PERFECT PELL POWERS

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In the thirty years since it was proved that 0, 1 and 144 were the only perfect squares in the Fibonacci sequence [**1, 9**], several generalisations have been proved, but many problems remain. Thus it has been shown that 0, 1 and 8 are the only Fibonacci cubes [**6**] but there seems to be no method available to prove the conjecture that 0, 1, 8 and 144 are the only perfect powers.

In a different direction, generalising the sequence to $P_n(a)$ defined by $P_0(a) = 0$, $P_1(a) = 1$, $P_{n+2}(a) = aP_{n+1}(a) + P_n(a)$ or to $p_n(a)$ defined by $p_0(a) = 0$, $p_1(a) = 1$, $p_{n+2}(a) = ap_{n+1}(a) - p_n(a)$, it has been shown that the problem of determining the squares in these sequences can be handled easily when a is odd, but only in exceptional cases when a is even [**2, 3, 4**]. In the case of the first of these with $a = 2$, we obtain the Pell sequence, $0, 1, 2, 5, \ldots, 169, \ldots$, to which we shall refer below simply as P_n. It has been shown by Ljunggren [**5**] that its only squares are 0, 1 and 169. However, the method of that paper was long and extremely complicated, involving relative units in a biquadratic field, and Mordell asked over 30 years ago [**7**] whether a simpler proof might not be available. There has indeed been another proof recently [**8**] which is quite different in conception, depending as it does on purely analytical ideas. Although that proof is a considerable achievement, whether it can be regarded as more *simple* is a matter of opinion, as it still seems to require tools and a mass of detail disproportionate to the apparent difficulty of the problem. Maybe what Mordell had in mind was a proof akin to that for Fibonacci squares, both short and technically elementary.

Despite this challenge, no such proof has appeared; it may therefore perhaps be of interest to present the following very simple proof of the fact that there are no *other* powers in the sequence, a result far exceeding the present state of knowledge of the corresponding problem for the Fibonacci sequence.

THEOREM. *The only solutions of* $P_n = x^k$ *with* $k > 2$ *are given by* $n = 0, 1$.

LEMMA. *The Diophantine equation* $y^2 - 2z^k = -1$ *with* $k > 2$ *has only the solutions* $y = z = 1$ *and* $y = 239$, $z = 13$, $k = 4$.

Proof of lemma. For $k = 4$ or a multiple of 4, the result is Ljunggren's. For other values, k must have an odd prime factor, and so without loss of generality may be taken to be odd, say $k = 2K + 1$. For any solution both y and z must be odd, and factorising in $Q[i]$ gives $(y + i)(y - i) = (1 + i)(1 - i)z^{2K+1}$. Since $(1 + i)$ and $(1 - i)$ are associates we find that $y + i = (1 + i)(a + ib)^{2K+1}$ and $z = a^2 + b^2$ for some suitable rational integers a and b, since any units, i.e. powers of i, can be absorbed into the $a + ib$. Thus we find $2i = (1 + i)(a + ib)^{2K+1} - (1 - i)(a - ib)^{2K+1}$ and so

$$\begin{aligned} 1 + i &= (a + ib)^{2K+1} + i(a - ib)^{2K+1} \\ &= (a + ib)^{2K+1} + (-1)^K(ia + b)^{2K+1}. \end{aligned}$$

Thus, if K is even, $(1 + i)$ is divisible by $(a + ib) + (ia + b) = (1 + i)(a + b)$ whence

$a+b=\pm 1$, and similarly, if K is odd, $a-b=\pm 1$. In either case we obtain $z=a^2+b^2=2a^2\pm 2a+1$, and so $2z=c^2+1$, where $c=|2a\pm 1|\geq 1$.

Our equation can now be rewritten in the form $y^2-(c^2+1)(z^K)^2=-1$, and since the general solution of the Pell equation $u^2-(c^2+1)v^2=-1$ is given by

$$u+v\sqrt{c^2+1}=(c+\sqrt{c^2+1})^{2m+1},$$

we find that

$$z^K=(\tfrac{1}{2}(c^2+1))^K=\sum_{r=0}^{m}\binom{2m+1}{2r+1}c^{2m-2r}(c^2+1)^r. \tag{1}$$

Now suppose that p is any prime dividing $\frac{1}{2}(c^2+1)$. Then $p\geq 5$. Let $p^\lambda \parallel \frac{1}{2}(c^2+1)$. Then from (1) we see that $p\mid(2m+1)$ and so if $p^\mu\parallel(2m+1)$, we see that the first term on the right hand side of (1) is divisible by p^μ precisely, whereas all the other terms are divisible by higher powers. Thus $\lambda K=\mu$, and since this holds for every prime factor of $\frac{1}{2}(c^2+1)$, it follows that $(\frac{1}{2}(c^2+1))^K$ divides $(2m+1)$ and so $2m+1\geq(\frac{1}{2}(c^2+1))^K$. On the other hand from (1) we see that $(\frac{1}{2}(c^2+1))^K>2m+1$ unless $m=0$ and $c=1$. Thus $z=1$.

Proof of theorem. For n odd, the result follows from the lemma and the identity $Q_n^2-2P_n^2=(-1)^n$ where the sequence Q_n satisfies the same recurrence relation as P_n but with initial conditions $Q_0=Q_1=1$. For n even, $n\neq 0$, let $n=2^h m$, where m is odd. Then it is found without difficulty that $h\geqslant 2$ and that $P_n=2^h P_m Q_m Q_{2m} X$, where the five factors on the right are pairwise coprime. It thus follows that if P_n is to be a perfect kth power, then each factor on the right must also be one. But by the lemma P_m can be a perfect kth power only if $m=1$, and then $Q_{2m}=3$ fails to be one, which concludes the proof.

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