Ascent and descent of going-down rings for integral extensions

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The author and Ira J. Papick have termed an integral domain R a going-down ring if $R \subset T$ satisfies going-down for each domain T containing R . The present paper investigates conditions which, for an integral extension $A \subset B$ of domains, imply that A (respectively B) is going-down whenever B (respectively A) is going-down. This explains the "descent" (respectively "ascent") in the title. Two typical results (the first about descent, the second about ascent) are given next.

THEOREM. *Let $R \subset T$ be an integral extension of domains such that T is going-down. If either T is quasi-local or R is integrally closed, then R is going-down.*

THEOREM. *Let R be an integrally closed domain with quotient field K , and let T be the integral closure of R in an algebraic field extension L of K . If the group of K-algebra automorphisms of the normal closure of L/K is finite of order $2^a 3^b$ and if R is going-down, then T is going-down.*

1. Introduction

Recall from [6] and [7] that a (commutative integral) domain R is called a going-down ring in case $R \subset T$ satisfies going-down for each domain T containing R . Examples of going-down rings are Prüfer domains, arbitrary domains of Krull dimension 1 , and the rings

253

constructed by iterated restrained power series in [6, Corollary 4.4]. In
this paper we investigate conditions which, for an integral extension
$A \subset B$ of domains, imply that A (respectively B) is going-down whenever
B (respectively A) is going-down; that is, conditions guaranteeing
descent (respectively ascent) of going-down rings for integral extensions.
An example of Heinzer and Ohm [10], summarized below in Section 2, shows
that, without further assumptions, both descent and ascent fail. The
extra conditions assumed in our results avoid certain features of the
Heinzer-Ohm example and, in the case of the second part of Theorem 2.4 and
Theorem 3.3, are motivated by the descent [9, Corollary 3] and ascent [17,
p. 31] results for Prüfer domains.

   A word about the propriety of motivating questions about going-down by
results about Prüfer domains seems to be in order. The characterization of
going-down rings in [7, Theorem 1] was in fact motivated by
characterizations of Prüfer domains involving flatness (cf. [18, Theorem 4]
and [6, Proposition 3.1]). Going-down has figured in the characterization
of several kinds of Prüfer domains (cf. [4, Corollary 4.3], [5, Corollaries
4 and 10], [6, Propositions 2.5 and 2.7]). Moreover, the class of Prüfer
domains and the class of going-down rings are each closed under
localization, and both fail to be closed under adjunction of
indeterminates.

   Our descent results are given in Section 2. With the aid of a recent
result of Brase [1], Corollary 2.5 leads to new examples of going-down
rings. Ascent, which is surprisingly elusive, is studied in Section 3,
whose main result depends on Theorem 2.4 about descent and on the structure
of the ambient Galois group.

   $\overline{D}$ will denote the integral closure of a domain D. Unexplained
terminology is standard, as in [8] and [11].

2. Descent

   Recall from [6] that a commutative ring A is said to be treed in
case spec(A), as a poset under inclusion, is a tree. By [6, Theorem
2.2], any going-down ring is treed. Another proof follows easily from [6,
Proposition 3.2, (i) $\Longleftrightarrow$ (iii)]. By using the ideas surrounding [13,
Theorem 2.5], Lewis has recently constructed a treed domain of Krull

dimension 2 which is not a going-down ring. A related construction of
Heinzer and Ohm [10, p. 6], originally developed for other reasons (*cf.*
[10, Example 2.2] and [2, p. 25]), will be summarized next in order to
indicate what sorts of results about descent and ascent may be possible.

EXAMPLE 2.1 (Heinzer-Ohm). Let $K = k(y, z)$ where y, z are
indeterminates over an algebraically closed field k of characteristic
zero. A rank 2 valuation of K over k may be given by sending y and
z to $(0, 1)$ and $(1, 0)$ respectively, and then taking the value of a
polynomial in $k[y, z]$ to be the infimum of the values of its
constituent monomials; let V be the corresponding valuation ring. Let
V^* be the integral closure of V in an algebraic closure of K . Set
$D = k + J$, where J is the Jacobson radical of V^* .

It is shown in [10] that D is a quasilocal G-domain (in the sense
of [11]) with infinitely many primes of height 1 . Thus D is not treed
and, in particular, D is not going-down. Of course, V^* is a Prüfer
domain by Prüfer's ascent result, so that both V and V^* are going-down.
As $D \subset V^*$ and $V \subseteq D$ are each integral, it follows that descent and
ascent both fail in general.

In order to motivate the hypotheses selected for our results, we next
isolate certain salient features of the preceding data. First, $D \subset V^*$
does not satisfy going-down: this suggests Corollary 2.3 below. Second,
V^* is not a valuation ring (that is V^* is not quasilocal; *cf*. Corollary
2.5 and the first part of Theorem 2.4). Third, D is not integrally
closed (*cf*. Theorem 3.3, Proposition 3.4, Corollary 3.5, and the second
part of Theorem 2.4). Moreover, applying the construction in [8, Appendix
2] to $V \subseteq D$ shows, by [7, Corollary], that an integral overring of a
(quasi-local) going-down ring need not itself be going-down (*cf*.
Proposition 3.4).

The next lemma will lead to our descent results. We use "unibranched"
in the sense of [4].

LEMMA 2.2. *Let* $A \subseteq B$ *be an integral extension of commutative
rings, with B treed. If each maximal ideal of A is unibranched in B ,
then A is unibranched in B and $A \subseteq B$ satisfies going-down.*

Proof. Let Q_1 and Q_2 be prime ideals of B contracting to a

prime P of A . Select a maximal ideal M of A containing P . Since $A \subset B$ satisfies going-up, there are primes N_i of B such that $Q_i \subset N_i$ and $N_i \cap A = M$ $(i = 1, 2)$; the unibranchedness hypothesis gives $N_1 = N_2$. Then Q_1 and Q_2 are comparable, since B is treed. As $A \subset B$ satisfies incomparability, it follows that $Q_1 = Q_2$, so that $A \subset B$ is unibranched. An easy application of lying over and going-up now shows that $A \subset B$ satisfies going-down.

COROLLARY 2.3. *Let R be a domain such that $\overline{R}$ is going-down. If $R \subset \overline{R}$ satisfies going-down (for example, by the lemma, if each maximal ideal of R is unibranched in $\overline{R}$), then R is going-down.*

Proof. By [7, Theorem 1], it is enough to show that $R \subset V$ satisfies going-down for each valuation overring V of R . Note that $\overline{R} \subset V$. As $R \subset \overline{R}$ and $\overline{R} \subset V$ each satisfy doing-down, transitivity completes the proof.

THEOREM 2.4. *Let $R \subset T$ be an integral extension of domains, with T being going-down. Assume that either T is quasi-local or R is integrally closed. Then R is going-down.*

Proof. As in the preceding proof, we need to show that $R \subset V$ satisfies going-down for each valuation overring V of R . First, observe that $R \subset T$ satisfies going-down, by Lemma 2.2, if T is quasi-local; by the going-down result of Krull [12], if R is integrally closed. Next, $T \subset VT$ satisfies going-down, since T is going-down, so that $R \subset VT$ satisfies going-down. However, $V \subset VT$ satisfies lying over since VT is integral over V . An application of [14, Lemma 1 (2)] implies that $R \subset V$ satisfies going-down, as required.

COROLLARY 2.5. *Let R be a domain such that $\overline{R}$ is a valuation ring. Then each overring of R (including R itself) is going-down.*

Proof. The integral closure of any overring of R is an overring of $\overline{R}$ and, hence, is a valuation ring. It therefore suffices to show that R is going-down. Apply either the first case of Theorem 2.4 or the case noted parenthetically in the statement of Corollary 2.3.

REMARKS 2.6. (a) The ploy of passing to VT in the above proof of Theorem 2.4 was suggested by McAdam's proof of [14, Theorem 1]. A slightly

different proof of Theorem 2.4 proceeds as follows. Let R and V be as
before, $P \subset M$ be primes of R, and N be a prime of V contracting to
M. Choose a valuation ring W of the quotient field of T such that

   (i) $V \subset W$ and

   (ii) N', the maximal ideal of W, contracts to N.

Since $T \subset W$, Lemma 2.2 or [12] supplies a prime Q of T which is
contained in $N' \cap T$ and contracts to P. Then, since T is going-down,
some prime N'' of W lies within N' and contracts to Q. Then $N'' \cap V$
is the required prime of V.

   (b) It follows from [6, Corollary 4.4 (i), (iii)] that the converse
of Corollary 2.5 (even replacing "valuation ring" by "Prüfer domain") is
false. By extending the above techniques, Papick [16] has recently
characterized the quasi-local i-domains (domains satisfying the hypotheses
in Corollary 2.5); also see Proposition 3.4 below. Along different lines,
Brase [1, Theorem] has characterized the quasi-local i-domains as the
domains for which the set of valuation ideals is closed under intersection.
By Corollary 2.5 and [6, Theorem 2.2], the set of prime ideals of any such
domain is linearly ordered by inclusion; Brase directly observed the
corresponding fact about the (larger) set of valuation ideals.

3. Ascent

   Our main interest in this section is the following question, which is
motivated by the ascent result of Prüfer [17, p. 31]. (For another proof
of Prüfer's result, see [8, Theorem 18.3].)

   QUESTION 3.0. *Let R be going-down and integrally closed, with
quotient field K. If T is the integral closure of R in a field
extension L of K, must T be going-down?*

   PROPOSITION 3.1. *The answer to Question 3.0 is affirmative in
general if it may be so answered in case R is quasi-local and L/K is
finite Galois with simple Galois group.*

   Proof. (i) Assume that L/K is finite Galois with simple Galois
group. To test whether T is going-down, take a domain B containing
T, primes $Q \subset N$ of T, and a prime I of B contracting to N. Let
$M = N \cap R$. By the hypothesis, $T_{R \backslash M}$ (which is the integral closure of

the quasi-local going-down ring R_M in L) is going-down. Thus $IB_{R\backslash M}$ contains a prime J of $B_{R\backslash M}$ contracting to $QT_{R\backslash M}$, whence $J \cap B$ is contained in I and contracts to Q . Thus $T \subset B$ satisfies going-down, to complete this case.

(ii)   Assume that L/K is finite Galois, with Galois group G . Induct on $|G|$, the order of G . Without loss of generality, G is neither simple nor trivial. Let F be the fixed field (in the sense of Galois theory) of a proper normal subgroup H of G . Now F/K is finite Galois, with Galois group isomorphic to G/H . As $|G/H| < |G|$, the induction hypothesis shows that S , the integral closure of R in F , is going-down. However, T is the integral closure of S in L , and L/F is finite Galois with Galois group H , so that the induction hypothesis applies again, and establishes this case.

(iii)   Assume that L/K is finite normal. Let F be the set of elements of L which are purely inseparable over K , and let S be the integral closure of R in F . Since L/F is finite Galois (with Galois group equal to the set of K-algebra automorphisms of L), it suffices to prove that S is going-down. This follows readily since $R \subset S$ is uni-branched [δ, Corollary 10.3] and R is going-down.

(iv)   The case of finite (algebraic) L/K follows by passing to the normal closure of L/K (to which (iii) applies) and then using an earlier descent result, the second part of Theorem 2.4.

(v)   General case. Deny. Then there exist primes $Q_1 \subset N$ of T and a prime I of an overring V of T such that I contracts to N and I contains no prime of V contracting to Q_1 . Since $R \subset V$ satisfies going-down, there is a prime J of V such that $J \subset I$ and $J \cap R = Q_1 \cap R$. As $Q_2 = J \cap T$ must be distinct from Q_1 , we may select α in $Q_1 \backslash Q_2$. Let $F = K(\alpha)$, and let $S = T \cap F$, the integral closure of R in F . By (iv), S is going-down, and hence treed. However, $Q_1 \cap S$ and $Q_2 \cap S$ are incomparable primes of S which are both contained in $N \cap S$. This contradiction completes the proof.

Theorem 3.3 will settle Question 3.0 for cases in which, so to speak,

the simple Galois groups arising from the proof of Proposition 3.1 are $Z/2Z$ and $Z/3Z$. First, we give a result of McAdam [15], whose proof is reproduced here with his kind permission.

PROPOSITION 3.2 (McAdam). *Let R be a quasi-local integrally closed going-down ring. If P is a prime ideal of R and b is an element of R, then P and Rb are comparable.*

Proof. Deny. Select c in $P \backslash Rb$, and set $u = cb^{-1}$. Observe that R contains neither u nor u^{-1}. Thus, if M is the maximal ideal of R, [19, Theorem 7] implies that $N = MR[u]$ is a prime of $R[u]$ which contracts to M. Since $R \subset R[u]$ satisfies going-down, N contains a prime Q contracting to P. As Q contains bu but not b, it follows that u is in Q. Then u is in N, so that u satisfies a polynomial over R with a unit coefficient. This contradicts the isomorphism $R[X]/MR[X] \to R[u]/N$ given in [19, Theorem 7], to complete the proof.

THEOREM 3.3. *The answer to Question 3.0 is affirmative if L/K is algebraic and the group of K-algebra automorphisms of the normal closure of L/K is finite of order $2^e 3^f$ (e, f are non-negative integers).*

Proof. The problem may be reduced by reworking parts of the proof of Proposition 3.1. As in (iv), an application of Theorem 2.4 reduces to the case L/K normal. As in (iii), replacing K by the purely inseparable part of L, we may take L/K finite Galois, with Galois group G of order $2^e 3^f$. Now G is solvable by Burnside's celebrated theorem [3, Theorem III], and so has a normal subgroup of index 2 or 3. By working with the corresponding fixed field via Galois theory as in (ii), an induction on $|G|$ reduces to the case $|G| = 2$ or 3. If the theorem is denied for this case, we argue as in (v) to obtain: incomparable primes Q_1 and Q_2 of T, both contained in a prime N of T; primes $P = Q_1 \cap R = Q_2 \cap R$ and $M = N \cap R$ of R; and an element α in $Q_1 \backslash Q_2$. As in (i), it is harmless to pass to $R_M \subset T_{R \backslash M}$, so that R may be taken quasi-local with maximal ideal M.

1: $|G| = 2$. Since R is integrally closed and $[L : K] = 2$, it follows from [20, Chapter V, Theorem 4] that there exist a, b in R such that

$$\alpha^2 + a\alpha + b = 0 \; .$$

Note that b is in $R \cap T\alpha \subset R \cap Q_1 = P$. Consequently, $\alpha(\alpha+a) = -b$ is in Q_2 , so that $\alpha + a$ is in Q_2 . Moreover, a is not in P ; otherwise, $\alpha^2 = -(a\alpha+b)$ would be in $PT \subset Q_2$, a contradiction. Thus a^2 is not in P , and an application of Proposition 3.2 shows that $P \subset Ra^2$; in particular, ba^{-2} is in R . Since

$$(\alpha a^{-1})^2 + \alpha a^{-1} + ba^{-2} = 0 \; ,$$

αa^{-1} lies in T . Now Q_1 contains $(\alpha a^{-1})a$ but not a ; so αa^{-1} is in Q_1 . Furthermore, Q_2 contains $(\alpha a^{-1}+1)a$ but not a ; so $\alpha a^{-1} + 1$ is in Q_2 . Thus $1 = (\alpha a^{-1}+1) - \alpha a^{-1}$ is in $Q_1 + Q_2 \subset N$, the desired contradiction.

    Case 2: $|G| = 3$. As above, [20, Chapter V, Theorem 4] provides a, b, c in R such that

$$\alpha^3 + a\alpha^2 + b\alpha + c = 0 \; .$$

By reasoning as above, c is in P and $\beta = \alpha^2 + a\alpha + b$ is in Q_2 .

    We claim that we may assume that b is in P . If not, β is in $Q_2 \backslash Q_1$. Moreover, one may laboriously check (see the remark following the proof) that β satisfies

$$\beta^3 - b\beta^2 + ac\beta - c^2 = 0 \; .$$

As ac (and $-c^2$) are in P , the claim is sustained merely by replacing α by β .

    Now that b is in P , the argument proceeds as in Case 1. Note that a is not in P . Then observe by Proposition 3.2 that ba^{-2} and ca^{-3} are in R . Since

$$(\alpha a^{-1})^3 + (\alpha a^{-1})^2 + (ba^{-2})(\alpha a^{-1}) + ca^{-3} = 0 \; ,$$

αa^{-1} is in T, and hence in Q_1. However $\alpha + a$ is in Q_2 (since $\alpha(\alpha+a)$ is). Thus $\alpha a^{-1} + 1$ is in Q_2, and once again we have the desired contradiction: 1 in N, to complete the proof.

REMARK. The key to generalizing Theorem 3.3 to certain groups whose orders involve primes other than 2 and 3 may lie in suitably identifying the coefficients of minimum polynomials of elements constructed, like β, from the minimum polynomial of α. The coefficients $-b$, ac, $-c^2$ of β's polynomial were, of course, obtained with the aid of α's polynomial by Cramer's rule from a system of three linear equations, whose coefficient matrix had (nonzero) determinant equal to c.

Besides Question 3.0, another ascent problem is to determine conditions on a going-down ring R which guarantee that $\overline{R}$ is going-down. (The extension $D \subset V^*$ in Example 2.1 shows that the corresponding descent problem is complicated; see, however, Corollary 2.5.) In this regard, we next offer a result which generalizes [6, Proposition 2.7] and [8, Proposition 25.13]. As usual ht, dim , and $\dim_v$ will denote height, Krull dimension, and valuative dimension, respectively.

PROPOSITION 3.4. *Let R be going-down. Assume that all the maximal ideals of R have the same finite height n. Let T be an integrally closed domain which is integral over R. Then T is Prüfer if and only if $\dim_v(R) = n$.*

Proof. The "only if" half is immediate since $\dim_v(R) = \dim_v(T)$, by [8, Proposition 25.12].

Conversely, assume that $\dim_v(R) = n$ and that T is not Prüfer. Choose a maximal ideal N of T such that T_N is not a valuation ring; set $M = N \cap R$. By integrality, M is maximal; so $ht(M) = n$.

Select u in the quotient field of T such that T_N contains neither u nor u^{-1}. By [19, Theorem 7], $Q = NT_N[u]$ is a nonmaximal

prime of $T_N[u]$ contracting to M . Since $R \subset T_N[u]$ satisfies going-down, we see that $ht(Q) \geq ht(M) = n$. Then

$$n = \dim_v (T) \geq \dim\bigl(T_N[u]\bigr) \geq 1 + ht(Q) \geq n + 1 ,$$

the desired contradiction.

Our final result is motivated by the observation that the rings V and D in Example 2.1 have valuative dimension 2 .

COROLLARY 3.5. *Let R be going-down, with $\dim_v (R) = 2$, and let T be an integrally closed domain which is integral over R . Then T is going-down.*

Proof. Let $Q \subset N$ be distinct primes of T , and let V be an overring of T with a prime contracting to N . Set $M = N \cap R$. To show that going-down obtains for the present data in $T \subset V$, we may suppose that $ht(M) = 2$. (Otherwise, $Q = 0$, a trivial case.) As it is enough to prove that $T_{R \setminus M} \subset V_{R \setminus M}$ satisfies going-down, it surely suffices to show that $T_{R \setminus M}$ is Prüfer. This, however, follows from Proposition 3.4, as applied to $R_M \subset T_{R \setminus M}$, since

$$2 = ht(M) = \dim\bigl(R_M\bigr) \leq \dim_v \bigl(R_M\bigr) \leq \dim_v (R) = 2 .$$

We close by observing that Proposition 3.4 leads easily to an amusing proof of Prüfer's ascent result for the case that the given Prüfer domain contains no primes of infinite height.

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