SYLOW THEORY FOR A CERTAIN CLASS OF OPERATOR GROUPS

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1. Introduction. In this paper we consider again the group-theoretic configuration studied in **(1)** and **(2)**. Let G be an additive group (not necessarily abelian), let M be a system of operators for G, and let ϕ be a family of admissible subgroups which form a complete lattice relative to intersection and compositum. Under these circumstances we call G an $M - \phi$ group. In **(1)** we studied the normal chains for an $M - \phi$ group and the relation between certain normal chains. In **(2)** we considered the possibility of representing an $M - \phi$ group as the direct sum of certain of its subgroups, and proved that with suitable restrictions on the $M - \phi$ group the analogue of the following theorem for finite groups holds: A group is the direct product of its Sylow subgroups if and only if it is nilpotent. Here we show that under suitable hypotheses (hypotheses (I), (II), and (III) stated at the beginning of § 3) it is possible to generalize to $M - \phi$ groups many of the Sylow theorems of classical group theorem. The most important of these is the existence theorem—Theorem 3.1.

2. Definitions and preliminary results. In order to make this paper as self-contained as possible we shall summarize in this paragraph the definitions and results which we shall use from the two previous papers **(1)** and **(2)**.

Let G be an $M - \phi$ group. The subgroups belonging to the lattice ϕ are called ϕ subgroups. The following notions are defined in the obvious manner: $M - \phi$ isomorphism, $M - \phi$ automorphism, $M - \phi$ homomorphism, the $M - \phi$ quotient group G/N (where N is a normal ϕ subgroup of G). The analogues of the Homomorphism Theorem and the Isomorphism Theorems hold (see **(1)** for a statement of these definitions and theorems).

Throughout this paper we shall assume that G possesses a ϕ composition series all of whose factors are abelian, that is, there is a chain of ϕ subgroups

$$(1) \qquad 0 = A_0 \subset \ldots \subset A_i \subset A_{i+1} \subset \ldots \subset A_n = G,$$

where A_i is normal in A_{i+1} for $i = 0, \ldots, n-1$, such that each factor A_{i+1}/A_i is abelian and ϕ simple, that is, has no proper normal ϕ subgroups ($\neq 0$). We call (1) a ϕ composition series of length n and we say that G is ϕ soluble. The analogue of the Jordan Hölder Theorem tells us that any two ϕ composition series have the same length and $M - \phi$ isomorphic factors. If

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the chain (1) consists of normal ϕ subgroups of G and if A_{i+1}/A_i contains no proper normal ϕ subgroups of G/A_i ($\neq 0$) we call (1) a principal ϕ series.

If a is an element of G, the intersection of all ϕ subgroups which contain a is called the ϕ cyclic subgroup generated by a. The $M - \phi$ group is said to be ϕ nilpotent if the upper central ϕ chain joins 0 and G (for a definition of ϕ centre and central ϕ chain see **(2)**, Definitions 5.1 and 5.2). The $M - \phi$ group P is said to be primary with characteristic F if it possesses a ϕ composition series all of whose factors are $M - \phi$ isomorphic to F. We shall make extensive use of the following theorem, which is proved in **(2)** (Theorem 7.2 under hypotheses (i), (ii′), and (iii)—see Remark after Corollary 7.1):

(A) Let P be a primary $M - \phi$ group with abelian characteristic and assume that

(i) Inner automorphisms are $M - \phi$ automorphisms.

(ii) The ϕ cyclic ϕ subgroups of P are abelian.

(iii) Any ϕ subgroup of P has a finite number of conjugates.

Then P is ϕ nilpotent.

We also need (Theorem 7.4 of **(2)**):

(A′) Let G be an $M - \phi$ group which possesses a ϕ composition series. Assume (i) of (A), and also that unitoral ϕ cyclic ϕ subgroups are primary. Then if G is ϕ nilpotent, G is the direct sum of primary ϕ subgroups. (An $M - \phi$ group is unitoral if it possesses a unique maximal normal ϕ subgroup.)

We shall also make use of the following result—the proof is an easy generalization of the argument used for ordinary groups (see, for example, **(3)**):

(B) Let G be an $M - \phi$ group which possesses a ϕ composition series; and let N be a minimal normal ϕ subgroup of G. Then N is the direct sum of a finite number of $M - \phi$ isomorphic ϕ simple ϕ subgroups.

The ϕ subgroup S of G is said to be a ϕ link if there is a normal ϕ chain connecting S and G, that is, if there exist ϕ subgroups S_i such that

(2) $S = S_0 \subset \ldots \subset S_i \subset S_{i+1} \subset \ldots \subset S_k = G$, where S_i is normal in S_{i+1}.

It is easy to see that if G possesses a ϕ composition series, the ϕ links satisfy the double chain condition. We shall need the following result concerning ϕ links (see Theorem 5.2 of **(2)**):

(C) If the $M - \phi$ group G is ϕ nilpotent, then any ϕ subgroup of G is a ϕ link.

The following notations will be used: If A and B are subgroups of the group G, $\{A, B\}$ denotes the compositum of A and B. If S is a subgroup of the group G and g an element of G, $S(g)$ denotes the conjugate subgroup $-g + S + g$. The notation $Z_\phi(G)$ is used for the ϕ centre of G. The symbol $\underset{(M-\phi)}{\cong}$ is used for $M - \phi$ isomorphism.

3. The existence theorem.

Throughout this paper we assume that G is a ϕ soluble $M - \phi$ group which satisfies the following hypotheses:

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(I) Inner automorphisms are $M - \phi$ automorphisms.

(II) ϕ cyclic ϕ subgroups of G are abelian, and unitoral ϕ cyclic ϕ subgroups are primary.

(III) Any ϕ subgroup has a finite number of conjugates.

Definition 3.1. Let G have a ϕ composition series which has F as a ϕ composition factor of multiplicity (exactly) m. Then if S is a primary ϕ subgroup of G with characteristic F and ϕ composition length m, S is called an F Sylow subgroup of G.

Definition 3.2. If K is a ϕ subgroup of G, the ϕ normalizer of K in G, $N_\phi(K)$ is the maximal ϕ subgroup of G in which K is normal.

THEOREM 3.1.[1] *For each ϕ composition factor F of G there exists an F Sylow subgroup S of G. Furthermore, the F Sylow subgroups of G are all conjugate; and if there is more than one, F is of finite order f, and the number of F Sylow subgroups is congruent to* 1 *modulo f.*

Proof. We use induction on $j(G)$, the ϕ composition length of G. If $j(G) = 1$, the theorem is obviously true. Assume the theorem true for all $M - \phi$ groups H with $j(H) < j(G)$; and assume that the ϕ simple abelian group F is a ϕ composition factor of multiplicity m for G.

We consider first the case where G contains a normal primary ϕ subgroup of characteristic F' not $M - \phi$ isomorphic to F and where G/N is primary of characteristic F. Let H be a maximal normal ϕ subgroup of G which contains N; hence

$$G/H \underset{(M - \phi)}{\cong} F.$$

Let s be an element of G not in H. Then if S is the ϕ cyclic ϕ subgroup of G generated by s,

$$S/S \cap H \underset{(M - \phi)}{\cong} S + H/H \underset{(M - \phi)}{\cong} G/H \underset{(M - \phi)}{\cong} F$$

so that S has F as a ϕ composition factor. Since S is ϕ cyclic, it is abelian and hence by (A′), $S = S_1 + S_2$, where char $(S_1) = F$ and char $(S_2) = F'$. Now $S_2 \subseteq N \subseteq H$, and hence S_1 is not contained in H since S is not contained in H.

If S_1 is an F Sylow subgroup of G, then the existence of an F Sylow subgroup for G is proved. Otherwise $N + S_1$ is a ϕ link and hence is contained in a maximal normal ϕ subgroup of G, say L.

Since H and L are proper ϕ subgroups of G, we know from the induction assumption that H contains an F Sylow subgroup T, and L an F Sylow subgroup W. Furthermore, $H = N + T$, $N \cap T = 0$; $L = N + W$, $N \cap W = 0$.

[1]The author would like to thank the referee for his suggestions regarding the proof of this theorem.

Form an ascending ϕ chain from N to G: $N = N_0 \subset N_1 \subset \ldots \subset N_i \subset N_{i+1} \subset \ldots \subset N_{m-1} = H \subset N_m = G$, where N_{i+1}/N_i is ϕ simple and is contained in the ϕ centre of G/N_i; such a chain can always be constructed since G/N is primary and hence ϕ nilpotent.

Define $U_i = N_i \cap T$, for $i = 0, \ldots, k + 1$. Then $0 = U_0 \subset \ldots \subset U_i \subset U_{i+1} \subset \ldots \subset U_{m-1} = T$ and $N_i = N + U_i$ for $i = 0, \ldots, m$, since $N \subseteq N_i \subseteq H$ and $H = N + T$.

We show that if $U_t \subseteq W$, then U_t is normal in W.

For if w and u_t are elements of W and U_t respectively, then since $N_t/N_{t-1} \subseteq Z_\phi(G/N_{t-1})$, $-w + u_t + w \equiv u_t \pmod{N_{t-1}}$. It follows that $-w + u_t + w = u_t + u_{t-1} + x$, where x and u_{t-1} are elements of N and U_{t-1} respectively, since $N_{t-1} = U_{t-1} + N$. Hence $x = -u_{t-1} - u_t - w + u_t + w$ is an element of $N \cap W = 0$. Thus $-w + u_t + w = u_t + u_{t-1}$ is an element of U_t so that U_t is normal in W.

It follows from the fact that $0 = U_0$ is contained in W but $T = U_{m-1}$ is not contained in W (since $H = N + T$ is not contained in L), that there exists an integer s such that U_{s-1} is contained in W but U_s is not contained in W. We assume that W is a maximal F subgroup of G and show:

(i) $W(u_s) = W$ for u_s in U_s if and only if u_s is in $W \cap U_s$.

(ii) F has finite order, say f. The number of conjugates $W(u_s)$ with u_s in U_s is $[U_s\colon W \cap U_s] \equiv 0 \pmod{f}$.

(iii) The total number of conjugates of W is $\equiv 0 \pmod{f}$.

Proof of (i). Assume that $W(u_s) = W$ for some element u_s of U_s. On the other hand, $-u_s + w + u_s \equiv w \pmod{N_{s-1}}$ since G and N_s commute, mod $N_{s-1} = U_{s-1} + N$. Hence $w' = -u_s + w + u_s = w + u_{s-1} + x$, where u_{s-1} and x are elements of U_{s-1} and N respectively. Thus $x = -u_{s-1} - w + w'$ is an element of $W \cap N = 0$; or $-u_s + w + u_s = w + u_{s-1}$.

Form

$$Q = \bigcap_{w \text{ in } W} U_s(w).$$

Then $Q(w) = Q$ for w in W. It follows that $\{Q, W\} = Q + W$ is an F subgroup of G. But by hypothesis W is a maximal F subgroup of G; hence Q is contained in W. But u_s is an element of Q; for $u_s = -w + u_s + w + u_{s-1}$ and hence is an element of $U_s(w)$ since $-w + u_s + w$ is in $U_s(w)$ and u_{s-1} is in $U_{s-1} = U_{s-1}(w) \subset U_s(w)$. Thus u_s is in $W \cap U_s$.

Proof of (ii). By hypothesis, W has a finite number of conjugates and by (i), $W(u_s') = W(u_s)$ if, and only if, $(u_s' - u_s)$ is in $W \cap U_s$. Therefore, the number of conjugates $W(u_s)$ is $[U_s\colon W \cap U_s]$. Now since U_s is ϕ nilpotent, $W \cap U_s$ is a ϕ link for U_s, and hence there exists a ϕ composition chain joining $W \cap U_s$ to U_s and all the ϕ composition factors are $M - \phi$ isomorphic to F. Hence F has finite order, say f, and $[U_s\colon W \cap U_s]$ is divisible by f.

Proof of (iii). Let W' be any conjugate of W; then there is an integer s' such that $U_{s'-1}$ is contained in W', but $U_{s'}$ is not contained in W'. We call s'

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the integer associated with the group W'. Since W is by hypothesis a maximal F subgroup of G, W' is also a maximal F subgroup and hence applying (ii) to the group W', we see that the number of conjugates $W'(u_{s'})$ with $u_{s'}$ in $U_{s'}$ is divisible by f.

Choose a conjugate W_1 of W so that the integer $s(1)$ associated with W_1 is as large as possible. Assume that the conjugates W_j have been defined for $j < i$ and that $s(j)$ is the integer associated with W_j. If the groups $W_j(u_{s(j)})$ for $u_{s(j)}$ in $U_{s(j)}$ do not exhaust the conjugates of W, choose W_i so that:

(a) W_i is different from $W_j(u_{s(j)})$ for $u_{s(j)}$ in $U_{s(j)}$.

(b) The integer $s(i)$ associated with W_i is as large as possible.

Since W has a finite number of conjugates, there are a finite number of groups W_i, say n. We show that $W_i(u_{s(i)}) \neq W_j(u_{s(j)})$ for $j < i$. For if $W_i(u_{s(i)}) = W_j(u_{s(j)})$, $W_i = S_j(u_{s(j)} - u_{s(i)})$, and since $s(i) \leqslant s(j)$ implies that $U_{s(i)} \subseteq U_{s(j)}$, $u_{s(j)} - u_{s(i)}$ is in $U_{s(j)}$. But it follows from the definition of W_i that this is impossible. Hence the groups $W_i(u_{s(i)})$ with $u_{s(i)}$ in $U_{s(i)}$ $(1 \leqslant i \leqslant n)$ are all distinct. But for fixed i, the number of conjugates $W_i(u_{s(i)})$ is divisible by f. Thus the total number of conjugates of W is divisible by f.

Hence we have shown that under the assumption that W is a maximal F subgroup of G, the number of conjugates of W is divisible by f. But any conjugate of W is an F Sylow subgroup of L, and by the induction assumption all the F Sylow subgroups of L are conjugates of W (in L and hence in G), and their number is congruent to 1 modulo f. Thus if W is a maximal F subgroup of G we have a contradiction. Hence G contains an F subgroup P which properly contains W. It is clear that P is an F Sylow subgroup for G. This proves the existence part of the theorem in the particular case we were treating—that is where G contains a normal primary ϕ subgroup of characteristic F' not $M - \phi$ isomorphic to F, and G/N is primary of characteristic F. We prove next the existence part of the theorem in the general case.

Let G be an $M - \phi$ group, one of whose ϕ composition factors is F, and assume that G is not primary. Let N be a minimal normal ϕ subgroup of G; by (B) N is primary. Either N is itself an F Sylow subgroup or else we have one of the following:

(a) N is primary of characteristic F' not $M - \phi$ isomorphic to F, and G/N is primary of characteristic F.

(b) G/N is not primary.

If (a) holds, we have the special case treated above. If on the other hand, G/N is not primary, by the induction assumption it contains an F Sylow subgroup K/N. Furthermore, since $K \neq G$ we can use induction again to obtain an F Sylow subgroup S of K. Clearly S is also an F Sylow subgroup of G. This completes the proof of the existence of an F Sylow subgroup in the general case.

We now turn to the second part of the theorem, still using induction on $j(G)$, the length of a ϕ composition series for G. Let H be a maximal normal ϕ

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subgroup of G. If G/H is not $M - \phi$ isomorphic to F, and T is an F Sylow subgroup of G, then T is contained in H. For otherwise $G = H + T$ and

$$T/H \cap T \underset{(M - \phi)}{\cong} G/H \text{ not } (M - \phi) \text{ isomorphic to } F$$

which is impossible. Hence in this case all the F Sylow subgroups of G are F Sylow subgroups H and conversely. Since $j(H) < j(G)$, the F Sylow subgroups are all conjugate in H and hence in G. If there is more than one, F is finite of order f and the number is congruent to 1 modulo f.

Suppose next

$$G/H \underset{(M - \phi)}{\cong} F,$$

and H does not have F as a ϕ composition factor. Let S be an F Sylow subgroup of G. Then

$$S \underset{(M - \phi)}{\cong} F,$$

and $G = H + S$, $H \cap S = 0$. If S is the unique F Sylow subgroup of G, then there is nothing to prove. Otherwise let T be an F Sylow subgroup of G distinct from S. We prove that if $T(s) = T$ for s in S, then $s = 0$.

Let t be an element of T. Then $-s + t + s = \bar{t}$ is an element of T. Now $-s + t + s - t$ is in H since G/H is abelian; and also $(-s + t + s) - t = \bar{t} - t$ is in T. Thus $-s + t + s - t$ is in $H \cap T = 0$. Hence for any t in T, $-s + t + s = t$ or $s = -t + s + t$ so that if $T(s) = T$, s is in $S(t)$ for every t in T. Let

$$Q = \bigcap_{t \in T} S(t);$$

$Q(t) = Q$ for t in T. Hence $\{Q, T\} = Q + T$ and therefore $\{Q, T\} = T$ or $Q \subseteq T$. Thus s is in T since s is in Q. But $S \cap T = 0$; therefore $s = 0$.

Thus if s_1 and s_2 are distinct elements of S, $T(s_1) \neq T(s_2)$ so that the number of conjugates $T(s)$ with s in S is equal to the order of F (since $S \cong F$). By hypothesis, T has a finite number of conjugates. Hence F has finite order f.

If the subgroups $T(s)$ with s in S do not exhaust the conjugates of T distinct from S, let T_1 be such a conjugate of T. Then there are f conjugates $T_1(s)$ with s in S. Continuing in this way, we find that the number of conjugates of T distinct from S is congruent to 0 modulo f. However, if S is not a conjugate of T, we may replace S by T', a conjugate of T, in the argument above and obtain the result that the number of conjugates of T is congruent to 1 modulo f. Thus if S is not a conjugate of T we have a contradiction. Therefore, S is a conjugate of T and the number of conjugates of T is congruent to 1 modulo f.

Finally, suppose

$$G/H \underset{(M - \phi)}{\cong} F,$$

and H contains F as a ϕ composition factor. Let T be an F Sylow subgroup of

G; then $S_1 = H \cap T$ is an F Sylow subgroup of H contained in T, and S_1 is normal in T. Furthermore, if S_2 is any F Sylow subgroup of H contained in T, then $\{S_1, S_2\} = S_1 + S_2$ is an F subgroup of H, but S_1 and S_2 are both maximal F subgroups of H; therefore, $S_1 + S_2 = S_1 = S_2$. On the other hand, if S is any F Sylow subgroup of H, by the induction hypothesis S is a conjugate of S_1, say $S = S_1(g) = (H \cap T)(g)$, and hence $S \subseteq T(g)$, an F Sylow subgroup of G. Thus every F Sylow subgroup of G contains one and only one F Sylow subgroup of H, and every F Sylow subgroup of H is contained in at least one F Sylow subgroup of G.

In particular, if H has an F Sylow subgroup S which is normal in G, it is the only F Sylow subgroup of H and hence is contained in every F Sylow subgroup of G. Thus in this case T is an F Sylow subgroup of G if and only if T/S is an F Sylow subgroup of G/S. Furthermore, T_1/S and T_2/S are conjugate in G/S if and only if T_1 and T_2 are conjugate in G. Hence we deduce the validity of our theorem in G from its validity in G/S, which we know from the induction assumption.

Assume, on the other hand, that H has an F Sylow subgroup S which is not normal in G. Let T be an F Sylow subgroup G such that $S \subset T$; then, as was shown above, $S = T \cap H$ and hence S is normal in T. Thus $N_\phi(S)$, the ϕ normalizer of S in G, contains any F Sylow subgroup T of G such that $S \subset T$; and also $N_\phi(S) \neq G$ since S is not normal in G.

Now let $S_1, \ldots, S_k$ be all the F Sylow subgroups of H. Then either $k = 1$ or F has finite order f and $k \equiv 1$ modulo f. Consider all the F Sylow subgroups of G which contain S_i. Since these are all contained in $N_\phi(S_i)$ there are a finite number of these, say

$$T_1^{(i)}, \ldots, T_{n_i}^{(i)};$$

furthermore, either $n_i = 1$ or n_i is congruent to 1 modulo f (if $n_i > 1$, F has finite order f) and the subgroups

$$T_1^{(i)}, \ldots, T_{n_i}^{(i)}$$

are all conjugate. The $T_j^{(i)}$ $(i = 1, \ldots, k;\ j = 1, \ldots, n_i)$ includes all F Sylow subgroups of G, since every F Sylow subgroup of G contains some F Sylow subgroup of H. Furthermore, they are all distinct since an F Sylow subgroup of G contains only one F Sylow subgroup of H. Thus the number of F Sylow subgroups of G is either 1 (if $k = 1$ and $n_1 = 1$) or is equal to $n_1 + \ldots + n_k \equiv k \equiv 1$ modulo f since each $n_i \equiv 1$ and $k \equiv 1$ modulo f. Also for fixed i the $T_j^{(i)}$ are conjugates since they are F Sylow subgroups of $N_\phi(S_i)$, and $T_j^{(i)}$ is conjugate to $T_s^{(r)}$ since $N_\phi(S_i)$ is conjugate to $N_\phi(S_r)$. Hence the F Sylow subgroups of G are all conjugate and if there is more than one, F has finite order f and their number is congruent to 1 modulo f.

Corollary 3.1. *If the ϕ composition factor F of G is infinite, G has just one F Sylow subgroup and it is normal.*

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4. Some further theorems on Sylow subgroups.

THEOREM 4.1. *If S is an F subgroup of G, S is contained in some F Sylow subgroup of G.*

Proof. We prove the theorem by induction on $j(G)$. If $j(G) = 1$, G is $M - \phi$ isomorphic to F and hence there is nothing to prove. Assume that the theorem is true for all H such that $j(H) < m$ and that $j(G) = m$. If G is primary the theorem is obvious so we assume that G is not primary. Let N be a minimal normal ϕ subgroup of G; we distinguish two cases:

(a) G/N is not primary. $N + S/N$ is an F subgroup of G/N and hence by the induction hypothesis there exists an F Sylow subgroup K/N of G/N which contains $N + S/N$; furthermore, $K \neq G$, since G/N is not primary. Now S is an F subgroup of K; using the induction hypothesis once again we conclude that S is contained in an F Sylow subgroup P of K, and it is easy to see that P is also an F Sylow subgroup for G.

(b) G/N is primary. If $N + S = G$, S is already an F Sylow subgroup for G, since it follows from the fact that G is not primary that $\text{char}(N) \neq F$. If $N + S \neq G$, $N + S$ is contained in a maximal normal ϕ subgroup K of G; for it follows from the ϕ nilpotency of G/N that $N + S$ is a ϕ link for G. Let P be any F Sylow subgroup for G, then by Lemma 4.1, $P \cap K$ is an F Sylow subgroup of K. By the induction hypothesis, S is contained in some F Sylow subgroup of K and hence, since the F Sylow subgroups are all conjugates, is contained in some conjugate $[P \cap K](k)$ of $P \cap K$. Hence S is contained in $P(k)$.

THEOREM 4.2. *If H is a ϕ subgroup of G and P_1 and P_2 are two F Sylow subgroups for H, they are not contained in the same F Sylow subgroup for G.*

Proof. If P_1 and P_2 are both contained in the F Sylow subgroup S of G, then $\{P_1, P_2\}$ is a ϕ subgroup of S and hence is primary with characteristic F. But $\{P_1, P_2\}$ is contained in H and P_1 is an F Sylow subgroup for H; hence $P_2 = P_1$.

THEOREM 4.3. *The ϕ normalizer N of an F Sylow subgroup P does not contain any conjugate of P distinct from P itself. Furthermore, N is its own ϕ normalizer.*

Proof. Assume that $P' = -g + P + g$ is contained in N. Then P' is an F Sylow subgroup for N and hence is conjugate to P in N so that there exists an element n in N such that $P' = -n + P + n$. But P is normal in N so that this implies $P' = P$. Let K be the ϕ normalizer of N in G. If k is in K, $-k + p + k \subseteq -k + N + k = N$ and hence $-k + P + k = P$. Thus P is normal in K so that $K = N$.

More generally we have:

THEOREM 4.4. *If H is a ϕ subgroup of G such that any ϕ composition factor of H has the same multiplicity for H as it does for G, then the ϕ normalizer of H is its own ϕ normalizer.*

Proof. Let N be the ϕ normalizer of H in G, and K the ϕ normalizer of N in G. Then if k is in K and $H' = H(k)$, $-n + H' + n = H'$ for n in N; for

$$-n + H' + n = -n - k + H + k + n = -k + (k - n - k) + H + (k + n - k) + k = -k + H + k$$

since $k + n - k$ is in N. Thus H and H' are both normal in N and $\{H, H'\} = H + H'$ has the same ϕ composition factors as H, since H' has the same ϕ composition factors as H. Hence $H = H'$ and H is normal in K. Therefore, $N = K$ and the theorem is proved.

THEOREM 4.5. *Let P_1 be the intersection of the F Sylow subgroups for G, P', and P'', and assume that if the ϕ subgroup S contains P_1, S is contained in no more than one F Sylow subgroup for G. Then*

(a) *The ϕ normalizers of P_1 in the F Sylow subgroups which contain it are all $M - \phi$ isomorphic.*

(b) *The ϕ normalizer of P_1 in G is not equal to the ϕ normalizer of P_1 in any F Sylow subgroup containing P_1.*

Proof. Let $P', P'', \ldots, P^{(s)}$ be the F Sylow subgroups (of G) which contain P_1; and let $N', N'', \ldots, N^{(s)}$ be the ϕ normalizers of P_1 in these groups. Since $P_1 \neq P^{(i)}$ and P_1 is a ϕ link for $P^{(i)}$ $(i = 1, \ldots, s)$, $P_1 \neq N^{(i)}$. Let N be the ϕ normalizer of P_1 in G. Then $N' = P' \cap N, \ldots, N^{(s)} = P^{(s)} \cap N$. Furthermore, $N^{(i)}$ is an F Sylow subgroup for N; for otherwise $N^{(i)}$ is properly contained in an F Sylow subgroup Q of N, which in turn is contained in $P^{(j)}$ for some $j \neq i$. Thus we would have $N^{(i)} \subset Q$, and $Q \subseteq P^{(j)}$ so that $N^{(i)}$ is contained in both $P^{(i)}$ and $P^{(j)}$, which contradicts the hypothesis since $P^{(i)}$ and $P^{(j)}$ are different. Hence the subgroups $N^{(i)}$ are F Sylow subgroups for N and so are $M - \phi$ isomorphic—which proves (a).

Now assume that $N = N^{(i)}$ for some i. Then N is an F group and $N' = N'' = \ldots = N^{(s)}$ so that $N' = N'' \subseteq P \cap P'' = P_1$, which is impossible. Hence (b).

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