sum_\alpha q^{(\alpha)}(x, \xi)\, D^\alpha_x(f(x)\, p(x, \xi))/\alpha! \tag{8}$$

where $q^{(\alpha)}$ *denote the* α-*th derivative of* q *in* ξ.

We extend the operators $p(x, D)$ to the more general class.

DEFINITION 3. By $L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ we denote the set of all continuous linear mappings $P: C^\infty_0(\Omega; C^\nu) \to C^\infty(\Omega; C^\mu)$ such that for all $f \in C^\infty_0(\Omega)$ there exists some $p_f \in S^m(\Omega; C^\nu, C^\mu)$ with $P(fu) = p_f(x, D)u$, for $u \in C^\infty_0(\Omega; C^\mu)$.

If $p(x, \xi) \in S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$, then $p(x, D) \in L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$.

To consider the multiplication of these operators, we shall impose the following condition.

DEFINITION 4. We shall say that $P \in L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ is compactly supported if for every compact set K in Ω there is another compact set K' in Ω such that if $u \in C^\infty_0(\Omega; C^\nu)$ and supp $u \subset K$ it follows that supp $Pu \subset K'$, and if $u \in C^\infty_0(\Omega; C^\nu)$ and u vansihes in K' it follows that Pu vanishes in K.

The multiplication of compactly supported operators is also compactly supported.

The following proposition is a representation formula for $P \in L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$.

PROPOSITION 6. *Let P be a compactly supported operator of $L^m(\Omega; C^\nu, C^\mu)$. Then one can find $p(x,\xi)\in S^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ such that $Pu = p(x,D)u$. We shall call $p(x,\xi)$ a symbol of P.*

Proof. This is shown easily by reducing to the scalar case, so we may omit the detail here.

We consider the adjoint of a pseudo-differential operator. In doing so, we write

$$(u,v) = \sum_{j=1}^{\mu}\int u_j(x)\, v_j(x)dx,\ u,v\in C_0^\infty(\Omega; C^\mu).$$

PROPOSITION 7. *Let P be a compactly supported operator of $L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ $(0\leqq\delta<\rho\leqq1)$. Then there is one and only one compactly supported operator $P^*\in L^m_{\rho,\delta}(\Omega; C^\mu, C^\nu)$ such that*

$$(Pu,v) = (u,P^*v),\ u\in C_0^\infty(\Omega; C^\nu),\ v\in C_0^\infty(\Omega; C^\mu);$$

the symbol $\sigma(P^*)$ of P^* is given by

$$\sigma(P^*)\sim\sum_\alpha D_x^\alpha p^{*(\alpha)}(x,\xi)/\alpha!, \tag{9}$$

where p^* is adjoint of the matrix $p(x,\xi)$.

We can easily see that if P is compactly supported, then p^* is so.

3. A sufficient condition on the hypoellipticity

To describe a sufficient condition for the hypoellipitcity of operators, let us give its definition.

DEFINITION 5. Let P be a compactly supported operator of $L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ $(0\leqq\delta<\rho\leqq1)$. Then we shall say that P is hypoelliptic if

$$\text{sing supp } Pu = \text{sing supp } u \quad \text{for } u\in\mathscr{D}'(\Omega, C^\nu). \tag{10}$$

We can assert the following main theorem on a sufficient condition for the hypoellipticity. We shall impose that all components of our operator are of the same order.

THEOREM. *Let P be a compactly supported operator of $L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ where $\mu>\nu$ and $p(x,\xi)$ be its symbol. If there exists a compactly supported operator Q in $L^{m'}_{\rho,\delta}(\Omega; C^\mu, C^\nu)$ with symbol $q(x,\xi)$ and if there exist two non-singular $\nu\times\nu$ matrices $A(x,\xi)$ and $B(x,\xi)$ satisfying following conditions* (I) $\sim$ (III), *then P is hypoelliptic.*

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(I) $$|A(x,\xi)\, q_{(\beta)}^{(\alpha)}(x,\xi)\, p_{(\beta')}^{(\alpha')}(x,\xi)\, B(x,\xi)| \leqq C_{\alpha,\alpha',\beta,\beta',K}(1+|\xi|)^{-\rho|\alpha+\alpha'|+\delta|\beta+\beta'|},$$
$$x \in K,$$

(II) $$|B(x,\xi)^{-1}(q(x,\xi)p(x,\xi))^{-1}A(x,\xi)^{-1}| \leqq C_K,$$
$$x \in K \text{ and } |\xi| > C_K,$$

(III) *there is a real number* m'' *such that* $|A(x,\xi)| + |B(x,\xi)| \leqq C_K|\xi|^{m''}$, $x \in K$ *and* $|\xi| > C_K$.

COROLLARY. *Let* P *be a compactly supported operator of* $L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ *where* $\mu > \nu$. *If the symbol* $p(x,\xi)$ *of* P *satisfies the following conditions* (I)$'$ ~ (III)$'$ *for two non-singular* $\nu\times\nu$ *matrices* $A(x,\xi)$ *and* $B(x,\xi)$, *then* P *is hypoelliptic.*

(I)$'$ $$|A(x,\xi)p^{*(\alpha)}_{(\beta)}(x,\xi)\, p_{(\beta')}^{(\alpha')}(x,\xi)\, B(x,\xi)| \leqq C_{\alpha,\alpha',\beta,\beta',K}(1+|\xi|)^{-\rho|\alpha+\alpha'|+\delta|\beta+\beta'|}$$
$$x \in K,$$

(II)$'$ $$|B(x,\xi)^{-1}(p^*(x,\xi)\, p(x,\xi))^{-1}A(x,\xi)^{-1}| \leqq C_K, \quad x \in K \text{ and } |\xi| > C_K,$$

(III)$'$ *there is a real number* m' *such that* $|A(x,\xi)| + |B(x,\xi)| \leqq C_K|\xi|^{m'}$, $x \in K$ *and* $|\xi| > C_K$.

Remark 1. By the corollary we can easily see that if P is elliptic in the sense of Komatsu [4] whose components are of the same order, then P is hypoelliptic.

Remark 2. The conditions (I) ~ (III) of the main theorem is invariant under diffeomorphisms of $\Omega (1-\rho \leqq \delta < \rho)$.

In the next section we shall prove the main theorem and the corollary.

4. Proof of the theorem

First we shall introduce the following:

DEFINITION 6. Let P be a compactly supported operator of $L^{m'}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ $(0 \leqq \delta < \rho \leqq 1)$. We shall say that P *has a left parametrix* if there exists a compactly supported operator $E \in L^{m'}_{\rho,\delta}(\Omega; C^\mu, C^\nu)$ for some real number m' such that the symbol of EP-I is identically equal to zero.

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Next we shall show that the existence of a left parametrix is a sufficient condition for the hypoellipticity.

PROPOSITION 8. (*cf.* [2]). *Let P be a compactly supported operator of $L^m_{\rho,\delta}$ $(\Omega;\, C^\nu, C^\mu)$ $(0 \leqq \delta < \rho \leqq 1)$. If P has a left parametrix, then P is hypoelliptic.*

Proof. First we shall prove that sing supp $Pu \subset$ sing supp u for $u \in \mathscr{D}'(\Omega, C^\nu)$. Let $x_0 \in (\text{sing supp } u)^C$. Since $(\text{sing supp } u)^c$ is open, we can choose a neighborhood U_1 of x_0 such that sing supp $u \cap U_1 = \phi$. We take a function $\varphi \in C_0^\infty(U_1)$ such that $\varphi \equiv 1$ in some neighborhood $U_2(\Subset U_1)$ of x_0. Then $u = \varphi u + (1-\varphi)u$. Since $\varphi u \in C_0^\infty(U_1, C^\nu)$, we see $P(\varphi u) \in C^\infty(\Omega;\, C^\mu)$. Hence, in particular, $P(\varphi u) \in C^\infty(U_1, C^\mu)$. So we consider only the second term $P(1-\varphi)u$. Taking some neighborhood U_3 of x_0 such that $U_3 \Subset U_2$, we consider the following bilinear form for $v \in C_0^\infty(U_3;\, C^\mu)$:

$$(P(1-\varphi)u, v) = ((1-\varphi)u, P^*v).$$

Since P^* has a pseudo-local property, i.e., (5), supp P^*v can be contained in some compact set K in Ω. We take $\chi \in C_0^\infty(\Omega)$ such that $\chi \equiv 1$ in some neighborhood of K. Then

$$((1-\varphi)u,\, P^*v) = ((1-\varphi)u,\, \chi P^*v) = (P\chi(1-\varphi)u, v).$$

Let ψ be in $C_0^\infty(U_2)$ and $\psi \equiv 1$ in some neighborhood of U_3. Then

$$(P(1-\varphi)u, v) = (P\chi(1-\varphi)u, \psi v) = (\psi P\chi(1-\varphi)u, v).$$

Since the symbol of $\psi P\chi(1-\varphi)u$ is identically zero, we have the following estimate

$$|(\psi P\chi(1-\varphi)u, v)| \leqq C\|v\|_s \quad \text{for} \quad v \in C_0^\infty(U_3;\, C^\mu),$$

where $\|v\|_s = ((2\pi)^{-n}\int(1+|\xi|^2)^s|\hat{v}(\xi)|^2 d\xi)^{1/2}$. Hence $P(1-\varphi)u \in H_{-s}(U_3, C^\mu)$. As s is an arbitrary real number, it follows that $P(1-\varphi)u \in C^\infty(U_3;\, C)$. Since x_0 is an arbitrary point in $(\text{sing supp } u)^c$, we see that sing supp $Pu \subset$ sing supp u.

Hence to prove the hypoellipticity, it is sufficient to show that sing supp $u \subset$ sing supp Pu.

Set $Pu = f$ and let E be a left parametrix for P. Then u can be expressed by

$$u = EPu - (EP - I)u = Ef - (EP - I)u.$$

Since E is a left parametrix for P, the symbol of $EP\text{-}I$ is equal to zero. Hence Proposition 1 implies that $EP\text{-}I$ has a C^∞-distribution kernel. So $(EP\text{-}I)u \in C^\infty(\Omega; C^\nu)$. For the first term Ef we can use the pseudo-local property for $\mathcal{D}'(\Omega, C^\nu)$ and we obtain that sing supp $Ef \subset$ sing supp $f =$ sing supp Pu. Thus we have the proposition.

Next due to Hörmander is a sufficient condition for the existence of a left parametrix.

PROPOSITION 9. (*cf.* [2]). *Let* $P \in L^m_{\rho,\delta}(\Omega; C^\nu, C^\mu)$ $(0 \leqq \delta < \rho \leqq 1)$ *and assume that there exist* $\nu \times \nu$ *matrices* $A(x,\xi)$ *and* $B(x,\xi)$ *such that the symbol matrix* $p(x,\xi)$ *of* P *satisfies the conditions*

$$|A(x,\xi)p^{(\alpha)}_{(\beta)}(x,\xi)B(x,\xi)| \leqq C_{\alpha,\beta,K}(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|}, \quad x \in K \text{ and } \xi \in R^n \tag{11}$$

for any compact subset K *of* Ω *and any multi-indices* α *and* β. *Also assume that*

$$|B(x,\xi)^{-1}p(x,\xi)^{-1}A(x,\xi)^{-1}| \leqq C_K, \quad x \in K \text{ and } |\xi| > C_K \tag{12}$$

and that there is a real number m' *satisfying*

$$|A(x,\xi)| + |B(x,\xi)| \leqq C_K|\xi|^{m'}, \quad x \in K \text{ and } |\xi| > C_K. \tag{13}$$

Then there exists a left parametrix $E \in L^{2m'}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$.

Now we can prove the main theorem and the corollary.

Proof of the main theorem.

Let P be an operator satisfying the conditions of the theorem. By Proposition 5 it follows that the symbol $r(x,\xi)$ of the operator QP is given by an asymptotic expansion

$$r(x,\xi) \sim \sum_\alpha q^{(\alpha)}(x,\xi)D^\alpha_x p(x,\xi)/\alpha!. \tag{14}$$

We shall show that $r(x,\xi)$ satisfies the conditions of Proposition 9. Then QP has a parametrix $F \in L^{2m''}_{\rho,\delta}(\Omega; C^\nu, C^\mu)$. Setting $E = FQ$, we have $E \in L^{2m''+m'}_{\rho,\delta}(\Omega; C^\mu, C^\nu)$ and it becomes a left parametrix for P. Hence by Proposition 8, P is hypoelliptic.

To show that (11) is satisfied for r, we can use the asymptotic expansion

$$(14)' \qquad R_N(x,\xi) = r(x,\xi) - \sum_{|\alpha|<N} q^{(\alpha)}(x,\xi) D_x^\alpha p(x,\xi)/\alpha!$$
$$\in S^{m+m'-N(\rho-\delta)}(\Omega; C^\nu, C^\nu).$$

Differentiate this equality α times in ξ and β times in x, we have, by the Leibniz formula,

$$r^{(\alpha)}_{(\beta)}(x,\xi) = \sum_{|\gamma|<N} C_{\alpha,\alpha',\beta,\beta'} C_\gamma q^{(\gamma+\alpha')}_{(\beta')}(x,\xi)\, p^{(\alpha-\alpha')}_{(\gamma+\beta+\beta')}(x,\xi)$$
$$+ R_N{}^{(\alpha)}_{(\beta)}(x,\xi),$$

where $C_{\alpha,\alpha',\beta,\beta'}$ and C_γ are constants depending only on their subscripts. Here $R_N{}^{(\alpha)}_{(\beta)}(x,\xi) \in S^{m+m'-N(\rho-\delta)-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}(\Omega; C^\nu, C^\nu)$. Operating $A(x,\xi)$ from the left and $B(x,\xi)$ from the right and considering Proposition 2, we have

$$|A(x,\xi) r^{(\alpha)}_{(\beta)}(x,\xi) B(x,\xi)|$$
$$\leqq \sum_{|\gamma|<N} |C_{\alpha,\alpha',\beta,\beta'} C_\gamma A(x,\xi) q^{(\gamma+\alpha')}_{(\beta')}(x,\xi) p^{(\alpha-\alpha')}_{(\gamma+\beta-\beta')}(x,\xi) B(x,\xi)|$$
$$+ |A(x,\xi) R_N{}^{(\alpha)}_{(\beta)}(x,\xi) B(x,\xi)|$$
$$\leqq C_{\alpha,\beta,N,K}(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|}$$
$$+ C_{\alpha,\beta,K}(1+|\xi|)^{m+m'+2m''-N(\rho-\delta)-\rho|\alpha|+\delta|\beta|}.$$

If we choose N sufficiently large, then we have

$$|A(x,\xi) r^{(\alpha)}_{(\beta)}(x,\xi) B(x,\xi)| \leqq C_{\alpha,\beta,K}(1+|\xi|)^{-\rho|\alpha|+\delta|\beta|},$$
$$x \in K.$$

It suffices to show that the conditions (12) and (13) in Proposition 9 are satisfied for $r(x,\xi)$. The condition (13) holds evidently, because the condition (III) of the main theorem are verified.

Finally we shall ascertain (12). Consider the asymptotic expansion (14)′ of $r(x,\xi)$, i.e.,

$$r(x,\xi) = \sum_{|\alpha|<N} q^{(\alpha)}(x,\xi) D_x^\alpha p(x,\xi)/\alpha! + R_N(x,\xi),$$

where $R_N(x,\xi) \in S^{m+m'-N(\rho-\delta)}_{\rho,\delta}(\Omega; C^\nu, C^\nu)$. Operating $A(x,\xi)$ from the right and $B(x,\xi)$ from the left, we have, by the triangle inequality that

$$(15) \qquad |A(x,\xi) r(x,\xi) B(x,\xi)| \geqq |A(x,\xi) q(x,\xi) p(x,\xi) B(x,\xi)|$$
$$- \Big| \sum_{0<|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} A(x,\xi) q^{(\alpha)}(x,\xi) p(x,\xi) B(x,\xi) \Big|$$
$$- |A(x,\xi) R_N(x,\xi) B(x,\xi)|.$$

The first term on the right hand side of (24) can be estimated as such as

$$(16)\qquad |A(x,\xi)q(x,\xi)p(x,\xi)B(x,\xi)| \geqq C_K, \quad x\in K \text{ and } |\xi| > C_K.$$

By the assumption (I) of the main theorem we can estimate the second term of (15) as follows.

$$\sum_{0<|\alpha|<N} |(i)^{|\alpha|}/\alpha!\, A(x,\xi)q^{(\alpha)}(x,\xi)p_{(\alpha)}(x,\xi)B(x,\xi)|$$
$$\leqq \sum_{0<|\alpha|<N} \frac{1}{\alpha!} |A(x,\xi)q^{(\alpha)}(x,\xi)p_{(\alpha)}(x,\xi)B(x,\xi)|$$
$$\leqq \sum_{0<|\alpha|<N} C_{\alpha,K}(1+|\xi|)^{-(\rho-\delta)N} \leqq C_{N,K}(1+|\xi|)^{-(\rho-\delta)},$$
$$x\in K \text{ and } |\xi| > C_K.$$

For the third term of (15), we have

$$(17)\qquad |A(x,\xi)R_N(x,\xi)B(x,\xi)| \leqq C_{N,K}(1+|\xi|)^{2m''+m+m'-(\rho-\delta)N}, \quad x\in K.$$

If we choose N sufficiently large and combine (15), (16) and (17), then we obtain

$$|A(x,\xi)r(x,\xi)B(x,\xi)| \geqq C_K - C'_K(1+|\xi|)^{-(\rho-\delta)},$$
$$x\in K \text{ and } |\xi| > C_K,$$

and we can choose C''_K so large that

$$C'_K(1+|\xi|)^{-(\rho-\delta)} < \frac{1}{2}C_K, \quad |\xi| > C''_K.$$

Consequently we have

$$|A(x,\xi)r(x,\xi)B(x,\xi)| \geqq C_K, \quad x\in K \text{ and } |\xi| > C''_K.$$

Hence the condition (12) of Proposition 9 is proved. Thus the theorem is established.

The corollary of the main theorem is proved in the same way.

Proof of Corollary.

Let P be as in the theorem. By Proposition 7 it follows that the symbol $q(x,\xi)$ of the adjoint operator P^* of P is given by an asymptotic expansion

$$(18)\qquad q(x,\xi) \sim \sum_{\alpha} D_x^\alpha p^{*(\alpha)}(x,\xi)/\alpha!,$$

i.e.,

$$R'_N(x,\xi) = q(x,\xi) - \sum_{|\alpha|<N} D_x^\alpha p^{*(\alpha)}(x,\xi)/\alpha! \in S^{m-(\rho-\delta)N}(\Omega;\, C^\mu, C^\nu). \tag{18$'$}$$

Substituting q in (14)$'$, we have

$$r(x,\xi) - \sum_{|\alpha|<N,\, |\beta|<N} D_x^\beta p^{*(\alpha+\beta)}(x,\xi) D_x^\alpha p(x,\xi)/\alpha!\beta! + \\ + \sum_{|\alpha|<N} R_N^{(\alpha)}(x,\xi) D_x^\alpha p(x,\xi)/\alpha!.$$

Since $R_N(x,\xi) \in S_{\rho,\delta}^{m-N(\rho-\delta)}(\Omega;\, C^\mu, C^\nu)$, by Proposition 2, we have $\sum_{|\alpha|<N} R_N^{(\alpha)}(x,\xi) D_x^\alpha p(x,\xi)/\alpha! \in S^{2m-N(\rho-\delta)}(\Omega;\, C^\nu, C^\nu)$. Since N is arbitrary, we obtain that

$$r(x,\xi) \sim \sum_{\alpha,\beta} D_x^\beta p^{*(\alpha+\beta)}(x,\xi)\, D_x^\alpha p(x,\xi)/\alpha!\beta!.$$

Using this asymptotic expansion instead of (14) we can easily see that $r(x,\xi)$ satisfies the conditions (11), (12) and (13) in Proposition 9, which proves the corollary.

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Nagoya University