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ON THE EXPECTED NUMBER OF VISITS OF A PARTICLE BEFORE ABSORPTION IN A CORRELATED RANDOM WALK

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§1. **Introduction.** Let a particle move along a straight line a unit distance during every interval of time τ. During the first interval τ it moves to the right with probability ρ_1 and to the left with probability $\rho_2=1-\rho_1$. Thereafter at the end of each interval τ, the particle with probability p continues its motion in the same direction as in the previous step and with probability $q=1-p$ reverses it. Goldstein [2], Gillis [1], Mohan [5], Gupta [3], Seth [6] and Jain [4] have considered various aspects of this random walk problem. In this paper we show that various probabilities may be calculated in terms of those which are related to the first passage problems. Finally we obtain the expected number of visits a particle makes to a certain point before getting absorbed. Such a number of visits is frequently required in many physical processes.

§2. **First passages, etc.** Let $a_{r,n}(;b_{r,n})\equiv$Probability that a particle arrives at r for the first time on the nth step given that initially it arrived at the origin from the left (from the right);

$p_{h,n}^{(m)}(;q_{h,n}^{(m)})$=Given that the first step has led a particle from the origin to the right (left), the probability of a particle returning for the mth time to $h(>0)$ on the nth step without having crossed it earlier;

$p_{h,n}^{(m)*}(;q_{h,n}^{(m)*})$=The above probabilities, without any restriction on the number of times it crosses h.

Supposing that the first passage through 1 occurs at the kth step and following Seth [6], we have the relations:

$$
\begin{aligned}
&a_{r,n}=\sum_k a_{1,k}a_{r-1,n-k}; \qquad b_{r,n}=\sum_k b_{1,k}a_{r-1,n-k}, \qquad r>1\\
\text{(2.1)}\qquad &a_{1,n}=p\delta_{n-1}+qb_{2,n-1}; \qquad b_{1,n}=q\delta_{n-1}+pb_{2,n-1},
\end{aligned}
$$

where $\delta_i=0$ $(i\neq 0)$ and $\delta_0=1$.

Multiplying (2.1) by s^n and summing over all n, we get the generating functions, on simplification

$$A_r(s)=\sum_n a_{r,n}s^n=A_r=A_1^r; \qquad B_r(s)=\sum_n b_{r,n}s^n=B_r=A_1^{r-1}B_1,$$

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where

$$A_1 = [1+s^2c-\{(1-s^2)(1-s^2c^2)\}^{1/2}]/2ps,$$
$$B_1 = [1-s^2c-\{(1-s^2)(1-s^2c^2)\}^{1/2}]/2qs, \qquad c = p-q.$$

It can also be seen that for $i>0$

$$A_i(s) = B_{-i}(s); \qquad B_i(s) = A_{-i}(s).$$

Using these generating functions, Seth [6] has given various unconditional probabilities of a particle returning to the origin in hypergeometric forms.

It will be seen in the sequel that the first passage probabilities can also be used to calculate various other probabilities.

Now transferring the origin to the position reached by the first step, we have

$$p_{h,n}^{(1)} = a_{k-1,n-1}; \qquad q_{h,n}^{(1)} = b_{k+1,n-1}.$$

It follows that

$$p_{h,n}^{(m)} = q\sum_k p_{h,k}^{(m-1)} b_{1,n-k-1}, \qquad h>0;$$

$$q_{h,n}^{(m)} = \sum_k q_{h,k}^{(m-1)} b_{1,n-k-1}, \qquad h \geq 0.$$

So the

$$P_h^{(m)}(s) = \sum_n p_{h,n}^{(m)} s^n = sA_1^{h-1}[qsB_1]^{m-1}; \qquad h>0, \qquad m \geq 1,$$

$$Q_h^{(m)}(s) = \sum_n q_{h,n}^{(m)} s^n = A_1^h[qsB_1]^m/q; \qquad h \geq 0, \qquad m \geq 1.$$

To express $p_{h,n}^{(m)*}$ in terms of $p_{h,n}^{(m)}$ we note that the particle in tracing such a path from 0 to h as contributed to $p_{h,n}^{(m)*}$ forms $(m-1)$ waves on the line $x=h$, some on the right of it and the remaining on the left. Consider the paths with i ($i=0, 1, \ldots, m-1$) crossings at h. Given the positions of the contacts with the barrier h, the i points of the crosses can be chosen in $\binom{(m-1)}{i}$ ways. Then in each of the paths contributing to $p_{h,n}^{(m)}$, there will be i more reversals and i less continuations of directions at h than in any path with i crossings at h and contributing to $p_{h,n}^{(m)*}$. Hence

$$p_{h,n}^{(m)*} = \sum_{i=0}^{m-1} \binom{m-1}{i} \left(\frac{p}{q}\right)^i p_{h,n}^{(m)} = \left(1+\frac{p}{q}\right)^{m-1} p_{h,n}^{(m)} = q^{1-m} p_{h,n}^{(m)}. \tag{2.4}$$

A similar reasoning shows that

$$q_{h,n}^{(m)*} = q^{1-m} q_{h,n}^{(m)}. \tag{2.5}$$

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The conditional probabilities of a particle moving from 0 to h in n steps without having crossed the line $x=h$ can be calculated as

$$(2.6)\quad p_{h,n} = \sum_m p_{h,n}^{(m)}, \qquad q_{h,n} = \sum_m q_{h,n}^{(m)}, \qquad p_{h,n}^{*} = \sum_m p_{h,n}^{(m)*}, \qquad q_{h,n}^{*} = \sum_m p_{h,n}^{(m)*}$$

The generating functions are found to be

$$P_h(s) = \sum_{m=1}^{\infty} P_h^{(m)}(s) = sA_1^{h-1}/(1-qsB_1),$$

$$Q_h(s) = \sum_{m=1}^{\infty} Q_h^{(m)}(s) = A_1^h sB_1/(1-qsB_1),$$

$$P_h^*(s) = \sum_{m=1}^{\infty} P_h^{(m)*}(s) = sA_1^{h-1}/(1-sB_1),$$

$$Q_h^*(s) = sA_1^h B_1/(1-sB_1).$$

The probability of a particle arriving at $-x$ on the nth step without returning to the origin where it initially arrived from the left can also be converted into first passage probability in the following manner:

Let this probability be $u_{x,n}$ and let $v_{x,n}(v_{x,n}^*)$=the probability $u_{x,n}$ when the nth step takes the particle to the position $-x$ from the left (from the right).

Evidently

$$(2.7)\qquad u_{x,n} = v_{x,n}+v_{x,n}^*.$$

If a path contributing to $v_{x,n}$ is rotated through 180° about the final position and the direction of each step is reversed [i.e., the motion in the kth ($k=1, 2, \ldots, n$) step is in the direction reverse to that in the $(n-k)$th step, or alternatively the path is reflected in the line representing the nth step], we obtain a corresponding path such that the particle takes the first step to the left and arrives at x for the first time on the nth step. The number of reversals and continuations of direction in this path are equal to those in the corresponding path contributing to $v_{x,n}$.

Now transferring the origin to the position reached by the particle on the first step in a congruent path and noting that the first step, viz. of -1 is due to a reversal and the path from -1 to x is a first passage in $(n-1)$ steps, we have

$$(2.8)\qquad v_{x,n} = qb_{x+1,n-1}.$$

Similarly rotating through 180° about the final position, a path contributing to $v_{x,n}^*$ (keeping the direction of motion unchanged) we get a congruent path, viz. one traced by a particle taking the first step to the right (with probability q) and arriving to $-x$ for the first time on the nth step, then

$$(2.9)\qquad v_{x,n}^* = qb_{-x+1,n-1}.$$

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Hence

$$(2.10)\qquad \mathrm{U}_x(s) = \sum_{n=0}^{\infty} \mathrm{u}_{x,n} s^n = qsA_1^{x-1}(1+A_1B_1).$$

§3. **Probabilites for a given number of reflections at or passages through h.** Defining by $\alpha_m^*(n, v)$, the probability of a particle reaching v from the left on the nth step after touching $h(>v)$ just m times and similarly $\beta_m^*(n, v)$ for the particle reaching v from the right, Gupta [3] showed by the image methods that:

$$(3.1)\qquad \alpha_m^*(n, v)/\beta_m^*(n, 2h-v) = \{1+(q-p)^m\}/\{1-(q-p)^m\}.$$

If $\gamma_m(n, v)$ be the probability of a particle moving to the lattice point $v(<h)$ in n steps having with h, m contacts, it also follows that

$$(3.2)\qquad \gamma_m(n, v)/\gamma_m(n, 2h-v) = \{1+(q-p)^m\}/\{1-(q-p)^m\},$$

where

$$\gamma_m(n, v) = \rho_1\alpha_m^*(n, v)+\rho_2\beta_m^*(n, v).$$

Further let $\alpha_m(n, v) \equiv$ Given that the first step takes the particle to $+1$; the probability of a particle moving to $v(<h)$ in n steps having with h, m contacts which are all reflections on the left; and similarly $\beta_m(n, v)$ when the first step takes the particle to -1. We shall prove that

$$(3.3)\qquad \alpha_m^*(n, v)/\alpha_m(n, v) = \beta_m^*(n, v)/\beta_m(n, v) = [1+(q-m)^m]/2q^m.$$

Each path contributing to $\alpha_m^*(n, v)$ has $(m-1)$ waves abutting upon the ordinate at h; let us first consider those with m_1 left waves and an even number $2k$ of crosses.

We identify the exponent of y with the number of crosses and that of t with the number of right waves. With regard to left waves we notice that before the first one or after the last one or between any pair of two consecutive ones, there is either (i) no crossing and no right wave (ii) 2 crossings and at least one right wave represented by the expression

$$y^0t^0+y^2(t+t^2+\cdots),$$

which raised to the power (m_1+1) will give the number of crosses by all possible placings of the $m-m_1-1$ right waves in between the m_1 left waves.

Hence the number of paths contributing to $\alpha_m^*(n, v)$ with $2k$ crosses and m_1 left waves with the relative positions amongst themselves of the waves along with their structures unchanged as also for the right waves and corresponding to a path contributing to $\alpha_m(n, v)$ is given by the coefficient of $y^{2k}t^{m-m_1-1}$ in the expression

$$\{1+y^2(t+t^2+\cdots)\}^{m_1+1} = \left[1+\frac{y^2t}{1-t}\right]^{m_1+1};$$

and is $\binom{m_1+1}{k}\binom{m-m_1-2}{k-1}$.

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Now in each one of such paths there will be $2k$ more continuations and $2k$ less reversals of direction than in its homologues contributing to $\alpha_m(n, v)$.

Hence the probability of a particle from 0 to $v(<h)$ in n steps with m contacts at h and crossing it at least once is

$$\alpha_m(n, v) \sum_{k=1}^{[1/2m]} \sum_{m_1=k-1}^{m-k-1} \binom{m_1+1}{k} \binom{m-m_1-2}{k-1} \frac{p^{2k}}{q^{2k}};$$

where $[j]$=the greatest integer not exceeding j.

Taking into account those paths of $\alpha_m^*(n, v)$ which do not cross h, we get

$$\begin{aligned}\alpha_m^*(n, v)/\alpha_m(n, v) &= 1+\sum_{k=1}^{[1/2m]} \sum_{m_1=k-1}^{m-k-1} \binom{m_1+1}{k} \binom{m-m_1-2}{k-1} \frac{p_{2k}}{q^{2k}} \\ &= 1+\sum_{k=1}^{[1/2m]} \binom{m}{2k} \frac{p^{2k}}{q^{2k}} \\ &= \frac{1}{2}\left\{\left(1+\frac{p}{q}\right)^m + \left(1-\frac{p}{q}\right)^m\right\} = [1+(q-p)^m]/2q^m.\end{aligned}$$

Similarly

$$\beta_m^*(n, v)/\beta_m(n, v) = [1+(q-p)^m]/]q^m.$$

The probabilities $\gamma_m(n, v)$ may thus be calculated from $\alpha_m(n, v)$ and $\beta_m(n, v)$ which in turn are computed from the following relations:

$$\alpha_m(n, v) = \sum_k p_{h,k}^{(m)} \mathrm{u}_{(h-v),n-k}; \qquad -\infty \leq v < h > 0$$

(3.4)

$$\beta_m(n, v) = \sum_k q_{h,k}^{(m)} \mathrm{u}_{(h-v),n-k}; \qquad h \geq 0.$$

The conditional probabilities of a particle moving from 0 to v in n steps without crossing but undergoing at least one reflection at h are

(3.5) $$\alpha(n, v) = \sum \alpha_m(n, v), \qquad \beta(n, v) = \sum_m \beta_m(n, v).$$

§4. **Passage in the presence of an absorbing barrier.** Let $r_{x,n}(;l_{x,n})\equiv$Given that the first step is to the right (left), the probability of a particle starting from 0 and reaching to x on the nth step without touching the barrier at a.

(i) case $a>x>0$

The set of paths contributing to $r_{x,n}$ (or $l_{x,n}$) can be divided into two disjoint subsets 1 and 11. The set 1 comprises those paths which lead from 0 to x in n steps without crossing x. The set 11 comprises those paths which lead from 0 to x in n steps after crossing x but not reaching a.

Hence

$$(3.6)\qquad \begin{cases} r_{x,n} = p_{x,n} + \sum_{h=x+1}^{a-1} \alpha(n, x), & x > 0; \\ 1_{x,n} = q_{x,n} + \sum_{h=x+1}^{a-1} \beta(n, x), & x \geq 0. \end{cases}$$

(ii) Case $x = -y$

Summing the probabilities of a particle starting from 0 and reaching $x(<0)$ without crossing $h(h=1, 2, \ldots, a-1)$ we get

$$(3.7)\qquad \begin{cases} r_{y,n} = \sum_{h=1}^{a-1} \alpha(n, -y), & y \geq 0; \\ 1_{y,n} = \sum_{h=0}^{a-1} \beta(n, -y), & y > 0. \end{cases}$$

Writing the generating functions for various probabilities we have

$$R_x(s) = \sum_{n=1}^{\infty} r_{x,n} s^n = A_1^{x-2}[1+qsA_1(1+A_1B_1)(1-A_1^{2a-2x-2})/(1-A_1^2)]/p, \qquad x > 0$$

$$L_x(s) = \sum_{n=1}^{\infty} 1_{x,n} s^n = A_1^{x-1}B_1[1+qsA_1(1+A_1B_1)(1-A_1^{2a-2x-2})/(1-A_1^2)]/p, \qquad x \geq 0,$$

$$R_y(s) = qs(1+A_1B_1)A_1^{y-1}(1-A_1^{2a-2})/p(1-A_1^2), \qquad y \geq 0$$

$$L_y(s) = qs(1+A_1B_1)A_1^{y-2}B_1(1-A_1^{2a})/p(1-A_1^2), \qquad y > 0.$$

Using L'Hospital's rule in finding the limits of the generating functions we easily get N_x, the expected number of visits a particle makes to the point $x(>0)$ from 0 before getting absorbed at a. These expected values are

$$N_x = \rho_1 R_x(1) + \rho_2 L_x(1) = 1 + q(2a-2x-1)/p,$$

$$N_y = \rho_1 R_y(1) + \rho_2 L_y(1) = 2q(a-\rho_1)/p.$$

Similarly

$$N_0 = \rho_1 R_0(1) + \rho_2 L_0(1) = \rho_2 + q(2a-1-\rho_1)/p.$$

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