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On the work of V. A. Rokhlin in ergodic theory

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Abstract. The impact of V. A. Rokhlin's work in ergodic theory is discussed with particular emphasis on his famous lemma and its generalizations and his foundational work on measurable partitions.

0. *Introduction*

Besides his other mathematical achievements, V. A. Rokhlin is one of the creators of modern ergodic theory. In the current literature his name appears most often in the phrases 'Rokhlin towers' and 'the Rokhlin lemma', while one of his most fundamental contributions is the theory of measurable partitions. This essay is a rather personal account of these ideas and the role they have played in the development of ergodic theory. In § 1 I will describe the Rokhlin lemma and give an account of its generalizations. The second section will be devoted to a sampling of the ways in which the lemma has proven useful. V. A. Rokhlin's basic work on measurable partitions will be described in § 3 followed by a discussion of the importance and usefulness of this theory. Here perhaps a personal confession is not out of order. The basic results in entropy theory were obtained in the Soviet Union in the schools of A. N. Kolmogorov–Ya. G. Sinai and V. A. Rokhlin and were presented in the language of measurable partitions. When I first learned of the theory I made a systematic attempt to finitize it and do everything in the framework of finite partitions, believing that measurable partitions were an unnecessary complication. Over the last decade, I've become more and more convinced of the power and fruitfulness of Rokhlin's theory and I hope to convince the reader of this.

1. *The Rokhlin Lemma*

In one of his first papers in ergodic theory (**R-1948**], Rokhlin proved that generically (with respect to a natural topology on the space of invertible measurable transformations of [0, 1] that preserve Lebesgue measure) a measure preserving transformation T is not mixing. Here *mixing* means that for all measurable $A, B \subset [0, 1]$

$$(*) \quad |T^{-n}A \cap B| \to |A| \cdot |B|,$$

where $|A|$ denotes the Lebesgue measure of A. Earlier, P. Halmos had shown that generically m.p.t.'s are *weakly mixing*, which means that $(*)$ holds along a sequence of n's of full density. The key to Rokhlin's proof is to be found in the lemma that bears his name which may be formulated as follows:

ROKHLIN'S LEMMA. *If $T:[0,1]\to[0,1]$ is an aperiodic invertible measurable transformation preserving Lebesgue measure then for all $n\in\mathbb{N}$ and $\varepsilon>0$ there exists a measurable set B such that:*

(i) *$B, TB, \ldots, T^{n-1}B$ are pairwise disjoint, and*

(ii) $\left|\bigcup_{i=0}^{n-1} T^iB\right|>1-\varepsilon$.

The aperiodicity of T means that the set of periodic points for T has zero measure. From the point of view of dynamics B is an approximate cross-section of the transformation T, and for large n such n-approximate cross-sections enable one to picture the short (with respect to n) range behavior of T on most of the space. This was not Rokhlin's point of view. He emphasized the following consequence of the lemma. Introduce a metric on the space of automorphisms of $[0,1]$ with Lebesgue measure by

$$d(S,T)=|\{x: Sx\neq Tx\}|.$$

There an immediate corollary to the lemma is:

THEOREM. *If T is aperiodic and $n, \varepsilon>0$ given, then there exists a periodic S with period n such that*

$$d(T,S)<\frac{1}{n}+\varepsilon.$$

In turn this result was used to establish the title theorem of the paper mentioned above.

In the crucial special case, when T is ergodic, (any invariant set has measure 0 or 1) a rapid proof of the lemma may be given using the 'skyscraper' of S. Kakutani over any set A of small positive measure. Indeed this skyscraper, whose kth floor, A_k, consists of $T^kA\backslash(\bigcup_0^{k-1} T^iA)$, $(k=0,1,2,\ldots)$ fills up a set of measure 1, by ergodicity, and the set

$$B=\bigcup_{j=1}^{\infty} A_{jn-1}$$

satisfies

(i) $T^iB\cap B=\varnothing$ for $0<i<n$

(ii) $B\cup TB\cup\cdots T^{n-1}B\supset \bigcup_{j=n-1}^{\infty} A_j$.

From (ii) it follows that

$$|(X\backslash(B\cup TB\cup\,..\,T^{n-1}B)|\leq\left|\left(\bigcup_0^{n-2} A_j\right)\right|\leq n|A|$$

and thus if $0<|A|<\varepsilon/n$ we are done.

One way to establish the general case is to show that if T is aperiodic but not necessarily ergodic then for any N there still exist sets A with $|A|<1/N$ and

$|\bigcup_0^\infty T^i A| = 1$. To see this let $\mathcal{M}$ denote the collection of measurable sets A with $T^i A \cap A = \emptyset$ for $0 < i < N$ and order $\mathcal{M}$ by $A_1 < A_2$ if $A_1 \subset A_2$ and $|A_2 \backslash A_1| > 0$. Any increasing chain is at most countable and so $\mathcal{M}$ has maximal elements. If A is a maximal element let $C = X \backslash \bigcup_0^\infty T^i A$. One checks that C is invariant (up to a set of measure zero) and thus if $|C| > 0$, the aperiodicity would give a subset $D \subset C$ with $T^i D \cap D = \emptyset$ for $0 < i < N$ and $A \cup D \in \mathcal{M}$ contradicting the maximality.

A more involved proof due to D. Ornstein may be found in the classic lectures by P. Halmos [H]. (I don't know what Rokhlin's original proof of the lemma was like since his note [**R-1948**] contains no proof of it.)

In the 1960s the importance of the lemma as a basic tool in studying measure preserving transformations became clear. Thus, it was only natural that when attempts were made to extend ergodic theory to the actions of groups other than Z or $\mathbb{R}$ these almost always included attempts to extend the Rokhlin lemma. During this period the connection between the Rokhlin lemma and the notion of hyperfiniteness, that was introduced by Murray and von Neuman in their classic work on Rings of Operators [**M-vN**] was also noticed and gradually came to the forefront.

To the reader unfamiliar with this circle of ideas the following brief remarks may not be out of place. A von-Neumann algebra is a weakly closed algebra of bounded operators on a Hilbert space. In their fundamental work on the structure of these algebras (which they called Rings), Murray and von Neumann introduced an important construction called the *group measure space* construction. This construction associates to a group G acting by non-singular transformations of a measure space $(X, \mathcal{B}, \mu)$ a von-Neumann algebra. Recall that a transformation T of a measure space is non singular if $\mu(T^{-1}A) = 0$ if and only if $\mu(A) = 0$. They also introduced a notion called *hyperfiniteness* of an algebra $\mathcal{A}$, which says that $\mathcal{A}$ can be approximated, in a certain sense, by finite-dimensional algebras. They proved that if the group G is countable and locally finite, i.e. can be expressed as an increasing union of finite groups, then the associated $\mathcal{A}$ is hyperfinite. They stated that the same theorem holds whenever G is abelian and write concerning this ([**M-vN**] lemma 5.2.3): 'The proof of the lemma is somewhat complicated. It requires rather deep results on the decomposition of mappings of measurable sets which will be published elsewhere. We shall not pursue this matter further on this occasion.'

In his two pioneering papers [**D1, D2**], H. Dye established the hyperfiniteness of the von Neumann algebra obtained by the group measure space construction whenever G is abelian or more generally of polynomial growth. In the case that the action of G preserves the finite measure μ and is ergodic then the resulting von Neumann algebra is a so-called type II_1 factor. Factor is a kind of irreducibility for a von Neumann algebra, corresponding to the ergodicity for non-singular transformation, while type II_1 is an arbitrary numbering of certain properties of these factors which need not concern us here. Another of the fundamental results in [**M-vN**] is that all type II_1 hyperfinite factors are isomorphic. When H. Dye proved that the factors arising by the group measure space construction from measure-preserving actions of abelian groups are hyperfinite he showed that the isomorphism between any two such factors actually arises from a more basic equivalence at the level of the measure-theoretic actions. He proved in fact that any two ergodic measure-preserving actions of abelian groups (or groups with polynomial growth) are *orbit equivalent*. This means the following:

The actions of G_i on $(X_i, \mathscr{B}_i, \mu_i)$ $i = 1, 2$ are said to be *orbit equivalent* if there is a one-to-one measure-preserving map $\theta: X_1 \to X_2$ that maps G_1-orbits onto G_2-orbits, i.e.

$$\theta(G_1 x) = G_2(\theta x) \qquad \text{for almost all } x \in X,$$

where as usual $Gx = \{gx: g \in G\}$.

To return to our main story since we have mentioned non-singular transformations we should point out that C. Linderholm [**IT**] extended the Rokhlin lemma to this case and it was put to use in extending the early work of Halmos and Rokhlin to this situation by A. Ionescu-Tulcea and R. Chacon and N. Friedman ([**CF**], [**IT**]). The connections between the work of Dye and the Rokhlin lemma were noticed by R. M. Belinskaya and A. M. Vershik [**B**] and independently by W. Krieger who used the connections between the two theories in a very fruitful way to classify type III transformations (those with no invariant measure, finite or infinite) up to orbit equivalence.

For the groups Z^ν, $\nu \geq 2$, the natural analogue of the lemma was proved independently by J. P. Conze [**C**] and I. Katznelson and me [**KW**].

Several years later A. Connes and W. Krieger [**CK**] extended Dye's result on orbit equivalence to solvable groups. As explained by Vershik [**V**], this is enough to imply the validity of the Rokhlin lemma itself for certain subsets of the group. After hearing a lecture by W. Krieger on his work with A. Connes in the summer of 1976, D. Ornstein and I proved directly the validity of the Rokhlin lemma for a large class of subsets in any solvable group. Further pursuit of these ideas led to our ultimate generalization of some version of the lemma for any amenable group (cf. [**OW**-1980, 1987]. A *countable amenable group* is a group G that has almost invariant finite sets, namely for any $g_1, \ldots, g_k \in G$ and any $\varepsilon > 0$ there is a finite set $F \subset G$ such that

$$|g_i F \cap F| > (1-\varepsilon)|F|, \quad 1 \leq i \leq k.$$

Any solvable group is amenable as is any group with polynomial growth. This latter fact is easy to see directly and doesn't rely on the beautiful characterization by M. Gromov of groups with polynomial growth. A precise formulation of all our results on amenable groups would be out of place here but I can give one of the easier ones that explains the situation.

A finite subset $F \subset G$ is said to be a *Rokhlin set* (or an R-set) if for any free action of G as measure-preserving transformations of a finite measure space $(X, \mathscr{B}, \mu)$ and any $\varepsilon > 0$, there is a measurable subset $B \in \mathscr{B}$ such that for $f_1, f_2 \in F$

(i) $f_1 B \cap f_2 B = \varnothing$

and

(ii) $\bigcup_{f \in F} fB = FB$ has measure at least $1 - \varepsilon$.

It is an easy exercise to see that if $F \subset G$ is an R-set then it *tiles* the group in the sense that there is subset $C \subset G$ with $\{Fc: c \in C\}$ forming a partition of the group. It takes some more work to establish the converse – for amenable groups. Namely, *if G is an amenable group then the finite set $F \subset C$ is an R-set if and only if it tiles the group.* I should emphasize that the amenability of the group is crucial to the proof that I know of this fact. There are tiles in the free group on two generators, but it is an open problem as to whether or not they are R-sets. Related results may be found in Vershik's appendix to the Russian translation of Greenleaf's monograph on amenable groups (for a translation into English of this appendix with further addenda see [**V**]).

There have been many generalizations of the Rokhlin lemma to other settings with more or less structure. In the work of Denker et al. (cf. [**DGS**]) on finding special kinds of topological models for ergodic transformations versions of the lemma occur in which the emphasis is on finding bases B for the so-called Rokhlin tower or R-tower $(B, TB, \ldots, T^{n-1}B)$ with further properties. Another type of variant is used by S. Alpern to sharpen the original category results (cf. [**A**] and references therein). A version was given a few years ago in which the quantification 'for all $\varepsilon > 0$' is replaced by 'for all sets A with $\mu(A) < 1$', and the conclusion that $\mu(\bigcup_0^{N-1} T^iB) > 1-\varepsilon$ is replaced by $\bigcup_0^{N-1} T^iB \supset A$. When N is a prime this result holds without further qualification, for N composite there are certain obstructions if T is not totally ergodic (see [**LW**]). Finally let me mention a version in which no measure appears at all. Here the context is that of a Borel space and a discrete group of Borel measurable maps and one is studying this action without fixing any measure. Motivated by the ε-free version, a kind of Rokhlin lemma is established in [**W**] that is strong enough to show, for example, the hyperfiniteness of the action in this category.

2. *Applications of the Rokhlin lemma*

We have already mentioned the original application made by Rokhlin of his lemma to show that any transformation can be approximated by periodic ones. In the hands of D. Ornstein, his co-workers and students the lemma became a powerful tool that enabled one to reduce many questions in ergodic theory to purely finitary combinatorial considerations. The most striking of these uses lay in the proof that Bernoulli shifts with the same entropy are isomorphic [**O**]. Almost every paper that followed that seminal work used in one form or another the Rokhlin lemma. In this connection, a very useful strengthening of the lemma was given in [**KW**], there for the case of Z^ν-actions, but the result was new even for $\nu = 1$. Since this version is not so widely known it is perhaps worth repeating here.

Given a finite partition $P = (P_1, \ldots, P_a)$ and n, $\varepsilon > 0$, there exists an R-tower $(B, TB, \ldots, T^{n-1}B)$ that fills at least $1-\varepsilon$ of the space, and *in addition* for each i, $0 \le i < n$ the set T^iB is *independent* of the partition P.

Another kind of use of the lemma is in performing constructions in arbitrary systems $(X, \mathscr{B}, \mu, T)$. Perhaps an example of this type of application will be more instructive than cataloging many papers. Let's begin with the observation that there

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is a uniformity in the mean ergodic theorem, namely for any $\{k_n\}$ one has

$$\lim_{n\to\infty}\left\|\frac{1}{n}\sum_{j=k_n+1}^{k_n+n} T^j f-\hat{f}\right\|_2=0,$$

where $f\in L^2(X,\mathscr{B},\mu)$ and $\hat{f}$ is the projection of f onto the subspace of invariant functions. Now one can ask, does a similar result hold for almost everywhere convergence, i.e. is it the case that for all k_n

$$(*)\quad \lim_{n\to\infty}\left|\frac{1}{n}\sum_{j=k_n+1}^{k_n+n} f(T^jx)-\hat{f}(x)\right|=0 \qquad \text{a.e. } x,$$

when f is a bounded function? Now it is easy to see that if f satisfies the following condition:

For every $c>0$

$$(**)\quad \sum_{n=1}^{\infty}\mu\left\{x:\left|\frac{1}{n}\sum_{0}^{n-1} f(T^jx)-\hat{f}(x)\right|>c\right\}<+\infty$$

then indeed $(*)$ holds. This is an easy consequence of the Borel-Cantelli lemma. *Conversely*, if the condition *fails* then one can show that for ergodic T there is a sequence $k_n\to\infty$ for which $(*)$ fails. Now the point is that functions for which $(*)$ fails are ubiquitous, in the sense that any function can be modified by very little, on a set of very small measure so that $(*)$ fails for some constant c. It is at this stage that one uses the Rokhlin lemma.

Since our purpose is to illustrate we will content ourselves with a sketch of the argument. To begin with suppose $f\equiv 0$. Let $B, TB,\dots,T^{N-1}B$ be an R-tower, and let B_{-1}, B_{+1} be two small subsets of B of equal measure. Putting $f=-\varepsilon$ on $\bigcup_0^{N-1} T^jB_{-1}$ and $f=+\varepsilon$ on $\bigcup_0^{N-1} T^jB_1$ we have a function of zero mean, and the contribution to the sum in $(**)$ with $c=\frac{1}{2}\varepsilon$ from points in T^jB_1 (with $n\le N-1-j$) is $(N-1-j)\cdot\mu(B_1)$ so that altogether we have a contribution on the order of

$$\tfrac{1}{2}N^2\cdot\mu(B_1).$$

We can keep $N\mu(B_1)$ quite small while this last expression is large. Our next step will be to use a much taller tower $(D, TD,\dots,T^{M-1}D)$ also quite thin which can be made disjoint from $\bigcup_0^{N-1} T^j(B_{-1}\cup B_{+1})$. To get this disjointness, choose $N\cdot(\mu(\bigcup_0^{M-1} T^jD))\ll\mu(B_{-1})$; then removing from B_{-1}, B_{+1} the part that gives some intersection doesn't change B_{-1}, B_{+1} by much and they are now disjoint. Once again take two disjoint subsets D_{-1}, D_{+1}, of D and put $f=\pm\varepsilon$ on $D_{\pm1}$ and we gain for $(**)$ another contribution of the order of $M^2\cdot\mu(\bigcup_0^{M-1} T^jD_1)$. This can be made large while still $N\cdot\mu(\bigcup_0^{M-1} T^jD)$ is small enough for one earlier consideration to be valid. Continuing this procedure indefinitely gives a function that differs from zero on a set of small measure; where it differs from zero it is still never more than ε, and $(**)$ diverges for $c=\frac{1}{2}\varepsilon$.

If we start with an arbitrary function we can do the same kind of thing. In order to know that the small change that we make along the tower gives rise to a systematic deviation at many levels for many averages one uses the individual ergodic theorem and the strengthened form of the Rokhlin lemma described above.

In yet another direction, Rokhlin towers serve to distinguish some basic classes of ergodic transformations. For example, $(X, \mathscr{B}, \mu, T)$ is said to be of *rank one*, if for any $B \in \mathscr{B}$, and any N_0, $\varepsilon > 0$, there is a tower of height $N \geq N_0$ and a subset of its levels whose union approximates B to within ε. In some sense these are the simplest transformations and now a great deal is known about them. For example, S. Kalikow [**K**] has proved that for these transformations mixing implies 3-fold mixing. Some beautiful structural results concerning them and their generalizations (finite rank) may be found in the work of J. King [**KI**]. To this kind of work belong the results of A. Katok and Stepin et al. [**K-S**] on the speed of approximation of transformations by periodic ones and constructions connected to this work.

It is fair to say that this tool is one of the most useful in ergodic theory and will continue to remain on the stage for the foreseeable future.

3. *Measurable partitions*

The basic example of a measure space is the unit interval with Lebesgue measure. For an example of a measure space and a sub σ-algebra one has only to turn to $X = [0, 1] \times [0, 1]$, the unit square with 2-dimensional Lebesgue measure and for $\mathscr{a} \subset \mathscr{B}$, the sub σ-algebra of sets that depend only on the first coordinate, so that

$$\mathscr{a} = \{B \times [0, 1]: B \in \mathscr{B}_1 \text{ - the measurable subsets of } [0, 1]\}.$$

In this situation, the 2-dimensional measure can be calculated by first integrating over the 'fibers' of the σ-algebra $\mathscr{a}$, i.e. the sets of the form $\{x\} \times [0, 1]$ and then integrating with respect to x. In fact this is precisely the content of Fubini's theorem. Rokhlin [**R-1949a**] gave conditions under which a similar type of disintegration of a measure into conditional measures on fibers can be carried out.

In the probability literature similar work was carried out under the terminology of regular conditional expectations (cf. Doob's book [**D**]). What made Rokhlin's work so important for ergodic theory was that he also adapted these ideas to the situation when a measurable transformation of the space was involved. In particular, if $T: X \to X$ is measure preserving on $(X, \mathscr{B}, \mu)$ and $\mathscr{A} \subset \mathscr{B}$ is also T-invariant, then under certain technical conditions (which in practice are almost always satisfied) there exists another measure space $(Y, \mathscr{C}, \nu)$, and maps $S: Y \to Y$, $\pi: X \to Y$ such that $S\pi = \pi T$, and $\pi^{-1}(\mathscr{C}) = \mathscr{a}$ modulo null sets. Furthermore, in the ergodic case, almost all fibers look like the same space (i.e. are either finite with the same cardinality, or uncountably infinite) and T can be represented as a skew product over Y, i.e. X as a measure space is isomorphic to $Y \times Z$, and

$$T(y, z) = (Sy, F_y(z)),$$

where F depends measurably on Y, and $F_y: z \to z$ is measure preserving.

When studying a factor of a transformation, one starts with the (Y, S) and $\Pi: X \to Y$ and then $\mathscr{a}$ is defined as $\Pi^{-1}(\mathscr{C})$, and this result still has content and gives a very useful description of how T is built up from S. In the work of Furstenberg on the ergodic theoretic approach to Szemeredi's [**F**] theorem this geometric way of going up to T from a factor plays a vital role in the inductive scheme by which the proof is carried out.

In Rokhlin's work on entropy theory a further extension of these technical devices entered in an important way. Here one deals with measurable partitions, or their associated σ-algebras which are not quite invariant under T but either increase or decrease $T^{-1}\alpha \supset \alpha$, $T^{-1}\alpha \subset \alpha$. In the work of Ornstein and his school all of this work, such as the beautiful theorem of Pinsker-Rokhlin-Sinai characterizing completely positive entropy, was repeated using finite partitions and thus avoiding many technical difficulties. However, it appears that a true understanding of the nature of the entropy for smooth dynamical systems (cf. the work of Pesin and its generalizations) requires going back again to the formalism of measurable partitions.

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