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GENERAL THEOREMS FOR UNIFORM ASYMPTOTIC STABILITY AND BOUNDEDNESS IN FINITELY DELAYED DIFFERENCE SYSTEMS

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Abstract. The paper deals with boundedness of solutions and uniform asymptotic stability of the zero solution. In our current undertaking, we aim to prove two open problems that were proposed by the author in his book [12]. Our approach centers on finding the appropriate Lyapunov functional that satisfies specific conditions, incorporating the concept of wedges.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}^d$ be the sets of integers, non-negative integers, real numbers, non-negative real numbers, and d-dimensional real space, respectively. This paper is concerned with the uniform asymptotic stability of the zero solution of the finite delay difference equation

$$x(n+1) = f(n, x_n), \tag{1.1}$$

where x_n is the segment of $x(s)$ for $n-h \leq s \leq n$, h is a nonzero positive integer. Here, the function f is continuous in x with $f : \mathbb{Z} \times C \to \mathbb{R}^d$ where C is the set of functions $\phi : \{n_0 - h, n_0 - h + 1, \cdots, n_0 - 1, n_0\} \to \mathbb{R}^d$, $h > 0$ and integer and $n_0 \geq 0$ is the initial time. Let

$$C(n) = \{\phi : \{n-h, n-h+1, \cdots, n-1, n\} \to \mathbb{R}^d\}.$$

It is to be understood that $C(n)$ is C when $n = 0$. Also ϕ_n denotes $\phi \in C(n)$ and $||\phi_n|| = \max_{n-h\leq s\leq n} |\phi(n)|$, where $|\cdot|$ is a convenient norm on $\mathbb{R}^d$. For $n = n_0$,

$$C(0) = \{\phi : \{-h, -h+1, \cdots, -1, 0\} \to \mathbb{R}^d\}.$$

We say that $x(n) \equiv x(n, n_0, \varphi)$ is a solution of (1.1) if $x(n)$ satisfies (1.1) for $n = n_0+1, n_0+2, \cdots$ and $x(n) = \varphi(n)$, $n = n_0 - h, n_0 - h + 1, \cdots, n_0 - 1, n_0$ where φ is a given initial sequence such that $\varphi : \{n_0 - h, n_0 - h + 1, \cdots, n_0 - 1, n_0\} \to \mathbb{R}^d$. If $x(n)$ is any solution of system (1.1), then the variation of the function V, where

$$V : \mathbb{Z}^+ \times C \to \mathbb{R}^+,$$

is defines as

$$\triangle V(x(n)) = V(f(n, x(n))) - V(x(n)) = V(x(n+1)) - V(x(n)).$$

Throughout this paper we assume that $f(n, 0) = 0$, for all $n \in \mathbb{Z}$, when we are considering the stability of the zero solution. In this paper we will prove two general theorems regarding the uniform asymptotic stability of the zero solution and the uniform boundedness of all solutions in terms of wedges by assuming the existence of a Lyapunov functional.
Delay discrete systems play a pivotal role in mathematical models that describe how a system changes over time, but with the added complexity of incorporating delays in the process. These delays represent the time it takes for a system to respond to inputs or changes in its environment.

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Studying the boundedness of solutions and stability of the zero solution is important since many real-world systems exhibit delays in their responses, such as control systems, biological processes, communication networks, and economic systems. Understanding and analyzing delay discrete systems help in designing and optimizing such systems for better performance. It's important to study the boundedness of solutions in delay discrete systems to ensure that the system's behavior remains manageable and doesn't diverge or go to infinity, which could lead to instability or unpredictable outcomes. Additionally, the analysis of the stability of the zero solution is as important as boundedness. The zero solution represents the equilibrium state where the system remains unchanged over time. Stability analysis of the zero solution helps in understanding whether small perturbations or disturbances in the system will die out over time (stable), grow indefinitely (unstable), or remain at a constant level (marginally stable). This information is crucial for ensuring the reliability and robustness of the system. For more reading we refer the interested reader to [3], [4], [5].

The use of Lyapunov functionals in the context of delay difference equations is rooted in stability analysis and control theory. Lyapunov methods provide a powerful tool for studying the stability and convergence properties of dynamic systems, including those described by delay difference equations. Aleksandr Lyapunov introduced the concept of Lyapunov functions in the late 19th century. His work laid the foundation for stability analysis in differential equations. Lyapunov methods were later extended to difference equations, which describe systems evolving in discrete time steps. As the study of systems with time delays gained prominence, researchers began applying Lyapunov methods to analyze stability in the presence of delays. The advantages and effectiveness of the use of Lyapunov functional can be seen in many areas including but not limited to stability analysis, and control design.

Lyapunov functionals provide a systematic way to analyze the stability of solutions to delay difference equations. They provide a mathematically rigorous framework for stability analysis. They allow researchers to derive explicit stability criteria and prove the convergence properties of systems described by delay difference equations. In summary, the use of Lyapunov functionals in delay difference equations has a rich history and remains a powerful and widely adopted methodology.

We assume the readers are familiar with the calculus of difference equations and for a comprehensive study of the calculus of difference equations, we refer to the books [2], [6], and for an excellent reference to the use of Lyapunov functionals in discrete systems we refer to the book [12].

In this paper, we refer to wedges as $W_i : [0, \infty) \to [0, \infty)$ that are continuous with $W_i(0) = 0$, $W_i(r)$ strictly increasing, and $W_i(r) \to \infty$ as $r \to \infty, i = 1, 2, 3, 4$.

Using Lyapunov functionals, in [14] the author proved general theorems regarding stability of the zero solution and the boundedness of all solutions of functional systems of difference equations of the form

$$x(n+1) = G(n, x(s);\ 0 \le s \le n) \stackrel{def}{=} G(n, x(\cdot)) \tag{1.2}$$

where $G : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous in x. During our analysis of (1.2), we encountered endless difficulties due to finding a suitable Lyapunov functional that satisfied the pair of inequalities

$$W_1(|x(n)|) \le V(n, x(\cdot)) \le W_2(|x(n)|) \tag{1.3}$$

and

$$\triangle V(n, x(\cdot)) \le -\rho W_3(|x(n)|) + K \tag{1.4}$$

for some positive constant ρ and non-negative constant K.

In this paper we try to close the gap and prove parallel theorems regarding the stability of the zero solution and the boundedness of all solutions of the finitely delayed functional difference

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equation (1.1). Those parallel theorems that we attempt to prove were proposed as open problems in the book [12].

This paper is organized as follows: In the introduction, we define our functional delay problem and provide relevant stability definitions, along with an overview of previous research conducted in this context. Section 2 focuses on stating and proving a comprehensive theorem concerning the uniform asymptotic stability of the zero solution for equation (1.1). The theorem necessitates the presence of a Lyapunov functional meeting specific conditions, which involve wedges. Similarly, Section 3 addresses another pivotal theorem, emphasizing the uniform boundedness of all solutions to equation (1.1). Once again, this theorem calls for a Lyapunov functional that satisfies certain conditions related to wedges. Finally, in Section 4, we present practical applications in the form of examples derived from our findings.

The next definition and theorem can be found in [13].

Definition 1 ([13])**.** *Let $x(t) = 0$ be a solution of* (1.1)*.*

(a) The zero solution of (1.1) *is stable if for each $\varepsilon > 0$ and $t_1 \geq t_0$ there exists $\delta > 0$ such that $[\phi \in C(t_1),\ \|\phi\| < \delta, t \geq t_1]$ imply that $|x(t, t_1, \phi)| < \varepsilon$.*

(b) The zero solution of (1.1) *is uniformly stable if it is stable and if δ is independent of $t_1 \geq t_0$.*

(c) The zero solution of (1.1) *is asymptotically stable if it is stable and if for each $t_1 \geq t_0$ there is an $\eta > 0$ such that $[\phi \in C(t_1),\ \|\phi\| < \eta]$ imply that $|x(t, t_1, \phi)| \to 0$ as $t \to \infty$. Note that if this is true for every $\eta > 0$, then $x = 0$ is asymptotically stable in the large or globally asymptotically stable.*

(d) The zero solution of (1.1) *is uniformly asymptotically stable (UAS) if it is uniformly stable and if there is an $\eta > 0$ such that for each $\gamma > 0$ there exist a $T > 0$ such that $[t_1 \geq t_0, \phi \in C(t_1),\ \|\phi\| < \eta, t \geq t_1 + T]$ imply that $|x(t, t_1, \phi)| \to 0$ as $t \to \infty$. We note also that if this is true for every $\eta > 0$, then $x = 0$ is uniformly asymptotically stable in the large.*

In [13] this author proved a general theorem of three parts in which stability, uniform stability and asymptotic stability were proven concerning the zero solution of the (1.1). However, the result concerning the uniform asymptotic stability was left open, which we solve, in this paper. The results of [13] are summarized in Theorem 1.1.

Theorem 1.1. *(*[13]*) Let $D > 0$ and there is a scalar functional $V(t, \psi_t)$ that is continuous in ψ and locally Lipschitz in ψ_t when $t \geq t_0$ and $\psi_t \in C(t)$ with $||\psi_t|| < D$. Suppose also that $V(t, 0) = 0$ and*

$$W_1(|\psi(t)|) \leq V(t, \psi_t). \tag{1.5}$$

(a) *If*

$$\triangle V(t, \psi_t) \leq 0 \text{ for } t_0 \leq t < \infty \text{ and } ||\psi_t|| \leq D, \tag{1.6}$$

then the zero solution of (1.1) *is stable.*

(b) *If in addition to (a),*

$$V(t, \psi_t) \leq W_2(||\psi_t||), \tag{1.7}$$

then the zero solution of (1.1) *is uniformly stable.*

(c) *If there is an $M > 0$ with $|F(t, \psi_t)| \leq M$ for $t_0 \leq t < \infty$ and $||\psi_t|| \leq D$, and if*

$$\triangle V(t, \psi_t) \leq -W_2(|\psi(t)|), \tag{1.8}$$

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then the zero solution of (1.1) *is asymptotically stable.*

2. General Theorem on (UAS)

In this section we state and solve the open problem that was posed in [12] and [13] regarding the (UAS) of the zero solution of (1.1).

Theorem 2.1. *(*[12]*) Let $D > 0$ and there is a scalar functional $V(n, \psi_n)$ that is continuous in ψ and locally Lipschitz in ψ_n when $n \geq n_0$ and $\psi_n \in C(n)$ with $||\psi_n|| < D$. In addition we assume if $x : [n_0 - h, \infty) \to \mathbb{R}^d$ is a bounded sequence, then $F(n, x_n)$ is bounded on $[n_0, \infty)$. Suppose there is a function V such that $V(n, 0) = 0$,*

$$W_1(|\psi(n)|) \leq V(n, \psi_n) \leq W_2(||\psi_n||),$$

and

$$\triangle V(n, \psi_n) \leq -W_3(|\psi(n)|),$$

then the zero solution of (1.1) *is (UAS).*

Proof. Find δ of the uniform stability from part (b) of Theorem 1.1 for the given $\varepsilon > 0$ where $\varepsilon = \min[1, \frac{D}{2}]$. For a given $\gamma > 0$, we need to find an integer $T > 0$ such that $[n_1 \geq n_0, \phi \in C(n_1), \|\phi\| < \eta, n \geq n_1 + T]$ imply that $|x(n, n_1, \phi)| < \gamma$. We determine a δ of uniform stability for this same γ, so that $[n_2 \geq n_0, ||\phi_{n_2}|| < \delta, n \geq n_2]$ imply that $|x(n, n_1, \phi)| < \gamma$. In order to distinguish this new δ from the prior δ, denote it by μ. Summing $\triangle V(n, \psi_n) \leq 0$, from $s = n_1$ to $n - 1$ leads to

$$V(n, x_n) \leq V(n_1, \phi_{n_1}) \leq W_2(||\phi_n||) < W_2(\eta). \tag{2.1}$$

For $n \in [n_2, n_3 - 1]$, suppose that $|x(n)| > \frac{\mu}{2}$. Then we have

$$\triangle V(n, x_n) \leq -W_3(\mu/2).$$

Consequently, by summing $\triangle V(n, x_n) \leq -W_3(|x(n)|)$, will end up with

$$\begin{aligned} 0 \leq V(n_3, x_{n_3}) &\leq V(n_2, x_{n_2}) - \sum_{s=n_2}^{n_3-1} W_3(\mu/2) \\ &\leq W_2(\eta) - (n_3 - n_2) W_3(\mu/2). \end{aligned}$$

This implies that

$$n_3 - n_2 < \Big\lfloor \frac{W_2(\eta)}{W_3(\mu/2)} \Big\rfloor. \tag{2.2}$$

Here the notation $\lfloor z \rfloor = \max\{m \in \mathbb{Z}^+ : m \leq z\}$. Moreover, it was previously mentioned that if $|x(n)| < \mu$ holds within the interval $[n_4, n_5 - 1]$, satisfying $n_5 - n_4 \geq h$, then it follows that

$$|x(n)| < \gamma \quad \text{for } \; n \geq n_5 - 1. \tag{2.3}$$

A final and crucial fact to know is that when $|x(n_6)| \leq \frac{\mu}{2}$ and $|x(n_7)| \geq \mu$ with $n_6 < n_7$, and given that $f(n, x_n)$ remains bounded in n, there exists a positive constant S such that

$$n_7 - n_6 > S. \tag{2.4}$$

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Thus, by summing $\triangle V(n, x_n) \leq -W_3(|x(n)|)$, on the interval $[n_6, n_7 - 1]$ we get

$$\begin{aligned} 0 \leq V(n_7, x_{n_7}) - V(n_6, x_{n_6}) &\leq -\sum_{s=n_6}^{n_7-1} W_3(|x|) \\ &\leq -(n_7 - n_6)W_3(\mu/2) \\ &\leq -SW_3(\mu/2), \end{aligned}$$

since W_3 is continuous and increasing. Thus, $V(n, x_n)$ decreases by the value $TW_3(\mu/2)$ on the interval $[n_6, n_7 - 1]$. Consequently, we are now able to find an integer N with

$$NSW_3(\mu/2) > W_2(\eta). \tag{2.5}$$

The information provided in equation (2.2) indicates the existence of an integer e_i within each interval of size $\left\lceil \frac{W_2(\eta)}{W_3(\mu/2)} \right\rceil$, where $|x(e_i)| \leq \frac{\mu}{2}$. Here the notation $\lceil z \rceil = \max\{m \in \mathbb{Z}^+ : m \geq z\}$. The assertion (2.3) implies the existence of an integer point E_i within each interval of length r for every n, such that $|x(E_i)| \geq \mu$. Otherwise, the magnitude of $|x(n)|$ will stay below γ. On the other hand, as a consequence of statement (2.4) we see that T time units pass between e_i and E_i. Let

$$K = r + \left\lfloor \frac{W_2(\eta)}{W_3(\mu/2)} \right\rfloor.$$

Then on each interval of length K we have that $V(n, x_n)$ decreases $SW_3(\mu/2)$ units. Consequently, the values $T = NK$ suffices and as a results we have

$$|x(n, n_1, \phi)| < \gamma \text{ for } n > n_1 + T.$$

This completes the proof. □

Fore more reading on the notion of stability by different approaches we refer to [7]-[13], [17], [18], [19], and [21].

3. General Theorem on Uniform Boundedness

Now we shift our focus to the study of boundedness of solutions of system (1.1). When Lyapunov functionals are used to study the behavior of solutions of functional difference equations with finite delays of the form of (1.1), we are likely to encounter a pair of inequalities of the form

$$V(n, x(\cdot)) = W_1(x(n)) + \sum_{s=n-r}^{n-1} C(n, s)W_2(x(s)), \tag{3.1}$$

$$\triangle V(n, x(\cdot)) \leq -W_3(x(n)) + F(n) \tag{3.2}$$

where V is a Lyapunov functional bounded below, x is the known solution of the functional difference equation, and K, F, and $W_i, i = 1, 2, 3$, are scalar positive functions.
Inequalities (3.1) and (3.2) are full of information that is not visible to the naked eye. Our job now is to prove a general theorem and try to extract boundedness of the solutions. The next theorem was proposed as an open problem in [12] so that equations of the form

$$x(n+1) = a(n)x(n) + \sum_{s=n-r}^{n-1} C(n, s)g(x(s)) + p(n) \tag{3.3}$$

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can be handled, where the function g is continuous. Before we state and prove the open problem, we state a definition regarding uniform boundedness of solutions of (1.1). For more on inequalities (3.1) and (3.2) we refer to [15].

Definition 2. *Solutions of* (1.1) *are uniformly bounded (UB) if for each* $B_1 > 0$ *there is* $B_2 > 0$ *such that* $\big[n_0 \geq 0, \phi \in C, ||\phi|| < B_1,\ n \geq n_0\big]$ *implies* $|x(n, n_0, \phi)| < B_2$.

Theorem 3.1. *Suppose there is a scalar and differentiable functional* $V(n, x_n)$ *that is defined for* $n \in \mathbb{Z}$. *Assume the delay in* (1.1) *is* r *instead of* h. *Let* $\phi : [n_0 - r, \infty) \to \mathbb{R}^d$. *Suppose every solution* $\phi(n)$ *of* (1.1) *satisfies*

$$W_4(|\phi(n)|) \leq V(n, \phi_n) \leq W_1(|\phi(n)|) + W_2\Big(\sum_{s=n-r}^{n-1} W_3(|\phi(s)|)\Big) \tag{3.4}$$

and

$$\triangle V(n, \phi_n) \leq -W_3(|\phi(n)|) + M \tag{3.5}$$

for some positive constant M. *Then solutions of* (1.1) *are (UB).*

Proof. For $n_1 \geq n_0$, and $\phi \in C(n_1)$, we let $||\phi|| \leq B_1$, for positive constant B_1. Let $x(n) = x(n, n_1, \phi)$. A summation of the inequality in (3.5) from $s = n - r$ to $s = n - 1$ with $n - 1 \geq n_1 - r$ gives

$$V(n, x_n) - V(n - r, x_{n-r}) \leq -\sum_{s=n-r}^{n-1} W_3(|x(s)|) + Mr.$$

This gives us the relation

$$\sum_{s=n-r}^{n-1} W_3(|x(s)|) \leq V(n - r, x_{n-r}) - V(n, x_n) + Mr. \tag{3.6}$$

Set $V(s) = V(s, x_s)$ on an arbitrary interval $[n_1, L]$ for any $L > n_1 + r$. Since V is continuous in x, it has a maximum. Hence, let $V(n^*) = \max_{n_1 \leq n^* \leq L} V(n)$. Suppose $n^* \leq n_1 + r$. Then by summing (3.5) from n_1 to $n^* - 1$ followed by the use of (3.4) gives

$$\begin{aligned} V(n) &\leq V(n^*) \leq V(n_1) - \sum_{s=n_1}^{n^*-1} W_3(|x(s)|) + (n^* - n_1)M \\ &\leq V(n_1) + (n^* - n_1)M \\ &\leq W_1(B_1) + W_2(rW_3(B_1)) + Mr. \end{aligned}$$

From the left side of (3.4) we have that $W_4(|x(n)|) \leq V(n)$, and hence the above inequality gives

$$|x(n)| \leq W_4^{-1}\big[W_1(B_1) + W_2(rW_3(B_1)) + Mr\big].$$

On the other hand if $n^* \in [n_1 + r, L]$, then $V(n^* - r, x_{n^*-r}) - V(n^*, x_n^*) \leq 0$, and hence from (3.6) we have that

$$\sum_{s=n^*-r}^{n^*-1} W_3(|x(s)|) \leq Mr.$$

We observe that for such n^*, $\triangle V(n^*) \geq 0$, and hence from (3.5) we have that $0 \leq -W_3(|\phi(n)|) + M$. This gives

$$|x(n^*)| \leq W_3^{-1}(M).$$

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Thus, for $n \in [n_1, L]$ we have from (3.4) that

$$W_4(|x(n)|) \le V(n) \le V(n^*) \le W_1\big(W_3^{-1}(M)\big) + W_2(Mr).$$

This yields the bound

$$|x(n)| \le W_4^{-1}\big[W_1\big(W_3^{-1}(M)\big) + W_2(Mr)\big].$$

The proof is concluded since L is arbitrary and by taking

$$B_2 = \max\Big\{W_4^{-1}\big[W_1(B_1) + W_2(rW_3(B_1)) + Mr\big],\ W_4^{-1}\big[W_1\big(W_3^{-1}(M)\big) + W_2(Mr)\big]\Big\}.$$

□

Fore more reading on the notion of boundedness we refer to [1], [4], [7], [15], and [20].

4. Applications

This section is devoted to applications of Theorems 2.1 and 3.1. Our applications will be presented in the forms of examples. We begin with the following example.

The next example is concerned with uniform boundedness of solutions of (3.3).

Example 4.1. *We consider the scalar nonlinear finitely delayed difference equation*

$$x(n+1) = b(n)h(x(n)) + a(n)g(x(n-r)) + c(n), \tag{4.1}$$

where $a, b, c : \mathbb{Z}^+ \to \mathbb{R}$, r is a positive integer. The functions g and h are considered to be continuous in x. Suppose there are three positive constants ζ_1, ζ_2, and ζ_3 such that $|h(x)| \le \zeta_1|x|, |g(x)| \le \zeta_2|x|$, and $|c(n)| \le \zeta_3$. Additionally, we assume that

$$\lim_{n\to\infty} \zeta_1|b(n)| \ne 1, \tag{4.2}$$

$$\zeta_1|b(n)| + \zeta_2|a(n+r)| - 1 \le -\zeta_2|a(n+r)|, \tag{4.3}$$

and

$$\sum_{n=0}^{\infty} |a(n)| < \infty. \tag{4.4}$$

Then all solutions of (4.1) are uniformly bounded.

Proof. We consider the Lyapunov functional $V(n) := V(n, x(n))$,

$$V(n) = |x(n)| + \sum_{s=n-r}^{n-1} |a(s+r)||g(x(s))|.$$

Then along solutions of (4.1) we have

$$\begin{aligned}
\triangle V(n) &= |x(n+1)| - |x(n)| + |a(n+r)||g(x(n))| - |a(n)||g(x(n-r))| \\
&\le |b(n)||h(x(n))| + |a(n)||g(x(n-r))| + \zeta_3 - |x(n)| \\
&+ |a(n+r)||g(x(n))| - |a(n)||g(x(n-r))| \\
&= \Big(\zeta_1|b(n)| + \zeta_2|a(n+r)| - 1\Big)|x(n)| + \zeta_3.
\end{aligned}$$

Now we make sure the requirements of Theorem 3.1 are met. We may take $W(|x(n)|) = W_1(|x(n)|) = |x(n)|$. From the definition of $V(n)$ and due to condition (4.4), we have that

$$V(n) = |x(n)| + \sum_{s=n-r}^{n-1} |a(s+r)||g(x(s))| \le |x(n)| + \zeta_2 \sum_{s=n-r}^{n-1} |a(s+r)||x(s)|.$$

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Thus, we take $W_3(|x(s)|) = \zeta_2|a(s+r)||x(s)|$. Consequently,

$$\begin{aligned} \triangle V(n) &\leq \Big(\zeta_1|b(n)| + \zeta_2|a(n+r)| - 1\Big)|x(n)| + \zeta_3 \\ &\leq -\zeta_2|a(n+r)||x(n)| + \zeta_3 \\ &= -W_3(|x(n)|) + \zeta_3. \end{aligned}$$

Thus all the requirements of Theorem 3.1 are satisfied and all solutions of (4.1) are uniformly bounded.

□

For example the nonlinear delay equation

$$x(n+1) = \frac{1}{2}\frac{n}{n+1}x(n) + \frac{1}{6}\frac{1}{n^2+1}\frac{x(n-r)}{x^2(n)+1} + \sin(n), \quad n \geq 0$$

satisfies conditions of Theorems 2.1 and 3.1 with

$$\zeta_1 = \zeta_2 = \zeta_3 = 1,\ |b(n)| \leq \frac{1}{2},\ |a(n)| \leq \frac{1}{6},\text{ and } \sum_{n=0}^{\infty}|a(n)| < \infty.$$

Remark 1. *In Example 4.1 the boundedness of solutions did not depend on the size of the delay.*

In the next example we use a Lyapunov functional, and show that all solution of equations of the form of (3.3) are (UB).

Example 4.2. *Assume $D(n,s) \neq 0$ for all $-r \leq s \leq n$, and there is a positive constant M such that $|p(n)| \leq M$ for all $n = 0, 1, 2, \cdots$ Then solutions of the scalar finitely delayed Volterra difference equation*

$$x(n+1) = a(n)x(n) + \sum_{s=n-r}^{n-1} D(n,s)x(s) + p(n), \tag{4.5}$$

are (UB) provided that

$$\lim_{n\to\infty}(-1 + |a(n)|) \neq 0, \tag{4.6}$$

$$-1 + |a(n)| + \sum_{u=n+1}^{\infty}|D(u,n)| \leq -\sum_{u=n}^{\infty}|D(u,n)|, \tag{4.7}$$

and

$$\sum_{s=n-r}^{n-1}\sum_{u=n}^{\infty}|D(u,s)| \leq L, \text{ for a positive constant } L. \tag{4.8}$$

Proof. Consider the Lyapunov functional

$$V(n, x_n) = |x(n)| + \sum_{s=n-r}^{n-1}\sum_{u=n}^{\infty}|D(u,s)||x(s)|. \tag{4.9}$$

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Then along the solutions of (4.5) we have

$$\begin{aligned}
\triangle V(n,x_n) &\leq \Big(|a(n)|-1\Big)|x(n)| + \sum_{s=n-r}^{n-1}|D(n,s)||x(s)| + M \\
&+ \sum_{s=n-r+1}^{n}\sum_{u=n+1}^{\infty}|D(u,s)||x(s)| - \sum_{s=n-r}^{n-1}\sum_{u=n}^{\infty}|D(u,s)||x(s)| \\
&\leq \Big(|a(n)| + \sum_{u=n+1}^{\infty}|D(u,n)|-1\Big)|x(n)| + \sum_{s=n-r}^{n-1}|D(u,s)||x(s)| + M \\
&+ \sum_{s=n-r+1}^{n-1}\sum_{u=n+1}^{\infty}|D(u,s)||x(s)| - \sum_{s=n-r}^{n-1}\sum_{u=n}^{\infty}|D(u,s)||x(s)|. \qquad (4.10)
\end{aligned}$$

By noting that

$$\begin{aligned}
\sum_{s=n-r}^{n-1}\sum_{u=n}^{\infty}|D(u,s)||x(s)| &= \sum_{s=n-r}^{n-1}\Big[|D(n,s)||x(s)| + \sum_{u=n+1}^{\infty}|D(u,s)||x(s)|\Big] \\
&= \sum_{s=n-r}^{n-1}|D(n,s)||x(s)| + \sum_{s=n-r}^{n-1}\sum_{u=n+1}^{\infty}|D(u,s)||x(s)| \\
&= \sum_{s=n-r}^{n-1}|D(n,s)||x(s)| + \sum_{u=n+1}^{\infty}|D(u,n-r)||x(n-r)|\Big] \\
&+ \sum_{s=n-r+1}^{n-1}\sum_{u=n+1}^{\infty}|D(u,s)||x(s)|.
\end{aligned}$$

Substituting back into (4.10), we arrive at the inequality

$$\begin{aligned}
\triangle V(n,x_n) &\leq \Big(|a(n)| + \sum_{u=n+1}^{\infty}|D(u,n)|-1\Big)|x(n)| - \sum_{u=n+1}^{\infty}|D(u,n-r)||x(n-r)| + M \\
&\leq \Big(|a(n)| + \sum_{u=n+1}^{\infty}|D(u,n)|-1\Big)|x(n)| + M
\end{aligned}$$

It is clear from the definition of $V(n)$ that $W_4(|x|) = W_1(|x|) = |x|$. As a consequence of (4.8), we take $W_3(|x(s)|) = \sum_{u=n}^{\infty}|D(u,s)||x(s)|$. Then from $\triangle V$ and (4.7) we have

$$\begin{aligned}
\triangle V(n,x_n) &\leq \Big(|a(n)| + \sum_{u=n+1}^{\infty}|D(u,n)|-1\Big)|x(n)| + M \\
&< -\sum_{u=n}^{\infty}|D(u,n)||x(n)| + M \\
&= -W_3(|x(n)|) + M.
\end{aligned}$$

Since all conditions of Theorem 3.1 are met, we conclude that all solutions of (4.5) are (UB). □

We end this study with an example showing the zero solution of a totally nonlinear difference equations is (UAS).

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Example 4.3. *Consider the higher order highly nonlinear and finitely delayed difference equation*

$$x(n+1) = b(n)x^3(n-r) + c(n)x^3(n), \tag{4.11}$$

where $b, c : \mathbb{Z}^+ \to \mathbb{R}$, *and* r *is a positive integer. Let*

$$Q = \{\psi \in C(n) : ||\psi_n|| = \max_{n-r \le s \le n} |\psi(n)| < 1\}.$$

Assume

$$\lim_{n\to\infty} (-1 + c^2(n)) \neq 0. \tag{4.12}$$

If there is an $\alpha \in (0,1)$ *such that*

$$c^2(n) + \beta - 1 \le -\alpha, \tag{4.13}$$

and for positive constant γ,

$$\alpha - \frac{b^2(n)c^2(n)}{\beta - b^2(n)} > \gamma, \tag{4.14}$$

with $\beta > b^2(n)$, *and* β *is to be defined shortly. Additionally, if*

$$\lim_{n\to\infty} \big(\alpha - \frac{b^2(n)c^2(n)}{\beta - b^2(n)}\big) \neq 0, \tag{4.15}$$

then the zero solution of (4.11) *is asymptotically stable.*

Proof. Let $x(n)$ be a solution of (4.11) with $x \in Q$ and consider the Lyapunov functional $V(n) := V(n, x(n))$,

$$V(n) = x^2(n) + \beta \sum_{s=n-r}^{n-1} x^6(s).$$

Then along solutions of (4.11) we have

$$\begin{aligned} \triangle V(n) &= x^2(n+1) - x^2(n) + \beta x^6(n) - \beta x^6(n-r) \\ &= b^2(n)x^6(n-r) + c^2(n)x^6(n) + 2b(n)c(n)x^3(n)x^3(n-r) \\ &+ \beta x^6(n) - \beta x^6(n-r) - x^2(n). \end{aligned}$$

Since $x(n) \in Q$, we have that $x^2(n) > x^6(n)$, and hence

$$\begin{aligned} \triangle V(n) &\le \Big[c^2(n) + \beta(n) - 1\Big]x^6(n) + \Big(b^2(n) - \beta\Big)x^6(n-r) \\ &+ 2b(n)c(n)x^3(n)x^3(n-r) \\ &\le -\alpha x^6(n) + \Big(b^2(n) - \beta\Big)x^6(n-r) \\ &+ 2b(n)c(n)x^3(n)x^3(n-r) \\ &= -\Big[\alpha - \frac{b^2(n)c^2(n)}{\beta - b^2(n)}\Big]x^6(n) \\ &- \Big[\frac{b(n)c(n)}{\sqrt{\beta - b^2(n)}}x^3(n) - \sqrt{\beta - b^2(n)}x^3(n-r)\Big]^2 \\ &\le -\gamma x^6(n). \end{aligned}$$

Next we verify all conditions of Theorem 2.1 are met. Let $W_1(|x(n)|) = x^2(n)$ and $W_3(|x(n)|) = \gamma x^6(n)$. Since $x(n) \in Q$, we obtain from $V(n)$ that

$$\begin{aligned} x^2(n) + \beta \sum_{s=n-r}^{n-1} x^6(s) &\leq |x(n)| + \beta \sum_{s=n-r}^{n-1} |x(s)| \\ &\leq \Big(1 + r\beta\Big)||x_n||. \end{aligned}$$

Thus, $W_2(||x_n||) = \Big(1 + r\beta\Big)||x_n||$, and the zero solution of (4.11) is uniformly asymptotically stable, by Theorem 2.1. For example, if we let

$$c^2 = \frac{1}{10}, b^2 = \frac{2}{10}, \beta = \frac{1}{4},$$

then all conditions of Example 4.3 are satisfied with $\alpha = \frac{13}{20}$. □

We end this paper by comparing our results with those of [8]. In [8], the authors prove the discrete analogue of continuous Halanay inequality and apply it to derive sufficient conditions for the global asymptotic stability of the equilibrium of certain generalized difference equations. However, their results regarding stability will not work for equations like (4.5) when $p(n) = 0$, for all $n \in \mathbb{Z}^+$. This is due to the fact that our kernel, $D(n, s)$, is not constant. Additionally, the right side of (6) of [8] must have the linear term $ax(n)$ for constant a in order to invert and conclude the results. Of course, our theorems do not ask for such a requirement.

References

[1] J, Cermák, *Difference equations in the qualitative theory of delay differential equations*, Proceedings of the Sixth International Conference on Difference Equations, 391–398, CRC, Boca Raton, FL, 2004.
[2] S. Elaydi, *An Introduction to Difference Equations*, Springer, New York, 1999.
[3] R. Bellman, K. L. Cooke and J. J. Hirsch Jr., *Differential-Difference Equations,* Academic Press, 1963.
[4] L. Jia, Z. Wang and L. Huang, *Stability and boundedness of delay difference equations with nonnegative coefficients,* Applied Mathematics and Computation, (2008) 205(1), 48-56.
[5] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics,* (1992), Kluwer Academic Publishers.
[6] W. Kelley and A. Peterson, *Difference equations an introduction with applications*, Academic Press, 2001.
[7] M. Islam and E. Yankson, *Boundedness and stability in nonlinear delay difference equations employing fixed point theory, Electron. J. Qual. Theory Differ. Equ.* 26 (2005).
[8] E. Liz and J. B. Ferreiro, *A note on the global stability of generalized difference equations*, Applied Mathematics Letters 15 (2002) 655-659.
[9] Y. N. Raffoul, *Stability and periodicity in discrete delay equations, J. Math. Anal. Appl.* 324 (2006) 1356-1362.
[10] Y. N. Raffoul, *Stability in neutral nonlinear differential equations with functional delays using fixed point theory,* Mathematical and Computer Modelling, 40(2004), 691-700.
[11] Y. N. Raffoul, *Stability in functional difference equations using fixed point theory,* Communications of the Korean Mathematical Society 29 (1), (2014), 195-204.
[12] Y. N. Raffoul, *Qualitative Theory of Volterra Difference Equations,* Springer Nature Switzerland AG, (2018).
[13] Y. N. Raffoul, *Lyapunov Functionals and Stability in Finite Delays Difference Equations,* Recent Progress in Difference Equations, Discrete Dynamical Systems and Applications, pp: 369-382 Springer (2019),Book Series.
[14] Y. N. Raffoul,*Periodicity in general delay non-linear difference equations using fixed point theory*, J. Difference Equ. Appl. 10 (2004), no. 13-15, 1229–1242.
[15] Y. N. Raffoul, *General theorems for stability and boundedness for nonlinear functional discrete systems*, J. Math. Analy. Appl., 279 (2003), pp. 639-650.
[16] W-T. Li and H-F. Huo, *Positive periodic solutions of delay difference equations and applications in population dynamics*, J. Comput. Appl. Math. 176 (2005), no. 2, 357–369.
[17] J. Migda, *Asymptotic behavior of solutions of nonlinear difference equations*, Math. Bohem. 129 (2004), no. 4, 349–359.

[18] C. Qian and Y. Sun, *On global attractivity of nonlinear delay difference equations with a forcing term*, J. Difference Equ. Appl. 11 (2005), no. 3, 227–243.
[19] D. C. Zhang and B. Shi, *Shi, B. Global behavior of solutions of a nonlinear difference equation*, Appl. Math. Comput. 159 (2004), no. 1, 29–35.
[20] G. Zhang and L. Zhang, *Periodicity and attractivity of a nonlinear higher order difference equation*, Appl. Math. Comput. 161 (2005), no. 2, 395–401.
[21] H. Zhu and L. Huang, *Asymptotic behavior of solutions for a class of delay difference equation.*, Ann. Differential Equations 21 (2005), no. 1, 99–105.

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