SELF-POLAR DOUBLE CONFIGURATIONS IN PROJECTIVE GEOMETRY

II. LINES AND SECUNDA [1]

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The principal theorem to be proved in this part is:

THEOREM II. *If in Π_n a normal rational curve, ρ, and a quadric primal S are such that there is a proper simplex inscribed in ρ and self-polar with regard to S, then there exist sets of N, $= \binom{n+1}{2}$, chords of ρ every two of which are conjugate with regard to S. A set can be constructed to contain any pair of chords of ρ which are conjugate with regard to S.*

The polar secunda of the N chords form with the chords a double-N which is self-polar with regard to S.

It should be remarked that the edges of the inscribed simplex provide a highly special set of N mutually conjugate chords; such a set is obtained only when the pair of chords selected meet the curve in the vertices of a tetrahedron which is self-polar with regard to the section to of S by the Π_3 in which it lies.

The approach to the theorem is by way of the double configurations determined, as in Part I, by matrices of linear forms, now to be taken of size $p = 3$, $q = n$.

For $n = 3$ the figure is the double-six of lines on the cubic surface, F^3, and for $n = 4$ it is the double-ten consisting of lines on a Bordiga surface, F^6, and planes which cut the surface in cubic curves.

In the case of the double-six there is always a "Schur" quadric, S, with regard to which the lines of one row are the polars of the lines of the other, i.e., the six lines of one row are mutually conjugate with regard to S. Moreover, these lines are chords to each curve of one of the two-parameter families of normal rational (cubic) curves on F^3.

In the case of the double-N there is in general no such quadric, but Coble [2], constructed a special White surface (White [3]), for which there exists a "Schur" quadric; there is on this surface a special normal rational curve to which the lines h_i are chords. The basis of the present

[1] The meanings of this and other terms are given in the Appendix to this Part.

discussion is the identity of Coble's surface in the case $n = 4$ with the surface F^6_S described in [1], p. 392.

The special form of matrix for the surface F_S in Π_n is

$$\begin{bmatrix} x_0 & x_1 & \cdots & x_{n-2} & u_1 \\ x_1 & x_2 & \cdots & x_{n-1} & u_2 \\ x_2 & x_3 & \cdots & x_n & u_3 \end{bmatrix} \equiv [x_{\alpha+\varepsilon},\ u_\alpha], \qquad \begin{matrix} \alpha = 1,\ 2,\ 3 \\ \varepsilon = -1, 0, \cdots, n-3 \end{matrix}$$

where

$$u_\alpha = g_{\alpha\delta} x_\delta,$$

i.e., the constants $a_{\alpha\beta\delta}$ in the forms $x_{\alpha\beta} = a_{\alpha\beta\delta} x_\delta$ of Part I are taken to be

$$\text{for} \quad \alpha = 1,\ 2,\ 3, \quad \beta = 1, 2, \cdots, n-1,$$

$$a_{\alpha\beta\delta} = \begin{cases} 1 & \text{for} \quad \delta = \alpha+\beta-2 \\ 0 & \text{for} \quad \delta \neq \alpha+\beta-2 \end{cases}$$

$$\text{for} \quad \alpha = 1,\ 2,\ 3, \quad \beta = n,$$

$$a_{\alpha n\delta} = g_{\alpha\delta}.$$

The equations

$$||x_{\alpha+\varepsilon},\quad u_\alpha|| = 0$$

are the result of the elimination of the set $\{\lambda_\alpha\}$ from the equations

$$\lambda_\alpha x_{\alpha+\varepsilon} = 0$$
$$\lambda_\alpha u_\alpha = 0.$$

The equations $\lambda_\alpha x_{\alpha+\varepsilon} = 0$, for every set $\{\lambda_\mu\}$, are the equations of a line which is a chord of the normal rational curve ρ, given parametrically by

$$\sigma x_\delta = \theta^\delta.$$

Conversely, every chord of ρ has equations of this form and the equations of the locus $\mathscr{R}$ of chords of ρ is

$$\mathscr{R}: \qquad ||x_{\alpha+\varepsilon}|| = 0.$$

Thus

THEOREM 1. *F_S lies on the locus $\mathscr{R}$ of chords of ρ.*

$\mathscr{R}$ is of dimension 3 and order $\binom{n-1}{2}$.

It is to be proved next that

THEOREM 2. *The freedom of F_S is $n(n+4)$.*

F_S is determined by:

(i) ρ (which determines $\mathscr{R}$): freedom n^2+2n-3,

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(ii) a tertium, $u_\alpha = 0$, which may be replaced by any tertium of the family $\mu_{\varepsilon+2}x_{\alpha+\varepsilon}+\mu_n u_\alpha = 0$: net freedom $3(n-2)-(n-1) = 2n-5$,

(iii) a set of 9 homogeneous constants $\{g_{\alpha\alpha'}\}$, since, given ρ and the tertium, the set of equations $||x_{\alpha+\varepsilon},\ g_{\alpha\alpha'}u_{\alpha'}|| = 0$ determines a different surface F_S for every set $\{g_{\alpha\alpha'}\}$: freedom 8, giving a total of n^2+4n.

Shortly a form of equations, referred to a coordinate system which is one of a finite set determined by F_S, is found which displays exactly the $2n$, $=(n^2+4n)-(n^2+2n)$, independent constants.

A chord of ρ is the intersection of a set of corresponding primes of the projectively related systems

$$\lambda_\alpha x_{\alpha+\varepsilon} = 0;$$

a line h_i on F_S is obtained when the corresponding prime of the system

$$\lambda_\alpha u_\alpha = 0$$

contains the chord. The coordinate system in which the equations of ρ are

$$\sigma x_\delta = \theta^\delta$$

is completely and uniquely determined by the selection of three distinct points of ρ to be the vertices ε_0, ε_n of the simplex of reference and the unit point $(1, 1, \cdots, 1)$. The coordinate system may therefore be selected so that one of the lines h_i, say h_0, joins the points ε_0, ε_n on ρ, and there will still be one constant available for manipulation (the points $\theta = 0$, $1/\theta = 0$ have been fixed, but the point for which $\theta = 1$ is arbitrary on the curve).

With this selection of coordinate system the equations

$$||x_{\alpha+\varepsilon},\ u_\alpha|| = 0$$

have to be satisfied by $x_1 = x_2 = \cdots = x_{n-1} = 0$, and consequently $x_0 x_n(g_{20}x_0 + g_{2n}x_n) = 0$, i.e., $g_{20} = g_{2n} = 0$. Now replace the matrix $[x_{\alpha+\varepsilon},\ u_\alpha]$ by

$$[x_{\alpha+\varepsilon},\ u'_\alpha] = [x_{\alpha+\varepsilon},\ u_\alpha]\begin{bmatrix} \mathbf{1}_{n-1} & g_{2\nu} \\ \mathbf{0}^T & -1 \end{bmatrix}, \qquad \nu = 1, \cdots, n-1,$$

so that $u'_2 \equiv 0$. F_S is unchanged, so that

THEOREM 3. *A coordinate system can be found in which the equations of F_S are*

$$||X_S|| = ||x_{\alpha+\varepsilon}u'_\alpha|| \equiv \left\| \begin{matrix} x_0 & x_1 & \cdots & x_{n-2} & v \\ x_1 & x_2 & \cdots & x_{n-1} & 0 \\ x_2 & x_3 & \cdots & x_n & w \end{matrix} \right\| = 0.$$

On F_S the line h_0 is $x_1 = x_2 = \cdots = x_{n-1} = 0$, and the paired secundum k_0 is $v = w = 0$.

Take

$$v \equiv b_\delta x_\delta, \ w = c_\delta x_\delta.$$

The linearly distinct quadratic forms given by the 2×2 minors in X_S are:

$$x_i x_{j+k} - x_j x_{i+k} \tag{1}$$

for all relevant sets i, j, k (these are a set of $\binom{n}{2}$ independent forms corresponding to quadrics which contain ρ), and

$$\left.\begin{array}{c} x_0 w, \ x_1 w, \cdots, x_{n-1} w \\ x_1 v, \cdots, x_n v \end{array}\right\}, \tag{2}$$

making a total $\frac{1}{2}n(n+3)$ linearly independent quadratic forms. Thus, by Theorem I of Part I,

THEOREM 4. *The double-N determined by X_S is self-polar with regard to a Schur quadric, S.*

In fact it is possible to write down almost explicitly the equation of S, since S is the tangential quadric apolar to all quadrics of the sets (1) and (2). Take the tangential equation of the quadric to be

$$G_{\delta\delta'} X_\delta X_{\delta'} = 0.$$

If this is apolar to the forms (1), then

$$G_{ij} = G_{i-k, j+k}$$

for all relevant i, j, k, so that we may take

$$G_{ij} = G_{i+j}, \qquad i, j = 0, \cdots, n.$$

The conditions of apolarity to the forms (2) then give

$$G_{r+\delta} c_\delta = 0, \qquad r = 0, \cdots, n-1$$

$$G_{s+\delta} b_\delta = 0, \qquad s = 1, \cdots, n.$$

These are two sets of n bilinear equations, so that

THEOREM 5. (i) *given the two sets $\{b_\delta\}$, $\{c_\delta\}$ the set of ratios of $\{G_{i+j}\}$ is uniquely determined.*

(ii) *given the set $\{G_{i+j}\}$ the two separate sets of ratios $\{b_\delta\}$, $\{c_\delta\}$ are uniquely determined,*

i.e., in case (ii) there is a one-parameter family of surfaces F_S which contain ρ and h_0 and have a given quadric S (of the family apolar to ρ) as Schur quadric. Alternatively the configuration is determined by:

ρ (freedom n^2+2n-3), a chord h_0 of ρ (freedom 2), an arbitrary secundum k_0 (freedom $2n-2$), a quadric of the apolar system such that h_0 and k_0 are polars (freedom 2), with one constant still disposable.

One numerical result is needed before the final construction of the mutually conjugate set of chords is undertaken. This is given in Theorem 8 to which Theorems 6 and 7 are preliminaries.

THEOREM 6. *The Π_{n-4} common to two secunda k_i, k_j meets $\mathscr{R}$ in $\binom{n-2}{2}$ points.*

$$k_0 \text{ meets } \mathscr{R} \text{ in the curve,}$$

$$||x_{\alpha+\varepsilon}|| = 0,\ v = w = 0,$$

i.e., in a curve which is a general secundum section of $\mathscr{R}$. Let k_1 be given by

$$l_\alpha x_{\alpha+\varepsilon} = 0,\ l_\alpha u'_\alpha = 0$$

so that $k_0 \cap k_1$ is given by

$$v = w = l_\alpha x_{\alpha+\varepsilon} = 0$$

where (taking account of $v = w = 0$) only $n-2$ of the last $n-1$ equations are independent. Thus only $n-2$ of the columns of $||x_{\alpha+\varepsilon}|| = 0$ are linearly independent, and it represents a locus determined by a $3 \times (n-2)$ matrix of linear forms in Π_{n-4}. This locus either consists of $\binom{n-2}{2}$ points or might exceptionally consist of a curve together with a smaller number of points. But $\mathscr{R} \cap k_0$ and $\mathscr{R} \cap k_1$ are both irreducible curves, so that the locus (i.e., the intersection $\mathscr{R} \cap k_0 \cap k_1$ must consist of $\binom{n-2}{2}$ points.

THEOREM 7. *The number of chords of ρ meeting both of two secunda in general position is $(n-1)^2$.*

This is the same as: the order of the surface generated by the chords of ρ which meet a general secundum is $(n-1)^2$. This numerical result is most simply obtained by considering the normal rational curve consisting of n lines with $n-1$ intersections (forming, e.g., an open chain). The surface of chords meeting a secundum then breaks up into $n-1$ planes and $\binom{n}{2}-(n-1)$ quadric surfaces, giving the total order $(n-1)^2$.

In the case of k_0, k_1 and ρ, $\binom{n-2}{2}$ of the chords pass through the points $k_0 \cap k_1 \cap \mathscr{R}$, leaving $(n-1)^2-\frac{1}{2}(n-2)(n-3) = \frac{1}{2}n(n+1)-2$ others. Thus

THEOREM 8. *There are $N-2$ chords of ρ which meet k_0 and k_1 in distinct points.*

So far the double-N configuration has been derived from a system consisting of the curve ρ, the chord h_0 joining ε_0, ε_n, and a pair of equations $v = 0$, $w = 0$ determining the secundum k_0. But we have seen (in Theorem 5) that these determine uniquely the Schur quadric S, and that conversely the ρ, h_0 and S determine, except for one proportionality factor, the complete configuration. On this second basis it is possible to give an explicit construction for a set of N mutually conjugate chords of a normal rational curve, namely,

THEOREM 9. *Given a normal rational curve, ρ, any chord, h_0, and any quadric S apolar to ρ, construct the polar secundum k_0 of h_0 with regard to S, and take P_1 any point on the curve $k_0 \cap \mathscr{R}$, where $\mathscr{R}$ is the chord-locus of ρ. Let h_1 be the chord through P_1 and k_1 its polar with regard to S, then there are $N-2$ chords of ρ which meet k_0 and k_1 in distinct points. These, with h_0, h_1, form the set of N mutually conjugate chords determined by ρ, S and the pair of conjugate chords h_0, h_1.*

This theorem leaves open the question of whether there exist sets of N mutually conjugate lines which are not chords of a normal rational curve.

Appendix

As the results of this paper may be of interest to workers whose field is not specifically projective geometry, it has been thought desirable to give the following definitions and explanations of the terms used.

Π_n: projective space of n dimensions, i.e., the space of points in which vectors $\boldsymbol{x}$, $=(x_0, x_1, \cdots, x_n)$, and $\rho\boldsymbol{x}$, for all $\rho \neq 0$, correspond to the same point. The field is not necessarily restricted to the complexes, but as many of the results are most simply stated in a form which requires that an equation of degree m should have m roots, it is convenient for the present purpose to do so.

GLOSSARY

Prime, secundum, tertium: in Π_n, spaces of dimensions $n-1$, $n-2$, $n-3$ respectively (dual to point, line, plane).

Dual or tangential coordinates: $\boldsymbol{X} = (X_0, X_1, \cdots, X_n)$ are the coordinates of the prime $\boldsymbol{X}^T\boldsymbol{x} = 0$.

(Proper) simplex: the configuration in Π_n determined by $n+1$ linearly independent points.

Normal rational curve: curve of order n in Π_n; coordinate systems can be found in which its parametric equations are $x_0 : x_1 : \cdots x_n = 1 : \theta : \cdots : \theta^n$.

Chord: line joining two points of a curve.

Chordal Π_r: Π_r joining $r+1$ points of a curve.

Primal: Locus of dimension $n-1$ in Π_n.

(Point-) Quadric: primal represented by the vanishing of a quadratic form; $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$.

Tangential quadric: quadric regarded as the family of its tangent primes. The tangential form of $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ is $\boldsymbol{X}^T(\text{adj } \boldsymbol{A})\boldsymbol{X}$.

Conjugate points with regard to a quadric: $\boldsymbol{p}$ and $\boldsymbol{q}$ are conjugate if $\boldsymbol{p}^T\boldsymbol{A}\boldsymbol{q} = 0$.

Polar prime of a point with regard to a quadric: the locus of points conjugate to the point. The polar prime of $\boldsymbol{p}$ is $\boldsymbol{p}^T\boldsymbol{A}\boldsymbol{x} = 0$.

Polar space, Π_{n-r-1}, of a space Π_r with regard to a quadric: space such that every point of Π_r is conjugate to every point of Π_{n-r-1}. (If $\boldsymbol{x}$ is taken as a vector in V_{n+1}, and orthogonality is defined in terms of a given quadratic – determining the isotropic cone – then a pair of "polar spaces" is a pair "orthogonal complements".)

Conjugate spaces, Π_r, with regard to a quadric: each meets (in a of point at least) the polar of the other.

Apolar quadrics: apolarity is a relation between a point quadric and a tangential quadric. $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ and $\boldsymbol{X}^T\boldsymbol{B}\boldsymbol{X}$ are apolar if trace $(\boldsymbol{AB}) = 0$. The geometrical condition for apolarity is that there exist proper simplexes inscribed in the point quadric (i.e., having all their vertices on the quadric) and self-polar with regard to the tangential quadric. If $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}$ reduces to a prime-pair the condition of apolarity is that the primes should be conjugate with regard to $\boldsymbol{X}^T\boldsymbol{B}\boldsymbol{X}$. (In the vector space, if orthogonality is defined in terms of the tangential quadratic form, the condition that a point quadratic form is apolar to the isotropic form is that it vanishes on each of the vectors of some orthogonal basis.)

Freedom: A locus or set of loci Φ has freedom $\mathfrak{f}$ if in the equations of Φ when referred to an arbitrary coordinate system there appear $\mathfrak{f}$ independent constants. Alternatively, if the coordinate system is selected from a finite family of coordinate systems geometrically determined by Φ, then the number of independent constants which appear in the equations is $\mathfrak{f}-(n^2+2n)$. (See, e.g., Room [**1**], pp. 25 and 475).

References

[1] Room, T. G., *Geometry of Determinantal Loci* (Cambridge U.P., 1938).

[2] Coble, A. B., *The Double-N_n configuration*, Duke Month. J. 9 (1942) 436.

[3] White, *On certain sets of plane curves*, Proc. Camb. Phil. Soc. 22 (1924) 216—227.

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