ON THE ADDITION OF RELATIONS IN AN ABELIAN CATEGORY

P. J. HILTON AND Y.-C. WU

1. Introduction. In [**1**] the category $\mathscr{A}^\sim$ of relations in an abelian category $\mathscr{A}$ was studied. A relation from A to B appeared as an equivalence class of pairs of $\mathscr{A}$-morphisms (φ, ψ),

$$A \xrightarrow{\varphi} X \xleftarrow{\psi} B.$$

Moreover, this equivalence class may be written as the composite $\bar{\psi}\varphi$, where $\mathscr{A}$ is canonically embedded in $\mathscr{A}^\sim$ and $\bar{\Gamma}$ is the image of Γ in the canonical involution on $\mathscr{A}^\sim$. Every equivalence class has an essentially unique minimal representative (φ_0, ψ_0), characterized by the property that

$$\langle \varphi_0, \psi_0 \rangle : A \oplus B \twoheadrightarrow X_0,$$

where

$$A \xrightarrow{\varphi_0} X_0 \xleftarrow{\psi_0} B;$$

and $\bar{\psi}\varphi = \alpha\bar{\beta}$ if and only if the square

$$\begin{array}{ccc} \cdot & \xrightarrow{\beta} & \cdot \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \cdot & \xrightarrow[\psi]{} & \cdot \end{array}$$

is exact. Further, every relation has an essentially unique expression as $\bar{\epsilon}\theta\bar{\mu}$ with ϵ epic, μ monic.

An addition was introduced into $\mathscr{A}^\sim(A, B)$, turning it into a commutative semigroup with zero. However, it was observed that composition does not always distribute over addition, and it was clear that $\mathscr{A}^\sim(A, B)$ is not a cancellation semigroup. We may prove, with respect to distributivity, the following result.

THEOREM 1. (i) *The left-distributive law*

$$(\Gamma_1 + \Gamma_2)\Delta = \Gamma_1\Delta + \Gamma_2\Delta$$

holds for all Γ_1, Γ_2 *if* $\Delta = \theta\bar{\mu}$.

(ii) *The right-distributive law*

$$\Delta(\Gamma_1 + \Gamma_2) = \Delta\Gamma_1 + \Delta\Gamma_2$$

holds for all Γ_1, Γ_2 *if* $\Delta = \bar{\epsilon}\theta$.

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In fact, (i) was proved in [**1**] and (ii) follows by duality.

In § 2 we recall the lattice structure in $\mathscr{A}^{\sim}$ (see [**2**]) and study its relationship to the distributive law. In § 3 we characterize the addition in $\mathscr{A}^{\sim}$ and then characterize $\mathscr{A}$ in terms of the distributive law. In § 4 we characterize $\mathscr{A}$ in terms of cancellation in $\mathscr{A}^{\sim}$.

Our notation is based throughout on that of [**1**].

2. The lattice structure in $\mathscr{A}^{\sim}$. We consider the following relation $\geqq$ on $\mathscr{A}^{\sim}(A, B)$. Let (φ_i, ψ_i) be minimal representatives for $\Gamma_i \in \mathscr{A}^{\sim}(A, B)$, $i = 1, 2$. Then $\Gamma_1 \geqq \Gamma_2$ if and only if

$$\langle \varphi_1, \psi_1 \rangle = \rho \langle \varphi_2, \psi_2 \rangle \tag{1}$$

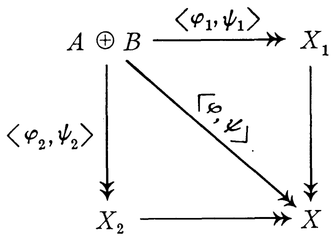
for some $\rho \in \mathscr{A}$. Of course, ρ in (1) is unique and is epic. It is clear that $\geqq$ is an order relation on $\mathscr{A}^{\sim}(A, B)$. Indeed, $\mathscr{A}^{\sim}(A, B)$ is then a lattice. The minimal element is represented by

$$A \xrightarrow{\iota_A} A \oplus B \xleftarrow{\iota_B} B$$

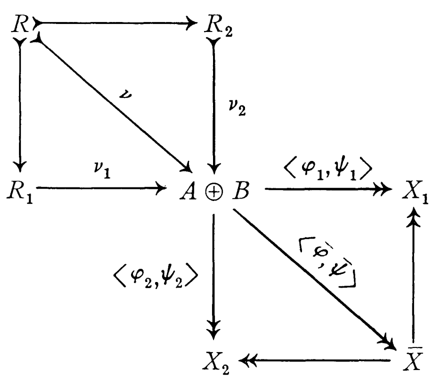
(so that $A \oplus B$ is characterized in $\mathscr{A}^{\sim}$) and the maximal element by

$$A \rightarrow 0 \leftarrow B$$

(so that 0 is characterized in $\mathscr{A}^{\sim}$). Moreover, if $\Gamma_1, \Gamma_2 \in \mathscr{A}^{\sim}(A, B)$ have minimal representatives (φ_1, ψ_1) and (φ_2, ψ_2), then $\mathrm{lub}(\Gamma_1, \Gamma_2)$ is represented by (φ, ψ), where $\langle \varphi, \psi \rangle$ is the diagonal of the cocartesian square



and $\mathrm{glb}(\Gamma_1, \Gamma_2)$ is represented by $(\bar{\varphi}, \bar{\psi})$ where, in the diagram



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we have $\langle \varphi_i, \psi_i \rangle \,||\, \nu_i$, $i = 1, 2$, ν is the diagonal of a cartesian square, and $\langle \bar{\varphi}, \bar{\psi} \rangle \,||\, \nu$. (The symbol $\alpha \,||\, \beta$ means that $\overset{\beta}{\rightarrowtail}\ \overset{\alpha}{\twoheadrightarrow}$ is exact.)

The order relation has the following properties, of which we omit the proofs of the first three.

PROPOSITION 1. *If* Γ_i *is represented by* (φ_i, ψ_i), $i = 1, 2$, *and if*

$$\langle \varphi_1, \psi_1 \rangle = \rho \langle \varphi_2, \psi_2 \rangle,$$

for some ρ, *then* $\Gamma_1 \geqq \Gamma_2$.

PROPOSITION 2. *If* $\Gamma_i \geqq \Gamma_2$, *then* $\bar{\Gamma}_1 \geqq \bar{\Gamma}_2$. *If* $\alpha, \beta \in \mathscr{A}(A, B)$ *and* $\alpha \geqq \beta$, *then* $\alpha = \beta$.

PROPOSITION 3. *If* $\Gamma_1 \geqq \Gamma_1'$, $\Gamma_2 \geqq \Gamma_2'$, *then* $\Gamma_1\Gamma_2 \geqq \Gamma_1'\Gamma_2'$.

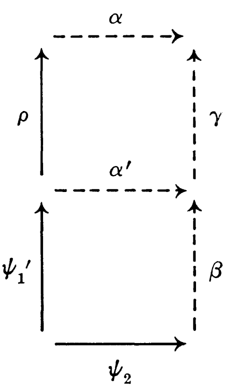
PROPOSITION 4. *If* $\Gamma_1 \geqq \Gamma_1'$, $\Gamma_2 \geqq \Gamma_2'$, $\Gamma_i, \Gamma_i' \in \mathscr{A}^{\sim}(A, B)$, $i = 1, 2$, *then*

$$\Gamma_1 + \Gamma_2 \geqq \Gamma_1' + \Gamma_2'.$$

Proof. It is sufficient to take $\Gamma_2 = \Gamma_2'$. Let (φ_1, ψ_1), (φ_1', ψ_1'), and (φ_2, ψ_2) be minimal representatives for Γ_1, Γ_1', and Γ_2, so that

$$\langle \varphi_1, \psi_1 \rangle = \rho \langle \varphi_1', \psi_1' \rangle.$$

Form cocartesian squares



Then, by definition of addition of relations,

$$\Gamma_1' + \Gamma_2 = \overline{\alpha'\psi_1'}(\alpha'\varphi_1' + \beta\varphi_2), \qquad \Gamma_1 + \Gamma_2 = \overline{\alpha\psi_1}(\alpha\varphi_1 + \gamma\beta\varphi_2).$$

However, $\alpha\psi_1 = \alpha\rho\psi_1' = \gamma\alpha'\psi_1'$, and $\alpha\varphi_1 = \alpha\rho\varphi_1' = \gamma\alpha'\varphi_1'$. Thus

$$\langle \alpha\varphi_1 + \gamma\beta\varphi_2, \alpha\psi_1 \rangle = \gamma \langle \alpha'\varphi_1' + \beta\varphi_2, \alpha'\psi_1' \rangle$$

so that $\Gamma_1 + \Gamma_2 \geqq \Gamma_1' + \Gamma_2$ by Proposition 1.

The relation of the order relation to the distributive law is given by the following theorem.

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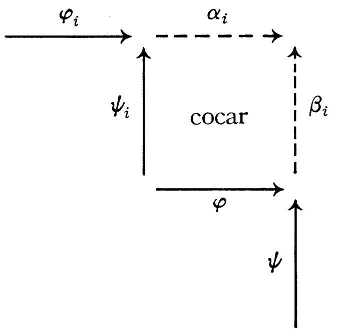
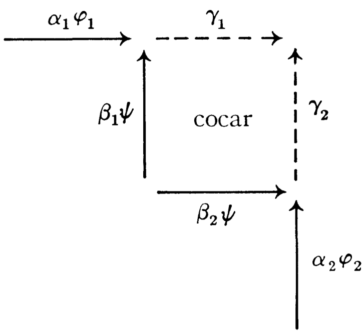
THEOREM 2. (i) $\Gamma_1\Delta + \Gamma_2\Delta \geqq (\Gamma_1 + \Gamma_2)\Delta$,
(ii) $\Delta(\Gamma_1 + \Gamma_2) \geqq \Delta\Gamma_1 + \Delta\Gamma_2$,
wherever the compositions make sense.

Proof. We prove (ii); (i) then follows by the dual argument.
We take $\Gamma_i = \bar{\psi}_i\varphi_i$, $i = 1, 2$, $\Delta = \bar{\psi}\varphi$. We form

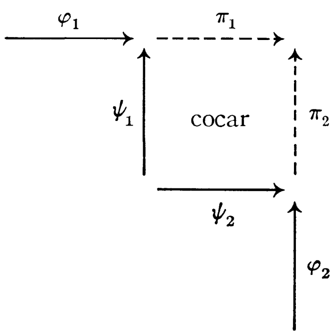


$i = 1, 2,$

so that $\Delta\Gamma_i = \overline{\beta_i\psi}\alpha_i\varphi_i$. We then form

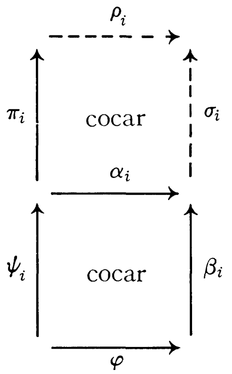


so that $\Delta\Gamma_1 + \Delta\Gamma_2$ is represented by $(\gamma_1\alpha_1\varphi_1 + \gamma_2\alpha_2\varphi_2, \gamma_1\beta_1\psi)$. We next form



so that $\Gamma_1 + \Gamma_2$ is represented by $(\pi_1\varphi_1 + \pi_2\varphi_2, \pi_1\psi_1)$.

Finally we form



$i = 1, 2,$

Remark that since $\pi_1\psi_1 = \pi_2\psi_2$, both ρ_i and $\sigma_i\beta_i$ are independent of i and we may write $\rho_i = \rho$. Also since $\sigma_1\beta_1\psi = \sigma_2\beta_2\psi$, there exists τ with $\sigma_i = \tau\gamma_i$, $i = 1, 2$.

Now $\Delta(\Gamma_1 + \Gamma_2)$ is represented by $(\rho(\pi_1\varphi_1 + \pi_2\varphi_2), \sigma_1\beta_1\psi)$ and

$$\rho(\pi_1\varphi_1 + \pi_2\varphi_2) = \sigma_1\alpha_1\varphi_1 + \sigma_2\alpha_2\varphi_2 = \tau(\gamma_1\alpha_1\varphi_1 + \gamma_2\alpha_2\varphi_2),$$
$$\sigma_1\beta_1\psi = \tau(\gamma_1\beta_1\psi).$$

Thus the theorem follows from Proposition 1.

COROLLARY 1. *We have the following inequalities:*

$$\Delta(\Gamma_1\Sigma + \Gamma_2\Sigma) \begin{array}{c} \geq \; \Delta\Gamma_1\Sigma + \Delta\Gamma_2\Sigma \; \geq \\ \\ \geq \; \Delta(\Gamma_1 + \Gamma_2)\Sigma \; \geq \end{array} (\Delta\Gamma_1 + \Delta\Gamma_2)\Sigma$$

3. Addition in $\mathscr{A}^\sim$ and distributivity. It follows from Theorem 1 that if $\theta \in \mathscr{A}$, then

$$(\Gamma_1 + \Gamma_2)\theta = \Gamma_1\theta + \Gamma_2\theta, \qquad \theta(\Delta_1 + \Delta_2) = \theta\Delta_1 + \theta\Delta_2, \tag{2}$$

wherever the compositions are defined. We will prove below (Theorem 4) that property (2) characterizes the morphisms of $\mathscr{A}$. However, we first take the opposite point of view and show that property (2), together with an extra property, characterizes the addition in $\mathscr{A}^\sim$.

We first introduce the following extra property.

PROPOSITION 5. *Let Γ_1 and Γ_2 be morphisms of $\mathscr{A}^\sim$ such that $\Gamma_1 + \Gamma_2 \in \mathscr{A}$. Then $\Gamma_1, \Gamma_2 \in \mathscr{A}$.*

Proof. A morphism $\bar{\psi}\varphi$ of $\mathscr{A}^\sim$ belongs to $\mathscr{A}$ if and only if ψ is monic and

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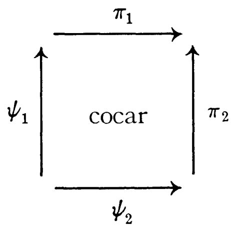
$\psi\alpha = \varphi$ for some α. Now if $\Gamma_i = \bar{\psi}_i\varphi_i$, $i = 1, 2$, then $\Gamma_1 + \Gamma_2 = \bar{\psi}\varphi$, where

(3) 

$\varphi = \pi_1\varphi_1 + \pi_2\varphi_2$, and $\psi = \pi_1\psi_1 = \pi_2\psi_2$. Thus if $\Gamma_1 + \Gamma_2 \in \mathscr{A}$, ψ is monic and

$$\pi_1\varphi_1 + \pi_2\varphi_2 = \pi_1\psi_1\alpha \quad \text{for some } \alpha.$$

But then ψ_1 and ψ_2 are monic so that (3) is bicartesian and, since

$$\pi_1(\psi_1\alpha - \varphi_1) = \pi_2\varphi_2,$$

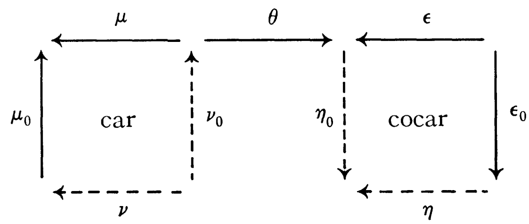
there exists τ with $\psi_2\tau = \varphi_2$. Thus $\Gamma_2 \in \mathscr{A}$ and similarly $\Gamma_1 \in \mathscr{A}$.

We may thus say that $\mathscr{A}$ is *isolated* in $\mathscr{A}^\sim$ under $+$. We now prove the following result.

THEOREM 3. *Let an addition* $\oplus$ *be defined in* $\mathscr{A}^\sim$ *such that*
(i) $\oplus$ *extends the addition in* $\mathscr{A}$*;*
(ii) *the morphisms of* $\mathscr{A}$ *distribute over* $\oplus$*;*
(iii) $\mathscr{A}$ *is isolated in* $\mathscr{A}^\sim$ *under* $\oplus$.
Then $\oplus$ *coincides with* $+$.

Proof. We know that $+$ has properties (i), (ii), and (iii). Now suppose that $\oplus$ also has these properties. We first observe the following fact, which we state as a lemma.

LEMMA 1. *Let* $\Gamma = \bar{\epsilon}\theta\bar{\mu}$. *If* $\epsilon_0\Gamma\mu_0 \in \mathscr{A}$, *then* ϵ *right-divides* ϵ_0 *and* μ *left-divides* μ_0.



Now $\epsilon_0\Gamma\mu_0 = \bar{\eta}\eta_0\theta\nu_0\bar{\nu}$. Thus if $\epsilon_0\Gamma\mu_0 \in \mathscr{A}$, η and ν are units, so that

$$\epsilon_0 = \eta^{-1}\eta_0\epsilon, \qquad \mu_0 = \mu\nu_0\nu^{-1}.$$

Returning to the theorem, let

$$\Gamma_i = \bar{\epsilon}_i\theta_i\bar{\mu}_i, \qquad i = 1, 2,$$
$$\Gamma_1 + \Gamma_2 = \bar{\epsilon}_0\theta_0\bar{\mu}_0, \qquad \Gamma_1 \oplus \Gamma_2 = \bar{\epsilon}\theta\bar{\mu}.$$

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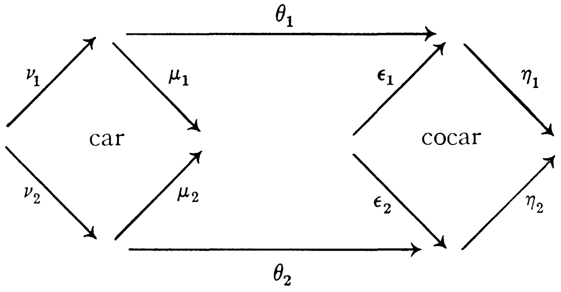
In fact, we have the following diagram:



and

$$\epsilon_0 = \eta_1\epsilon_1 = \eta_2\epsilon_2, \qquad \mu_0 = \mu_1\nu_1 = \mu_2\nu_2, \qquad \theta_0 = \eta_1\theta_1\nu_1 + \eta_2\theta_2\nu_2.$$

Now $\theta = \epsilon(\Gamma_1 \oplus \Gamma_2)\mu = \epsilon\Gamma_1\mu \oplus \epsilon\Gamma_2\mu$ by (ii). Thus $\epsilon\Gamma_1\mu, \epsilon\Gamma_2\mu \in \mathscr{A}$ by (iii) and $\theta = \epsilon\Gamma_1\mu + \epsilon\Gamma_2\mu$ by (i). Further,

$$\epsilon(\Gamma_1 + \Gamma_2)\mu = \epsilon\Gamma_1\mu + \epsilon\Gamma_2\mu$$

by (2), so that $\epsilon(\Gamma_1 + \Gamma_2)\mu \in \mathscr{A}$ and ϵ_0 right-divides ϵ, μ_0 left-divides μ, by Lemma 1. Now

$$\begin{aligned} \epsilon_0(\Gamma_1 \oplus \Gamma_2)\mu_0 &= \epsilon_0\Gamma_1\mu_0 \oplus \epsilon_0\Gamma_2\mu_0 \quad \text{by (ii)} \\ &= \eta_1\theta_1\nu_1 \oplus \eta_2\theta_2\nu_2 \\ &= \eta_1\theta_1\nu_1 + \eta_2\theta_2\nu_2 \quad \text{by (i)} \\ &= \theta_0. \end{aligned}$$

Thus, again invoking Lemma 1,

$$\epsilon \text{ right-divides } \epsilon_0, \qquad \mu \text{ left-divides } \mu_0.$$

Thus we may take $\epsilon = \epsilon_0$, $\mu = \mu_0$. But then

$$\theta = \epsilon\Gamma_1\mu + \epsilon\Gamma_2\mu = \epsilon_0\Gamma_1\mu_0 + \epsilon_0\Gamma_2\mu_0 = \theta_0,$$

and the theorem is proved.

Remark. It would be interesting to know whether a weaker property than (iii) can be used to characterize the addition in $\mathscr{A}^\sim$. Of course, properties (i) and (ii) are extremely natural requirements on the addition. It is of interest that no use is made in this theorem of the commutativity or associativity of the addition in $\mathscr{A}^\sim$, which thus appears as a consequence of properties (i), (ii), and (iii).

We now show how the distributive property picks out the morphisms of $\mathscr{A}$. In fact, we prove the converse of Theorem 1.

THEOREM 4. (i) *The left-distributive law*

$$(\Gamma_1 + \Gamma_2)\Delta = \Gamma_1\Delta + \Gamma_2\Delta$$

holds for all Γ_1, Γ_2 *if and only if* $\Delta = \theta\bar{\mu}$.

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(ii) *The right-distributive law*

$$\Delta(\Gamma_1 + \Gamma_2) = \Delta\Gamma_1 + \Delta\Gamma_2$$

holds for all Γ_1, Γ_2 *if and only if* $\Delta = \bar{\epsilon}\theta$.

Proof. We are content to prove (ii), (i) then following by duality. We already know that the condition is sufficient; hence we prove it necessary. That is, we prove that if $\Delta = \bar{\epsilon}\theta\bar{\mu}$ and if

$$\Delta(\Gamma_1 + \Gamma_2) = \Delta\Gamma_1 + \Delta\Gamma_2 \tag{4}$$

always holds, then μ is a unit. Now if (4) holds, then certainly

$$\theta\bar{\mu}(\Gamma_1 + \Gamma_2) = \theta\bar{\mu}\Gamma_1 + \theta\bar{\mu}\Gamma_2. \tag{5}$$

For $\theta\bar{\mu}(\Gamma_1 + \Gamma_2) = \epsilon\Delta(\Gamma_1 + \Gamma_2)$
$= \epsilon\Delta\Gamma_1 + \epsilon\Delta\Gamma_2$ by (4) and Theorem 1 $= \theta\bar{\mu}\Gamma_1 + \theta\bar{\mu}\Gamma_2$.

Thus we prove that (5) implies that μ is a unit. Let $\eta \,||\, \mu$,

$$C \xleftarrow{\eta} A \xleftarrow{\mu} X \xrightarrow{\theta} B,$$

and set $\Gamma_1 = \bar{\eta}$, $\Gamma_2 = \overline{-\eta}$. Then $\bar{\eta} + \overline{-\eta} = \bar{\eta}(1 + (-1)) = \bar{\eta}0_{CC}$, so that

$$\theta\bar{\mu}(\bar{\eta} + \overline{-\eta}) = \theta\bar{\mu}\bar{\eta}0_{CC} = \theta\bar{0}_{XC}0_{CC} = \theta p_1\bar{p}_2,$$

where p_1: $X \oplus C \to X$, p_2: $X \oplus C \to C$ are the projections.

On the other hand,

$$\theta\bar{\mu}\bar{\eta} + \theta\bar{\mu}\overline{-\eta} = \theta\bar{0}_{XC} + \theta\bar{0}_{XC} = \theta(\bar{0}_{XC} + \bar{0}_{XC}) = \theta\langle 1, 1\rangle\bar{0}_{X\oplus X,C} = \theta\bar{0}_{XC}.$$

Thus the distributive law implies the existence of a diagram

$$\begin{array}{ccccc} C & \xleftarrow{p_2} & X \oplus C & \xrightarrow{\theta p_1} & B \\ \| & & \uparrow \{\alpha,\beta\} & & \| \\ \| & & Y & & \| \\ \| & & \downarrow \gamma & & \| \\ C & \xleftarrow{0} & X & \xrightarrow{\theta} & B \end{array}$$

But then $0 = p_2\{\alpha, \beta\} = \beta$. Thus

$$Y \xrightarrow{\alpha} X \xrightarrow{\{1, 0\}} X \oplus C$$

is epic so that

$$X \xrightarrow{\{1, 0\}} X \oplus C.$$

This, however, implies that $C = 0$, so that $\eta = 0$ and μ is a unit.

COROLLARY 2. *Let* $\Gamma \in \mathscr{A}^{\sim}$. *Then* Γ *distributes on the right and left over addition if and only if* $\Gamma \in \mathscr{A}$.

4. Cancellation. In this section we prove the following theorem.

THEOREM 5. *Let* $\Gamma \in \mathscr{A}^{\sim}(A, B)$. *Then* $\Gamma + \Gamma_1 = \Gamma + \Gamma_2 \Rightarrow \Gamma_1 = \Gamma_2$ *if and only if* $\Gamma \in \mathscr{A}(A, B)$.

Proof. Let $\Gamma = \theta$, $\Gamma_i = \bar{\epsilon}_i \theta_i \bar{\mu}_i$, $i = 1, 2$. Then

$$\Gamma + \Gamma_i = \bar{\epsilon}_i(\epsilon_i \theta \mu_i + \theta_i)\bar{\mu}_i, \qquad i = 1, 2.$$

Thus if $\Gamma + \Gamma_1 = \Gamma + \Gamma_2$, it follows that there are units ω, ω' such that

$$\epsilon_2 = \omega \epsilon_1, \qquad \mu_1 = \mu_2 \omega', \qquad \omega(\epsilon_1 \theta \mu_1 + \theta_1) = (\epsilon_2 \theta \mu_2 + \theta_2)\omega'.$$

But this implies that $\omega\theta_1 = \theta_2\omega'$, so that $\Gamma_1 = \Gamma_2$.

Conversely, let $\Gamma \in \mathscr{A}^{\sim}(A, B)$ have the cancellation property, $\Gamma = \bar{\epsilon}\theta\bar{\mu}$,

$$A \overset{\mu}{\longleftarrow\!\!\!\prec} X \overset{\theta}{\longrightarrow} Y \overset{\epsilon}{\twoheadleftarrow} B.$$

Then

$$\begin{aligned} \bar{\epsilon}\theta\bar{\mu} + 0_{XB}\bar{\mu} &= (\bar{\epsilon}\theta + 0_{XB})\bar{\mu}, \quad \text{by Theorem 1} \\ &= \bar{\epsilon}\theta\bar{\mu} \\ &= \bar{\epsilon}\theta\bar{\mu} + 0_{AB}. \end{aligned}$$

Thus $0_{XB}\bar{\mu} = 0_{AB}$. This plainly implies that μ is epic, hence μ is a unit. Similarly, $\bar{\epsilon}\theta\bar{\mu} + \bar{\epsilon}0_{AY} = \bar{\epsilon}(\theta\mu + 0_{AY}) = \bar{\epsilon}\theta\bar{\mu} = \bar{\epsilon}\theta\bar{\mu} + 0_{AB}$; thus $\bar{\epsilon}0_{AY} = 0_{AB}$, and ϵ is monic, ϵ is a unit, $\Gamma \in \mathscr{A}$.

Added in proof. The authors' attention has been drawn to an article by H.-B. Brinkmann (*Addition von Korrespondenzen in Abelsche Kategorien*, to appear). In this article, Brinkmann obtains very elegant characterization of addition based on the statement which constitutes Theorem 2 of this paper.

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