BULL. AUSTRAL. MATH. SOC. 46B20
VOL. 42 (1990) [95–100]

THE NORMAL STRUCTURE OF JAMES QUASI REFLEXIVE SPACE

DARYL TINGLEY

It is shown that weakly compact sets of James quasi reflexive space have normal structure.

1. NORMAL STRUCTURE AND FIXED POINTS

Let K be a nonempty bounded convex subset of a Banach space X. The diameter of K, denoted $\operatorname{diam}(K)$, is defined by $\operatorname{diam}(K) = \sup\{\|x-y\| : x, y \in K\}$. (Here K need not be convex.) The set K is said to have normal structure if for every bounded convex set $H \subset K$ with $\operatorname{diam}(H) > 0$, there is $x_0 \in H$ such that $\operatorname{diam}(H) > \sup\{\|x - x_0\| : x \in H\}$. Normal structure was introduced by Brodskii and Milman [2] and has since been used extensively (see [11] for a survey) in the study of fixed point properties of nonexpansive maps. (A map $T\colon K \to K$ is said to be nonexpansive if $\|T_x - T_y\| \leqslant \|x - y\|$ for all $x, y \in K$.) Indeed, Kirk [5] has shown that if K is a weakly compact convex set that has normal structure, then every non-expansive map $T\colon K \to K$ has a fixed point. (However, Karlovitz [4] has shown that normal structure is not a necessary condition.)

The set K is said to have the fixed point property (f.p.p.) if every nonexpansive map $T\colon K \to K$ has a fixed point. If every weakly compact set $K \subset X$ has the f.p.p., then X is said to have the f.p.p. Thus X has the f.p.p. if weakly compact sets have normal structure, in which case we say that X has weak normal structure.

2. JAMES QUASI-REFLEXIVE SPACE

James space, J, consists of all the real sequences (α_n) for which $\lim \alpha_n = 0$ and $\|(\alpha_n)\|_J < \infty$ where

$$(1) \qquad \|(\alpha_n)\|_J = \sup\{[(\alpha_{p_1} - \alpha_{p_2})^2 + \ldots + (\alpha_{p_{n-1}} - \alpha_{p_n})^2 + (\alpha_{p_n} - \alpha_{p_1})^2]^{1/2}\},$$

the supremum being taken over all finite increasing sequences of positive numbers $\{p_1, p_2, \ldots, p_n\}$. It is easily shown (see [9], Vol. I, p.25) that J is a Banach space.

Received 12 September 1989

https://doi.org/10.1017/S0004972700028185 Published online by Cambridge University Press

The space J has been used to disprove several conjectures about Banach spaces [9] but perhaps its most striking property is that for which James [3] originally introduced it: J is not reflexive, the canonical image of J having codimension 1 in J^{**}, but J is nevertheless isometrically isomorphic to J^{**}.

If the term $(\alpha_{p_n} - \alpha_{p_1})^2$ is dropped from (1), another Banach space, sometimes called J_1, is obtained. J_1 and J are isomorphic, however, the norm $\|\cdot\|_J$ is needed to show that J is isometric and not just isomorphic to J^{**}. In [6] Khamsi showed that weakly compact sets in J_1 have normal structure and hence J_1 satisfies the f.p.p. He mentions there that it is unclear whether or not J has weak normal structure. In [7], Khamsi shows that J also has the f.p.p. The method used does not depend on normal structure and indeed was developed to show certain spaces that lack normal structure have the f.p.p. [**1**, **8**, **10**].

In this paper we show in a direct fashion that J has weak normal structure. Combining this with the theorem of Kirk, this gives another proof that J has the f.p.p.

3.

We say that a Banach space satisfies property (*) if every sequence (x_n) that converges weakly to 0 satisfies

$$(*) \quad \sup_m\{\limsup_n \|x_m - x_n\|\} > \liminf_n \|x_n\|.$$

It is easily seen that (*) is equivalent to

$$(*') \quad \sup\{\limsup_n \|x - x_n\| \ :\ x \in \operatorname{co}(x_1, x_2, \ldots)\} > \liminf_n \|x_n\|,$$ where $\operatorname{co}(x_1, x_2, \ldots)$ is the convex hull of the set $\{x_n\}$.

Theorem 1. *If a Banach space X satisfies property (*), then X has weak normal structure.*

Proof: Suppose that $K \subset X$ is weakly compact and does not have normal structure. Then a result of Brodskii and Milman [2] shows that there is a sequence $\{x_n\} \subset K$ such that

$$(2) \qquad \lim d(x_{n+1}, \operatorname{co}(x_1, \ldots, x_n)) = \operatorname{diam}\{x_n\},$$

where $\operatorname{diam}\{x_n\} = \sup\{\|x_n - x_m\| : n, m \geqslant 1\}$ and

$$d(x_{n+1}, \operatorname{co}(x_1, \ldots, x_n)) = \inf\{\|x_{n+1} - x\| : x \in \operatorname{co}(x_1, \ldots, x_n)\}.$$

Since any subsequence of $\{x_n\}$ also satisfies (2), and since both (2) and normal structure are invariant under translations, we may suppose that $\{x_n\}$ converges weakly to $0 \in K$. From (2) it follows that $\lim \|x - x_n\| = \operatorname{diam}\{x_n\}$ for all $x \in \operatorname{co}(x_1, x_2, \ldots)$ and hence it is easily seen that $\{x_n\}$ does not satisfy (*).

Although I have no counterexample, it would seem unlikely that (*) is equivalent to weak normal structure for a Banach space X.

The following provides a partial converse of Theorem 1 and indicates why counterexamples are not so easy to find. □

Theorem 2. *Suppose that a Banach space X has weak normal structure and satisfies the so called Opial condition (see* [11], *p.213): For every sequence $\{x_n\}$ that converges weakly to 0, and for every $x \in X$,*

$$\liminf \|x - x_n\| \geqslant \liminf \|x_n\|. \tag{3}$$

Then X satisfies (*).

Remark. Any Banach space that has an unconditional basis with unconditional basis constant 1 (see [9]) satisfies the Opial condition.

Proof: Suppose not. Then (from (*')) there is a sequence $\{x_n\}$ that converges weakly to 0 such that

$$\sup\{\limsup_n \|x - x_n\| : x \in \operatorname{co}(x_1, x_2, \ldots)\} \leqslant \liminf_n \|x_n\|. \tag{4}$$

Combining (3) and (4) it follows that

$$\lim_n \|x - x_n\| = \liminf_n \|x_n\| \text{ for every } x \in \operatorname{co}(x_1, x_2, \ldots).$$

Thus X does not have weak asymptotic normal structure (see [11], pp.209, 210). Since weak asymptotic normal structure and weak normal structure are equivalent [11], this contradicts the hypothesis that X has weak normal structure. □

Theorem 3. *The Banach space J satisfies condition (*) and hence has weak normal structure and the fixed point property.*

Proof: Let $\{x_n\} \subset J$ be any sequence that converges weakly to 0. Notice that if any subsequence of $\{x_n\}$ satisfies (*), then $\{x_n\}$ satisfies (*).

Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ where the 1 is in the nth position. Then $\{e_n\}$ is a basis for J ([9], p.25) and we may write $x_n = \sum_{i=1}^{\infty} \alpha_i^n e_i$ for each n. Since $\{x_n\}$ converges weakly to 0, $\alpha_i^n \underset{n}{\to} 0$ for each i. Thus, possibly by extracting a subsequence of $\{x_n\}$, we may choose a sequence of integers $\{b_n\}$ such that if $u_n = \sum_{i=n+1}^{b_n} \alpha_i^n e_i$ then

$$\|x_n - u_n\| \to 0. \tag{5}$$

https://doi.org/10.1017/S0004972700028185 Published online by Cambridge University Press

Write $u_n = \sum_{i=1}^{\infty} \beta_i^n e_i$. Obviously $\beta_i^n = \alpha_i^n$ for $n+1 \leqslant i \leqslant b_n$ and $\beta_i^n = 0$ otherwise. (In particular $\beta_n^n = 0$.) For each n, only finitely many β_i^n are non-zero so the supremum in (1) is obtained for some finite sequence $p_1, \ldots, p_k$. That is

$$\|u_n\|^2 = \sum_{i=1}^{k} \left(\beta_{p_i}^n - \beta_{p_{i+1}}^n\right)^2 + \left(\beta_{p_k}^n - \beta_{p_1}^n\right)^2. \tag{6}$$

The set $\{p_i\}_{i=1}^k$ depends on n. It is understood that when working with u_n the corresponding $\{p_i\}_{i=1}^k$ is being used. The set $\{p_i\}$ may be chosen so that none of the terms in (6) is 0 and

$$n \leqslant p_1 \leqslant \ldots \leqslant p_k \leqslant b_n. \tag{7}$$

If both $\beta_{p_1}^n$ and $\beta_{p_k}^n$ are non-zero they must have different signs, for otherwise $p_1 \neq n$ and the indices $n, p_1, \ldots, p_k$ show that $\|u_n\|$ is greater than that given by (6). By one final extraction of a subsequence of $\{x_n\}$ (and the corresponding subsequences of $\{u_n\}$ and $\{b_n\}$) we suppose that either

$$\beta_{p_1}^n \leqslant 0 \leqslant \beta_{p_k}^n \tag{8}$$

or $\beta_{p_k}^n \leqslant 0 \leqslant \beta_{p_1}^n$ for all n. As the proofs are similar, it is assumed that (8) holds.

We have now constructed sequences $\{x_n\}$ and $\{u_n\}$ so that (5) – (8) hold.

Let

$$M = \sup\{\alpha_i^n : i \geqslant 1, n \geqslant 1\}, \quad m = \inf\{\alpha_i^n : i \geqslant 1, n \geqslant 1\}. \tag{9}$$

Notice that $m \leqslant \beta_i^n \leqslant M$ for each i and n, $m \leqslant 0 \leqslant M$, and at least one of m and M is non-zero. We have two cases to consider.

$$M \geqslant -m \text{ and } M > 0 \tag{10}$$

or $-m \geqslant M$ and $-m > 0$. ▯

CASE I. $M \geqslant -m$ and $M > 0$.

Let ε, $0 < \varepsilon < M$ be given, and let q, t satisfy

$$\alpha_t^q > M - \varepsilon. \tag{11}$$

Let $r > t$ be arbitrary and let $s = r - 1$. Then

$$\|x_q - x_r\| \geqslant \|P_s x_q - u_r\| - \|(I - P_s)x_q\| - \|u_r - x_r\|. \tag{12}$$

https://doi.org/10.1017/S0004972700028185 Published online by Cambridge University Press

(Here $P_s x_q = \alpha_1^q e_1 + \ldots + \alpha_s^q e_s$).

The indices $t, r, p_1, \ldots, p_k$ show that (recall $\beta_r^r = 0$)

$$\begin{aligned}\|P_s x_q - u_r\|^2 &\geqslant (\alpha_t^q)^2 + (\beta_{p_1}^r)^2 + \|u_r\|^2 - \left(\beta_{p_k}^r - \beta_{p_1}^r\right)^2 + \left(\beta_{p_k}^r + \alpha_t^q\right)^2 \\ &= \|u_r\|^2 + 2\beta_{p_k}^r(\beta_{p_1}^r + \alpha_t^q) + 2(\alpha_t^q)^2.\end{aligned}$$

From (8), (9), (10) and (11)

$$\|P_s x_q - u_r\|^2 \geqslant \|u_r\|^2 - 2M\varepsilon + 2(M-\varepsilon)^2.$$

Thus (12) becomes

$$\|x_q - x_r\| \geqslant \sqrt{\|u_r\|^2 - 2M\varepsilon + 2(M-\varepsilon)^2} - \|(I - P_s)x_q\| - \|u_r - x_r\|.$$

Since r can be arbitrarily large and $s = r - 1$ taking limits in r gives

$$\limsup_r \|x_q - x_r\| \geqslant \sqrt{\limsup_r \|u_r\|^2 - 2M\varepsilon + 2(M-\varepsilon)^2}.$$

Since $\varepsilon > 0$ was arbitrary,

$$\sup_q\{\limsup_r \|x_q - x_r\|\} \geqslant \sqrt{\limsup_r \|u_r\|^2 + 2M^2} = \sqrt{\limsup_r \|x_r\|^2 + 2M^2}$$

and since $M > 0$, $\{x_n\}$ satisfies (*).

CASE II. $M < -m$ and $m < 0$.

Let ε be given with $0 < \varepsilon < -m$, and choose q, t so that $\alpha_t^q < m + \varepsilon$. Let r be such that $t < r$ and let $s = r - 1$. Then (12) is satisfied and in a manner similar to Case I, the indices t, $p_1, \ldots, p_k$, $b_r + 1$ show that

$$\|P_s x_q - u_r\|^2 \geqslant \|u_r\|^2 + 2(m+\varepsilon)^2 + 2m\varepsilon.$$

Substituting this into (12) it follows that

$$\sup_q\{\limsup_r \|x_q - x_r\|\} \geqslant \sqrt{\limsup_r \|x_r\|^2 + 2m^2}$$

and hence $\{x_n\}$ satisfies (*).

Since $\{x_n\} \subset J$ was an arbitrary sequence converging weakly to 0, we have shown that J satisfies property (*).

References

[1] J.M. Borwein and B. Sims, 'Nonexpansive mappings on Banach lattices and related topics', *Houston J. Math.* **10** (1984), 339–355.

[2] M.S. Brodskii and D.P. Milman, 'On the center of a convex set', *Dokl. Akad. Nauk SSSR* **59** (1948), 837–840.

[3] R.C. James, 'A non-reflexive Banach space isometric with its second conjugate space', *Proc. Nat. Acad. Sci. U.S.A.* **37** (1951), 134–177.

[4] L.A. Karlovitz, 'Existence of fixed points for nonexpansive mappings in a space without normal structure', *Pacific J. Math.* **66** (1976), 153–159.

[5] W.A. Kirk, 'A fixed point theorem for mappings, which do not increase distances', *Amer. Math. Monthly* **72** (1965), 1004–1006.

[6] M.A. Khamsi, 'Normal structure for Banach spaces with Schauder decomposition', *Canadian Math. Bull.* **32** (1989), ??–??.

[7] M.A. Khamsi, 'James quasi reflexive space has the fixed point property', *Bull. Austral. Math. Soc.* **39** (1989), 25–30.

[8] P.K. Lin, 'Unconditional bases and fixed points of nonexpansive mappings', *Pacific J. Math.* **116** (1985), 69–76.

[9] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Vol. I and II (Springer-Verlag, Berlin, Heidelberg, New York, 1977 and 1979).

[10] B. Maurey, *Points fixes des contractions sur un convexe ferme de L_1 : Seminaire d'analyse fonctionelle* (Ecole Polytechnique, Palaiseau, Exposé No. VIII, 1980/81).

[11] S. Swaminathan, 'Normal structure in Banach spaces and its generalization', *Contemp. Math.* **18**, 201–215. (A.M.S., Providence, R.I.).

Department of Mathematics and Statistics
University of New Brunswick
Fredericton, N. B.
Canada E3B 5A3