

EXTENSIONS OF REGULAR ORTHOGROUPS BY GROUPS

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(Received 24 July 1992; revised 6 November 1992)

Communicated by P. G. Trotter

Abstract

A common generalization of the author's embedding theorem concerning the E -unitary regular semigroups with regular band of idempotents, and Billhardt's and Ismael's embedding theorem on the inverse semigroups, the closure of whose set of idempotents is a Clifford semigroup, is presented. We prove that each orthodox semigroup with a regular band of idempotents, which is an extension of an orthogroup K by a group, can be embedded into a semidirect product of an orthogroup K' by a group, where K' belongs to the variety of orthogroups generated by K . The proof is based on a criterion of embeddability into a semidirect product of an orthodox semigroup by a group and uses bilocality of orthogroup bivarieties.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 20M10, 20M19.

Introduction

McAlister's P -theorem [5] describes the structure of E -unitary inverse semigroups. O'Carroll [6] noticed that the E -unitary inverse semigroups are just the inverse semigroups embeddable into a semidirect product of a semilattice by a group. This gave the author the idea to raise the question in [7] whether each E -unitary regular semigroup is embeddable into a semidirect product of a band by a group. This question is still open. A partial answer was given in [7] by proving that each E -unitary regular semigroup S with regular band of idempotents is embeddable into a semidirect product of a band B by a group where B can be chosen from the band variety generated by the band of idempotents in S .

Recently, another embedding theorem was obtained independently by Billhardt [1] and Ismael [3]. They proved that if $E\omega$, the closure of the set of idempotents in an inverse semigroup S , is a Clifford semigroup, then S is embeddable into a semidirect product of a Clifford semigroup by a group.

The aim of the present paper is to generalize these embedding theorems in Section 2 as follows. If S is an orthodox semigroup with a regular band of idempotents and θ is a group congruence on S whose kernel $\ker \theta$ is an orthogroup, then S is embeddable into a semidirect product of an orthogroup K by S/θ , where K belongs to the variety of orthogroups generated by $\ker \theta$. Because of the latter condition, our result, in fact, strengthens the embedding theorems in [1] and [3].

The proof of our main result is based on the criterion of embeddability into a semidirect product of an orthodox semigroup by a group presented in [9]. An important tool in the proof is that the orthogroup bivarieties are bilocal as is proved in [8].

In order to be self-contained, Section 1 summarizes these results together with some notions from [10] and specializations of results in [4] needed in Section 2. At the end of Section 1, we clear up the connection between the embeddability criterion and bilocality.

1. Preliminaries

In this section we summarize the notions and results needed in the following section. For undefined notions and notation the reader is referred to [2].

A completely regular semigroup which is orthodox is called an *orthogroup*. A *regular band* (*left regular band*, *right regular band*) is a band satisfying the identity $abaca \hat{=} abca$ ($aba \hat{=} ab$, $aba \hat{=} ba$). If an orthodox semigroup or an orthogroup has a (left, right) regular band of idempotents then, for brevity, we will term them a (*left*, *right*) *regular orthodox semigroup* or a (*left*, *right*) *regular orthogroup*, respectively. Notice that the left (right) regular orthodox semigroups are also called in the literature left (right) inverse semigroups and \mathcal{R} -(\mathcal{L})-unipotent semigroups.

Given an orthodox semigroup S , its band of idempotents is denoted by E_S or, if it causes no confusion, simply by E . The identity element of a group is denoted by 1.

If S is an orthodox semigroup and θ is a congruence on S then, as usual, we denote its *kernel* $\{s \in S: s \theta e \text{ for some } e \in E\}$ by $\ker \theta$ and its *trace* $\theta|E$ by $\text{tr} \theta$. It is well known that $\ker \theta$ is an orthodox subsemigroup in S . Obviously, we have $E \subseteq \ker \theta$. In particular, if θ is a group congruence, then $\ker \theta = \{s \in S: s \theta = 1\}$. If \mathbf{K} is a group variety then we denote by $\sigma_{\mathbf{K}}$ the least congruence θ on S such that $S/\theta \in \mathbf{K}$. In particular, as usual, σ is the least group congruence on S .

If $\phi: S \rightarrow T$ is a homomorphism then the congruence on S induced by ϕ is denoted by \equiv_{ϕ} .

Firstly we formulate the facts on left regular orthodox semigroups which are needed in the paper.

The following characterization is from [11].

RESULT 1.1. *The following two conditions are equivalent for a semigroup S .*

- (i) S is a left regular orthodox semigroup.
- (ii) Each \mathcal{R} -class in S contains a unique idempotent.

By making use of the basic properties of orthodox semigroups ([2, VI.1 and 2]) and specializing them to left regular orthodox semigroups, we obtain (i) and (ii) in the following result.

RESULT 1.2. *Let S be a left regular orthodox semigroup. Then*

- (i) $\gamma \cap \mathcal{R}$ is the identity relation on S ,
- (ii) for every $a, b \in S$, we have $a \mu b$ if and only if

$$(\exists a' \in V(a))(\exists b' \in V(b))(\forall e \in E) aea' = beb'$$

or, equivalently, if and only if

$$(\forall a' \in V(a))(\forall b' \in V(b))(\forall e \in E) aea' = beb',$$

- (iii) $(\mu \vee \gamma) \cap \mathcal{R} = \mu$.

PROOF. (iii) It is obvious that $(\mu \vee \gamma)/\mu$ is the least inverse semigroup congruence on S/μ . Since $\mu \subseteq \mathcal{R}$, one can easily see that \mathcal{R}/μ is Green's \mathcal{R} -relation on S/μ . Thus the equality to be proved follows by applying (i) to S/μ .

Now we recall some notions and results from [4].

The free semigroup with involution on an alphabet A will be denoted by A^\oplus . We represent A^\oplus as the free semigroup on the alphabet $\bar{A} = A \cup A^*$ where $A \cap A^* = \emptyset$ and a bijection $*$: $A \rightarrow A^*$, $a \mapsto a^*$ ($a \in A$) is given. The involution on A^\oplus , also denoted by $*$, is the unique involution extending this bijection. However, we will consider A^\oplus as a usual semigroup (without a unary operation). In particular, the congruences on A^\oplus will not be required to be compatible with the involution $*$. By writing $u(a_1, a_1^*, \dots, a_n, a_n^*)$ for a word $u \in A^\oplus$, we mean that at most the elements $a_1, \dots, a_n \in A$ occur in u , either with or without $*$.

A bi-identity in the alphabet A is a pair of words $u \hat{=} v$ with $u, v \in A^\oplus$. An orthodox semigroup S satisfies a bi-identity $u(a_1, a_1^*, \dots, a_n, a_n^*) \hat{=} v(a_1, a_1^*, \dots, a_n, a_n^*)$ if, for every $s_1, \dots, s_n \in S$ and every $s'_1 \in V(s_1), \dots, s'_n \in V(s_n)$, we have $u(s_1, s'_1, \dots, s_n, s'_n) = v(s_1, s'_1, \dots, s_n, s'_n)$ in S .

On classes of orthodox semigroups we consider the operations of forming direct products, regular subsemigroups, homomorphic images and isomorphic images, and denote them by P , S_r , H and I , respectively.

A bivariety of orthodox semigroups is a class of orthodox semigroups closed with respect to P , S_r and H . In particular, the bivarieties of inverse semigroups and those of orthogroups are just the varieties of inverse semigroups and of orthogroups,

respectively. Therefore, in these cases, we will often speak simply about varieties, for example group varieties and orthogroup varieties, and not about bivarieties.

For later use, we introduce notation for the following bivarieties:

- O** — orthodox semigroups
- LROG** — left regular orthogroups,
- RROG** — right regular orthogroups,
- SG** — semilattices of groups,
- G** — groups,
- LRB** — left regular bands,
- RRB** — right regular bands,
- S** — semilattices.

It is proved in [4] that the bivarieties of orthodox semigroups are just the classes defined by bi-identities. Moreover, the notion of a *bi-invariant congruence* is introduced, and a one-to-one correspondence is found between the bivarieties of orthodox semigroups and the bi-invariant congruences on an infinite alphabet. Given a bivariety **V** of orthodox semigroups and an alphabet *A*, the bi-invariant congruence on A^\oplus corresponding to **V** is

$$\rho(\mathbf{V}, A) = \{(u, v) \in A^\oplus \times A^\oplus : \text{the bi-identity } u \hat{=} v \text{ is satisfied in } \mathbf{V}\}.$$

When describing a property of the bivariety **V**, we will use $\rho(\mathbf{V})$ to denote the bi-invariant congruence corresponding to **V** on an infinite alphabet.

Note that if **V** is an inverse semigroup variety then the bi-invariant congruences corresponding to **V** coincide with the respective fully invariant congruences. However, with orthogroups this is not the case.

The notion of a *bifree object* is also defined and $A^\oplus / \rho(\mathbf{V}, A)$ turns out to be the bifree object in **V** on *A*.

Now we reformulate the results of [4] concerning the bi-invariant congruences $\rho(\mathbf{V}, A)$ of orthogroup varieties **V** in the special case of left regular orthogroup varieties.

For any word $u \in A^\oplus$, we introduce the following notation:

- $A(u)$ — the *content* of *u*, that is, the set of all elements in *A* such that *a* or *a** occurs in *u*,
- $0(u)$ — the longest initial segment *v* of *u* such that $|A(v)| = |A(u)| - 1$ (in particular, if $|A(u)| = 1$ then $0(u)$ is the empty word),
- $\bar{0}(u)$ — the element $a \in \bar{A}$ such that $0(u)a$ is an initial segment of *u*,
- $h(u)$ — the *head* of *u*, that is, the element of \bar{A} occurring first in *u* from the left.

We need to iterate the operations 0 and $\bar{0}$ as follows: for $2 \leq k \leq |A(u)|$, define

$$0^k(u) = 0(0^{k-1}(u)) \quad \text{and} \quad \bar{0}^k(u) = \bar{0}(0^{k-1}(u)).$$

Dually to $0^k(u)$, $\bar{0}^k(u)$ and $h(u)$, we define $1^k(u)$, $\bar{1}^k(u)$ and the tail $t(u)$, respectively. Notice that if $k = |A(u)|$ then $\bar{0}^k(u) = h(u)$ and $\bar{1}^k(u) = t(u)$.

If Q is any of A , h and t then define the equivalence relation

$$\{(u, v) \in A^\oplus \times A^\oplus : Q(u) = Q(v)\}$$

and denote it by Δ , h and t , respectively. Moreover, put

$$\Delta' = \{(u, v) \in A^\oplus \times A^\oplus : |A(u)| = |A(v)| \text{ and } |A(u) - A(v)| \leq 1\}.$$

Given a congruence ρ on A^\oplus , we define two relations ρ_0 and ρ_1 as follows:

$$\rho_0 = \{(0(u), 0(v)) : u, v \in A^\oplus, |A(u)|, |A(v)| \geq 2 \text{ and } u \rho v\}$$

and, dually, we define ρ_1 . Obviously, $\rho \subseteq \rho_0, \rho_1$.

We restate [4, Proposition 2.11].

RESULT 1.3. *For any variety \mathbf{V} of orthogroups, exactly one of the following conditions holds:*

- (i) $(\rho(\mathbf{V}))_0 = \Delta'$,
- (ii) $(\rho(\mathbf{V}))_0 = h \cap \Delta'$,
- (iii) $(\rho(\mathbf{V}))_0 \subseteq \Delta$, in which case $(\rho(\mathbf{V}))_0$ is a bi-invariant congruence and $((\rho(\mathbf{V}))_0)_0 = (\rho(\mathbf{V}))_0$.

In case (iii), the variety of orthogroups corresponding to the bi-invariant congruence $(\rho(\mathbf{V}))_0$ is denoted by \mathbf{V}_0 .

REMARK 1.4. If $(\rho(\mathbf{V}))_0 \subseteq \Delta$ then one can easily see that $(\rho(\mathbf{V}))_0 \subseteq h$ is also valid.

The following statement describes the bi-invariant congruence $\rho(\mathbf{V})$ for the varieties \mathbf{V} with $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{LROG}$ and $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{SG}$.

RESULT 1.5. *Let \mathbf{V} be a variety of orthogroups. Then*

- (i) $\mathbf{S} \subseteq \mathbf{V}$ if and only if $\rho(\mathbf{V}) \subseteq \Delta$,
- (ii) $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{LROG}$ if and only if $\rho(\mathbf{V}) \subseteq \Delta$ and $(\rho(\mathbf{V}))_1 = \Delta'$,
- (iii) $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{SG}$ if and only if $\rho(\mathbf{V}) \subseteq \Delta$ and $(\rho(\mathbf{V}))_0 = (\rho(\mathbf{V}))_1 = \Delta'$.

PROOF. (i) The ‘if’ part is obvious; the ‘only if’ part follows by [4, Proposition 2.14].

(ii) Firstly suppose that $\mathbf{V} \subseteq \mathbf{LROG}$. Since each left regular orthogroup satisfies the bi-identity $aa^*bb^* \hat{=} aa^*bb^*aa^*$, we clearly have $(\rho(\mathbf{V}))_1 \not\subseteq t$. The dual of Result 1.3 implies $(\rho(\mathbf{V}))_1 = \Delta'$. Conversely, if $(\rho(\mathbf{V}))_1 = \Delta'$ then a bi-identity $uab \hat{=} vba$ holds in \mathbf{V} where $u = u(a, a^*, b, b^*)$ and $v = v(a, a^*, b, b^*)$. Hence, if a band E

belongs to \mathbf{V} and $e, f \in E$ then $efu(e, e, f, f)ef = efv(e, e, f, f)fe$ in E which implies that $ef = efe$. Thus E is left regular, completing the proof of the inclusion $\mathbf{V} \subseteq \mathbf{LROG}$.

(iii) Since $\mathbf{S} = \mathbf{LRB} \cap \mathbf{RRB}$, we have $\mathbf{SG} = \mathbf{LROG} \cap \mathbf{RROG}$, and so the statement is immediate from (ii) and its dual.

The solution of the word problem in [4, Proposition 2.13] reduces to the following.

RESULT 1.6. *Let \mathbf{V} be a variety of orthogroups with $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{LROG}$, and let A be a non-empty set. Then, for any $u, v \in A^\oplus$, we have $u \rho(\mathbf{V}, A) v$ if and only if the following conditions are satisfied:*

- (i) $A(u) = A(v)$,
- (ii) $u \rho(\mathbf{V} \cap \mathbf{G}, A) v$,
- (iii) if $(\rho(\mathbf{V}))_0 = h \cap \Delta'$ then $h(u) = h(v)$,
- (iv) if $(\rho(\mathbf{V}))_0 \subseteq \Delta$ then $\bar{0}(u) = \bar{0}(v)$ and, in case $|A(u)| \geq 2$, also $0(u) \rho(\mathbf{V}_0, A) 0(v)$.

REMARK 1.7. By Remark 1.4, conditions (iii) and (iv) can be substituted by the following ones:

- (iii)' if $(\rho(\mathbf{V}))_0 \subseteq h$ then $h(u) = h(v)$,
- (iv)' if $(\rho(\mathbf{V}))_0 \subseteq \Delta$ and $|A(u)| \geq 2$, then $\bar{0}(u) = \bar{0}(v)$ and $0(u) \rho(\mathbf{V}_0, A) 0(v)$.

REMARK 1.8. In particular, if $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{SG}$, then, by Result 1.5(iii), we have $u \rho(\mathbf{V}, A) v$ if and only if conditions (i) and (ii) in Result 1.6 are satisfied.

We will need also [4, Lemma 2.10]:

RESULT 1.9. *Let \mathbf{V} be a variety of orthogroups with $\rho(\mathbf{V}), (\rho(\mathbf{V}))_0 \subseteq \Delta$, and let A be a non-empty set. Then, for any $u, v \in A^\oplus$, we have $u \rho(\mathbf{V}, A) \mathcal{R} v \rho(\mathbf{V}, A)$ in $A^\oplus / \rho(\mathbf{V}, A)$ if and only if conditions (i) and (iv) in Result 1.6 are satisfied.*

Now we recall the basic notions on graphs and semigroupoids needed later, and formulate the main results in [8].

A graph X consists of a set of objects denoted by $\text{Obj}(X)$ and, for every pair $i, j \in \text{Obj}(X)$, a set of arrows from i to j which is denoted by $X(i, j)$ and is called a hom-set. The different hom-sets are supposed to be disjoint. If $a \in X(i, j)$ then we also write that $\alpha(a) = i$ and $\omega(a) = j$. The set of all arrows will be denoted by $\text{Arr}(X)$. The arrows a, b are called coterminial if $a, b \in X(i, j)$ for some $i, j \in \text{Obj}(X)$ and are termed consecutive provided $\omega(a) = \alpha(b)$.

A graph with involution consists of a graph X and a mapping $*$ assigning to any arrow $a \in X(i, j)$ ($i, j \in \text{Obj}(X)$) an arrow $a^* \in X(j, i)$ such that $(a^*)^* = a$.

By a *subgraph of a graph* X we mean a graph Y such that $\text{Obj}(Y) \subseteq \text{Obj}(X)$ and $Y(i, j) \subseteq X(i, j)$ for every $i, j \in \text{Obj}(Y)$. A *subgraph of a graph X with involution* is defined to be a graph Y with involution which is a subgraph of the graph X and the involution on Y is the restriction of the involution on X .

Notice that if $B \subseteq \text{Arr}(X)$ for some graph X then we have

$$\text{Obj}(Y) \supseteq \{\alpha(b), \omega(b) : b \in B\} \quad \text{and} \quad Y(i, j) = B \cap X(i, j) \quad (i, j \in \text{Obj}(Y))$$

for every subgraph Y in X with $\text{Arr}(Y) = B$. The subgraph Y whose set of objects is the smallest possible will be termed the *subgraph of X determined by B* . If X is a graph with involution and $B \subseteq \text{Arr}(X)$ then by the *subgraph of X determined by B* we mean the subgraph of X whose underlying graph is determined by $\{b, b^* : b \in B\}$.

A *semigroupoid* is a graph C equipped with a composition which assigns to every pair of consecutive arrows $a \in C(i, j)$, $b \in C(j, k)$ an arrow $ab \in C(i, k)$ such that the composition is associative, that is, for any arrows $a \in C(i, j)$, $b \in C(j, k)$ and $c \in C(k, l)$, we have $(ab)c = a(bc)$.

Observe that $C(i, i)$, which, for brevity, will be denoted also by $C(i)$, is either empty or a semigroup for each $i \in \text{Obj}(C)$. For the sake of unicity, we will consider the empty set also as a semigroup, and we term $C(i)$ the *local semigroup of C at i* .

A *semigroupoid with involution* is a graph C with involution equipped with a composition which makes C a semigroupoid and which possesses the property that $(ab)^* = b^*a^*$ for every $a \in C(i, j)$, $b \in C(j, k)$ ($i, j, k \in \text{Obj}(C)$).

In particular, a set A can be considered as a graph with one object whose unique hom-set is A and, similarly, a semigroup S can be considered as a semigroupoid with one object whose unique hom-set is S .

A semigroupoid C is called *regular* if, for every $a \in \text{Arr}(C)$, there is $x \in \text{Arr}(C)$ with $axa = a$. In the same way as in the case of a semigroup, one can show that if such an x exists then $a' = xax$ has the property that $aa'a = a$ and $a'aa' = a'$. If the latter equalities hold for some a and a' then we say that a' is an *inverse* of a . The set of inverses of an element a is denoted by $V(a)$. An *idempotent* in a semigroupoid C is an arrow satisfying $e^2 = e$. Clearly, each idempotent belongs to a local semigroup. A regular semigroupoid is termed *orthodox* if the product of two idempotents is idempotent. Clearly, a regular semigroupoid is orthodox if and only if each of its local semigroups is orthodox.

The notion of the *product* and *coproduct* (=disjoint union) of a family of graphs and of a family of semigroupoids can be introduced. For the details the reader is referred to [10] (cf. also [8]).

We say that a graph X is *symmetric* if $X(j, i) \neq \emptyset$ provided $X(i, j) \neq \emptyset$. Notice that a graph with involution is always symmetric. A symmetric graph X is termed *connected* if, for any different $i, j \in \text{Obj}(X)$, there exists a sequence of consecutive arrows from i to j . It is well known that each symmetric graph is a coproduct of its

connected components.

Let X, Y be two graphs. A *graph function* $f: X \rightarrow Y$ consists of an *object function* $f: \text{Obj}(X) \rightarrow \text{Obj}(Y)$ and, for every $i, j \in \text{Obj}(X)$, a *hom-set function* $f: X(i, j) \rightarrow Y(if, jf)$. If C, D are semigroupoids then by a *morphism of semigroupoids* $\phi: C \rightarrow D$ we mean a graph function ϕ such that $a\phi \cdot b\phi = (ab)\phi$ for every pair a, b of consecutive arrows in C . If the hom-set functions are injective then ϕ is termed a *faithful morphism*. If the object function is bijective and the hom-set functions are surjective then ϕ is said to be a *quotient morphism*. If the object and the hom-set functions are bijective then ϕ is called an *isomorphism*.

By a *congruence* γ on a semigroupoid C we mean a family

$$\gamma = \{\gamma(i, j): i, j \in \text{Obj}(C)\}$$

of equivalence relations $\gamma(i, j)$ on $C(i, j)$ such that, for every $a \in C(i, j), c, d \in C(j, k)$ and $b \in C(k, l)$ ($i, j, k, l \in \text{Obj}(C)$), the relation $c\gamma(j, k)d$ implies $ac\gamma(i, k)ad$ and $cb\gamma(j, l)db$. For simplicity, we will often write γ instead of $\gamma(i, j)$. Note that $\gamma(i, i)$, which we will denote also by $\gamma(i)$, is a congruence on the local semigroup $C(i)$. If $a \in C(i, j)$ then the equivalence class containing a will be denoted by $a\gamma(i, j)$ or, simply, by $a\gamma$.

Given a congruence γ on a semigroupoid C , we can define the *quotient semigroupoid* C/γ as follows: $\text{Obj}(C/\gamma) = \text{Obj}(C)$, $(C/\gamma)(i, j)$ is the set of all $\gamma(i, j)$ -classes of $C(i, j)$ and the composition rule is given by

$$a\gamma \cdot b\gamma = (ab)\gamma \quad a \in C(i, j), b \in C(j, k).$$

Obviously, the congruence γ on C determines a quotient morphism γ^{\natural} whose object function is identical and whose hom-set functions assign the respective γ -class to each arrow. Conversely, if $\phi: C \rightarrow D$ is a quotient morphism then the family of equivalence relations on the hom-sets in C determined by the hom-set functions is a congruence on C which we will denote by \equiv_{ϕ} . Moreover, ϕ induces an isomorphism $\iota: C/\equiv_{\phi} \rightarrow D$ such that $\phi = \equiv_{\phi}^{\natural} \iota$.

Let X be a graph. A *non-empty path* in X is a sequence of consecutive arrows in X . If $p = e_1 e_2 \cdots e_n$ ($n \geq 1$) where $\alpha(e_1) = i$ and $\omega(e_n) = j$ then we say that p is a non-empty (i, j) -path. If p is a non-empty (i, j) -path and q is a non-empty (j, k) -path for some $i, j, k \in \text{Obj}(X)$ then their concatenation pq is a non-empty (i, k) -path.

Given any graph X , we will consider the *free semigroupoid with involution* on X and denote it by X^{\oplus} . It can be represented in the following way. Firstly let \bar{X} be a graph with involution defined as follows:

$$\text{Obj}(\bar{X}) = \text{Obj}(X) \quad \text{and} \quad \bar{X}(i, j) = X(i, j) \cup X^*(i, j)$$

where $X(i, j) \cap X^*(i, j) = \emptyset$, $*$: $X(i, j) \rightarrow X^*(j, i)$ is a bijection and $*$: $X^*(j, i) \rightarrow X(i, j)$ is its inverse for every $i, j \in \text{Obj}(X)$. Then

$$\text{Obj}(X^\oplus) = \text{Obj}(\overline{X}), \quad X^\oplus(i, j) = \{p: p \text{ is a non-empty } (i, j)\text{-path in } \overline{X}\},$$

the composition being given by concatenation, and the involution $*$ on \overline{X} being extended to an involution of X^\oplus in the usual way: for every path $p = e_1e_2 \cdots e_n$ in \overline{X} , we set $p^* = e_n^*e_{n-1}^* \cdots e_1^*$. However, we will consider X^\oplus as a usual semigroupoid (without a unary operation). In particular, the congruences on X^\oplus will not be required to be compatible with the involution $*$.

Notice that any non-empty path p in X^\oplus spans a subgraph in the graph \overline{X} with involution. We will denote this subgraph by $[p]$. Namely, if $p = e_1e_2 \cdots e_n$ ($n \geq 1$, $e_k \in \text{Arr}(\overline{X})$, $1 \leq k \leq n$) then

$$\text{Obj}([p]) = \{\alpha(e_k): 1 \leq k \leq n\} \cup \{\omega(e_n)\}$$

and

$$[p](i, j) = \{e_1, e_1^*, \dots, e_n, e_n^*\} \cap X(i, j) \quad (i, j \in \text{Obj}([p])).$$

Clearly, a subgraph spanned by a path in X^\oplus is connected.

Let X be a graph and C an orthodox semigroupoid. A graph function $\vartheta: \overline{X} \rightarrow C$ is termed *matched* if $x\vartheta$ and $x^*\vartheta$ are mutual inverses in C for all $x \in \text{Arr}(X)$. The uniquely determined extension of ϑ to a morphism $X^\oplus \rightarrow C$ will be denoted by $\tilde{\vartheta}$.

By an *inverse operation* on an orthodox semigroupoid C we mean a mapping $\prime: \text{Arr}(C) \rightarrow \text{Arr}(C)$ possessing the property that $a' \in V(a)$ for every $a \in \text{Arr}(C)$. Obviously, an inverse operation $\prime: \text{Arr}(C) \rightarrow \text{Arr}(C)$ determines a matched graph function $\vartheta: \overline{C} \rightarrow C$ by defining the object function as the identity mapping and by setting $a\vartheta = a$ and $a^*\vartheta = a'$ for every $a \in \text{Arr}(C)$.

Combining the ideas in [4, Section 1] and [10, Section 9 and Appendix B], the notion of the bivariety of orthodox semigroupoids is introduced and studied in [8]. In particular, an analogue of the bi-invariant congruence corresponding to a bivariety of orthodox semigroups is obtained which we need later.

Roughly speaking, the analogue of the bi-invariant congruence is defined in the following manner. Given a graph X and a bivariety \mathcal{V} of orthodox semigroupoids, we consider the ‘intersection’ of all congruences ϱ on X^\oplus for which $X^\oplus/\varrho \in \mathcal{V}$ and $\overline{X} \rightarrow X^\oplus/\varrho$, $y \mapsto y\varrho$ ($y \in \text{Arr}(\overline{X})$) is a matched graph function. We denote this congruence by $\varrho(\mathcal{V}, X)$. It is also shown that $X^\oplus/\varrho(\mathcal{V}, X)$ is the *bifree object* in \mathcal{V} on the graph X .

To any bivariety \mathbf{V} of orthodox semigroups, we can associate two bivarieties of orthodox semigroupoids:

- $\ell\mathbf{V}$ — the class of all orthodox semigroupoids whose local semigroups belong to \mathbf{V} ; and

$g\mathbf{V}$ — the bivariety of orthodox semigroupoids generated by \mathbf{V} .

Note that $\ell\mathbf{V}$ and $g\mathbf{V}$ are, respectively, the greatest and the least bivarieties of orthodox semigroupoids whose intersection with \mathbf{O} is \mathbf{V} .

A bivariety \mathbf{V} of orthodox semigroups is termed *bilocal* if $\ell\mathbf{V} = g\mathbf{V}$ (or, equivalently, $\ell\mathbf{V} \subseteq g\mathbf{V}$).

The following result shows how one can obtain $\varrho(g\mathbf{V}, X)$ by means of a bi-invariant congruence corresponding to \mathbf{V} . Let X be any graph. Denote $\text{Arr}(X)$ by A . Clearly, there is a natural morphism η from X^\oplus to the free semigroup A^\oplus with involution which maps all objects of X^\oplus to the single object of A^\oplus and which maps each path in X^\oplus to the corresponding word in A^\oplus . For notational convenience, we will often denote the word $p\eta$ corresponding to the path p also by p . Obviously, η is a faithful morphism which also respects the involution. As we have mentioned, $A^\oplus/\rho(\mathbf{V}, A)$ is the bifree object in \mathbf{V} on A . The composite morphism $\eta\rho(\mathbf{V}, A)^\natural: X^\oplus \rightarrow A^\oplus/\rho(\mathbf{V}, A)$ determines a congruence on X^\oplus denoted by $\rho(\mathbf{V}, A)|X^\oplus$ which identifies only the coterminal paths p, q for which $(p, q) \in \rho(\mathbf{V}, A)$ in A^\oplus .

RESULT 1.10. *Let X be a graph and \mathbf{V} a bivariety of orthodox semigroups. We have $\varrho(g\mathbf{V}, X) = \rho(\mathbf{V}, \text{Arr}(X))|X^\oplus$. Consequently, \mathbf{V} is bilocal if and only if, for every graph X , the inclusion $\rho(\mathbf{V}, \text{Arr}(X))|X^\oplus \subseteq \varrho(\ell\mathbf{V}, X)$ holds.*

The main result in [8] is the following.

RESULT 1.11. *Each orthogroup variety is bilocal.*

Finally, we recall the definition and the basic properties of the semidirect product of an orthodox semigroup by a group, formulate the embeddability criterion presented in [9] and relate it to bilocality.

Let S be an orthodox semigroup and G a group. Suppose that G acts on S by automorphisms on the left, that is, for every $g \in G$, an automorphism of S is given, denoted by $g: S \rightarrow S, s \mapsto gs$, such that $h(gs) = (hg)s$ holds for every $g, h \in G$ and $s \in S$. Briefly, we will say only that G acts on S . The *semidirect product* $S * G$ is defined on the underlying set $S \times G$ by the multiplication

$$(s, g)(\bar{s}, \bar{g}) = (s \cdot g\bar{s}, g\bar{g}) \quad (s, \bar{s} \in S, g, \bar{g} \in G).$$

The following properties of the semidirect product are straightforward.

RESULT 1.12. *Let S be an orthodox semigroup and G a group acting on S .*

- (i) *The semidirect product $S * G$ is an orthodox semigroup with $E_{S * G} = \{(e, 1): e \in E_S\}$, and $E_{S * G}$ is isomorphic to E_S .*

- (ii) *The second projection $\pi_2: S * G \rightarrow G$, $(s, g) \mapsto g$ is a homomorphism of $S * G$ onto G with $\ker(\equiv_{\pi_2}) = \{(s, 1) : s \in S\}$ which is isomorphic to S .*

Turning to the embeddability criterion, we introduce first an auxiliary notion, and then present the main construction in [9].

Let S, L be orthodox semigroups and H a group acting on L . Let θ be a group congruence on S . A homomorphism $\xi: S \rightarrow L * H$ is called a *homomorphism over θ* if $\equiv_{\xi\pi_2} \supseteq \theta$. In particular, the attribute ‘over σ ’ can be omitted since $\equiv_{\xi\pi_2} \supseteq \sigma$ holds for every homomorphism ξ .

Let \mathbf{V} be a bivariety of orthodox semigroups, S an orthodox semigroup and θ a group congruence on S . For brevity, denote S/θ by G . Firstly we define a graph C as follows:

$$\text{Obj}(C) = G, \quad C(g, h) = \{(g, s) \in G \times S : g \cdot s\theta = h\} \quad (g, h \in G).$$

One can equip C with the following multiplication: if $(g, s) \in C(g, h)$ and $(h, t) \in C(h, i)$ then $(g, s) \circ (h, t) = (g, st)$. Clearly, $(g, st) \in C(g, i)$ and this multiplication is associative. Thus $C = (C; \circ)$ constitutes a semigroupoid. It is easily checked that C is an orthodox semigroupoid whose local semigroups are isomorphic to $\ker \theta$. Note that this semigroupoid is closely related to the derived semigroupoid of the homomorphism $\theta^b: S \rightarrow G$, (cf. [10]).

Consider the free semigroupoid C^\oplus with involution on the graph C and the faithful morphism $\eta: C^\oplus \rightarrow A^\oplus$ where $A = \text{Arr}(C)$. Let us choose and fix an inverse operation $'$ on S . This determines an inverse operation $'$ on C by setting $(g, s)' = (h, s')$ provided $(g, s) \in C(g, h)$. Denote by ϑ the matched graph function $\bar{C} \rightarrow C$ determined by this inverse operation. The unique extension $\tilde{\vartheta}: C^\oplus \rightarrow C^\oplus$ of ϑ to C^\oplus is a quotient morphism. It thus determines a congruence $\equiv_{\tilde{\vartheta}}$ on the semigroupoid C^\oplus . Denote the image of $\equiv_{\tilde{\vartheta}}$ under η , namely, $\{(p\eta, q\eta) : (p, q) \in \equiv_{\tilde{\vartheta}}\}$, by ν . Consider the congruence τ_V^θ on A^\oplus generated by $\rho(\mathbf{V}, A) \cup \nu$.

REMARK 1.13. It is easily seen that τ_V^θ is obtained in the following way. For any words x, y in A^\oplus , we have $x \tau_V^\theta y$ if and only if there exists a finite sequence of words $x = w_0, w_1, \dots, w_n = y$ such that, for any i ($0 \leq i < n$), the word w_{i+1} is obtained from w_i by one of the following rules:

- (S1) $w_{i+1} \rho(\mathbf{V}, A) w_i$,
- (S2) $w_i = ua^*v, w_{i+1} = ua'v$ for some $u, v \in (A^\oplus)^1$ and $a \in A$,
- (S2') $w_i = ua'v, w_{i+1} = ua^*v$ for some $u, v \in (A^\oplus)^1$ and $a \in A$,
- (S3) $w_i = uabv, w_{i+1} = uc v$ for some $u, v \in (A^\oplus)^1$ and $a, b, c \in A$ with $a \circ b = c$ in C ,
- (S3') $w_i = uc v, w_{i+1} = uabv$ for some $u, v \in (A^\oplus)^1$ and $a, b, c \in A$ with $a \circ b = c$ in C .

Put $K_V^\theta = A^\oplus / \tau_V^\theta$. Since $\rho(\mathbf{V}, A) \subseteq \tau_V^\theta$, we have $K_V^\theta \in \mathbf{V}$. Define a homomorphism $\kappa_V^\theta: S \rightarrow K_V^\theta * G$ over θ by $s\kappa_V^\theta = ((1, s)\tau_V^\theta, s\theta)$ ($s \in S$). It is shown that K_V^θ is, up to isomorphism, independent of the choice of the inverse operation $'$. Moreover, the homomorphism κ_V^θ turns out to be universal among the homomorphisms of S over θ into a semidirect product of a member in \mathbf{V} by a group. Therefore we call it the *canonical homomorphism* of S over θ into a semidirect product of a member in \mathbf{V} by a group. If κ_V^θ is injective then we say that S is *canonically embeddable over θ* into a semidirect product of a member in \mathbf{V} by a group.

The embeddability criterion in [9] is the following.

RESULT 1.14. *Let \mathbf{V} be a bivariety of orthodox semigroups, S an orthodox semigroup and θ a group congruence on S . Then S is embeddable over θ into a semidirect product of a member in \mathbf{V} by a group if and only if S is canonically embeddable, or, equivalently, if and only if the following condition is satisfied:*

$$(1.1) \quad s \theta t \quad \text{and} \quad (1, s)\tau_V^\theta(1, t) \quad \text{imply} \quad s = t \quad \text{for every } s, t \in S.$$

Let \mathbf{V} be a bivariety of orthodox semigroups and \mathbf{K} a group variety. The *semidirect product* of \mathbf{V} by \mathbf{K} is the class

$$\mathbf{V} * \mathbf{K} = \{S * G: S \in \mathbf{V}, G \in \mathbf{K} \text{ and } G \text{ acts on } S\}$$

and the *Mal'cev product* $\mathbf{V} \circ \mathbf{K}$ of \mathbf{V} by \mathbf{K} (within the class of regular semigroups) is the class consisting of all regular semigroups T which possess a congruence θ such that $T/\theta \in \mathbf{K}$ and $\ker \theta \in \mathbf{V}$. It is easily seen that, in fact, we have

$$\mathbf{V} \circ \mathbf{K} = \{T \in \mathbf{O}: \ker \sigma_K \in \mathbf{V}\}.$$

Result 1.12 implies that $IS_r(\mathbf{V} * \mathbf{K}) \subseteq \mathbf{V} \circ \mathbf{K}$. Hence we immediately infer the following corollary by Result 1.14.

RESULT 1.15. *Let \mathbf{V} be a bivariety of orthodox semigroups and \mathbf{K} a variety of groups. Then we have $\mathbf{V} \circ \mathbf{K} = IS_r(\mathbf{V} * \mathbf{K})$ provided (1.1) holds with $\theta = \sigma_K$ for every $S \in \mathbf{O}$ possessing the property that $\ker \sigma_K \in \mathbf{V}$.*

We conclude this section by clearing up the connection between the embeddability of S into a semidirect product of a member in \mathbf{V} by a group and the bilocality of \mathbf{V} .

Suppose that the orthodox semigroup S is embeddable over a group congruence θ into a semidirect product of a member in a bivariety of orthodox semigroups \mathbf{V} by a group. Then $\ker \theta \in \mathbf{V}$ and $C \in \ell\mathbf{V}$ because the local semigroups in C are isomorphic to $\ker \theta$. This implies that $\varrho(\ell\mathbf{V}, C) \subseteq \equiv_{\bar{\vartheta}}$ where ϑ is the matched graph function used in the definition of τ_V^θ . On the other hand, the embeddability condition given in Result 1.14 can be rewritten in the form $\tau_V^\theta|C^\oplus \subseteq \equiv_{\bar{\vartheta}}$. Here $\rho(\mathbf{V}, A) \subseteq \tau_V^\theta$ and so, by Result 1.10, we have $\varrho(g\mathbf{V}, C) = \rho(\mathbf{V}, A)|C^\oplus \subseteq \tau_V^\theta|C^\oplus$. Thus we obtain

PROPOSITION 1.16. *Let \mathbf{V} be a bivariety of orthodox semigroups, S an orthodox semigroup and θ a group congruence on S . If S is embeddable over θ into a semidirect product of a member in \mathbf{V} by a group then the conditions $\ker \theta \in \mathbf{V}$ and $\rho(\mathbf{V}, A)|C^\oplus \subseteq \equiv_{\tilde{\theta}}$ necessarily hold. If \mathbf{V} is bilocal then the first condition implies the second one.*

Note that the second condition can be visualized in the following way: whenever p, q are paths in the graph C which can be interpreted as ‘substitutions’ into the different sides of a valid bi-identity in \mathbf{V} , then the products of p and q in the semigroupoid C are equal. By a ‘substitution’ here we mean that certain arrows a_1, a_2, \dots, a_n are substituted into the variables x_1, x_2, \dots, x_n of the bi-identity and a'_1, a'_2, \dots, a'_n into $x_1^*, x_2^*, \dots, x_n^*$, respectively, where $'$ is the inverse operation defining ϑ .

2. The main result

This section is devoted to proving the following embeddability theorem:

THEOREM 2.1. *Let \mathbf{V} be a variety of regular orthogroups. If S is an orthodox semigroup and θ is a group congruence on it such that $\ker \theta \in \mathbf{V}$ then S is embeddable over θ into a semidirect product of a member in \mathbf{V} by a group.*

The following corollary can be easily deduced from this theorem by Results 1.14 and 1.15

COROLLARY 2.2. *For every regular orthogroup variety \mathbf{V} and group variety \mathbf{K} , we have $\mathbf{V} \circ \mathbf{K} = IS_r(\mathbf{V} * \mathbf{K})$.*

Observe that if \mathbf{V} is an orthogroup variety with $\mathbf{S} \not\subseteq \mathbf{V}$ then \mathbf{V} is a variety of rectangular groups. Each orthodox semigroup S having a group congruence θ with $\ker \theta$ a rectangular group is itself a rectangular group. In this case, S is a direct product $E \times G$ of a rectangular band E and a group G , and θ is determined by a normal subgroup N in G as follows: $(e_1, g_1)\theta(e_2, g_2)$ if and only if $g_1 \in Ng_2$. It is well known that G can be embedded into a semidirect product $M * (G/N)$ where M belongs to the group variety generated by N . Such an embedding can be obviously extended to an embedding of $E \times G$ into a semidirect product $(E \times M) * (G/N)$ which proves Theorem 2.1 in case $\mathbf{S} \not\subseteq \mathbf{V}$. Therefore, in the rest of the section, we will suppose that $\mathbf{S} \subseteq \mathbf{V}$.

In order to prove Theorem 2.1, we can restrict ourselves to left regular orthogroup varieties \mathbf{V} . For, suppose that Theorem 2.1 holds for left regular orthogroup varieties \mathbf{V} . By Result 1.14, this implies that if S is an orthodox semigroup and θ is a group congruence on it with $\ker \theta \in \mathbf{V}$ then the canonical homomorphism κ_V^θ is injective.

Dually, we see that the same holds provided \mathbf{V} is a right regular orthogroup variety. Now let \mathbf{V} be a regular orthogroup variety, S an orthodox semigroup and θ a group congruence on it such that $\ker \theta \in \mathbf{V}$. Since $E \subseteq \ker \theta$, E is necessarily regular. It is well known from [12] that, in this case, S possesses a left regular orthodox semigroup congruence ϵ_0 and a right regular orthodox semigroup congruence ϵ_1 such that $\epsilon_0, \epsilon_1 \subseteq \gamma$ and $\epsilon_0 \cap \epsilon_1$ is identical. Thus we also have $\epsilon_0, \epsilon_1 \subseteq \theta$, and S is a subdirect product of $S_0 = S/\epsilon_0$ and $S_1 = S/\epsilon_1$. Moreover, $\theta_i = \theta/\epsilon_i$ ($i = 0, 1$) is a group congruence on S_i such that S_i/θ_i is isomorphic to S/θ , and we have $\ker \theta_0 \in \mathbf{V} \cap \mathbf{LROG} = \mathbf{V}^{(0)}$ and $\ker \theta_1 \in \mathbf{V} \cap \mathbf{RROG} = \mathbf{V}^{(1)}$. Hence, by assumption, the canonical homomorphism $\kappa_{\mathbf{V}^{(i)}}^{\theta_i}$ of S_i ($i = 0, 1$) is injective. Identifying S_i/θ_i ($i = 0, 1$) with S/θ , we can define an action of S/θ on $K_{\mathbf{V}^{(0)}}^{\theta_0} \times K_{\mathbf{V}^{(1)}}^{\theta_1}$ componentwise. The mapping

$$\begin{aligned} \kappa: S_0 \times S_1 &\rightarrow (K_{\mathbf{V}^{(0)}}^{\theta_0} \times K_{\mathbf{V}^{(1)}}^{\theta_1}) * (S/\theta), \\ (s_0, s_1) &\mapsto (((s_0\kappa_{\mathbf{V}^{(0)}}^{\theta_0})\pi_1, (s_1\kappa_{\mathbf{V}^{(1)}}^{\theta_1})\pi_1), (s_0\kappa_{\mathbf{V}^{(0)}}^{\theta_0})\pi_2 = (s_1\kappa_{\mathbf{V}^{(1)}}^{\theta_1})\pi_2), \end{aligned}$$

where π_1 and π_2 denote the first and the second projection, respectively, of a cartesian product, is obviously an embedding over θ . Since $K_{\mathbf{V}^{(0)}}^{\theta_0} \times K_{\mathbf{V}^{(1)}}^{\theta_1} \in \mathbf{V}$, the reduction of Theorem 2.1 to the case of left regular orthogroup varieties \mathbf{V} is complete.

Thus our goal is to prove that $\kappa_{\mathbf{V}}^{\theta}$ is injective provided $\mathbf{S} \subseteq \mathbf{V} \subseteq \mathbf{LROG}$. Throughout the section, we fix \mathbf{V} , S and θ . Moreover, we define two further congruences on S :

$$\delta_0 = \mu \cap \theta \quad \text{and} \quad \gamma_0 = \delta_0 \vee \gamma.$$

We will need the following modification of Result 1.2(iii).

LEMMA 2.3. *We have $\gamma_0 \cap \mathcal{R} = \delta_0$.*

PROOF. By Result 1.2(iii), we obtain that $(\delta_0 \vee \gamma) \cap \mathcal{R} \subseteq (\mu \vee \gamma) \cap \mathcal{R} = \mu$. On the other hand, we have $\gamma_0 \cap \mathcal{R} \subseteq \gamma \subseteq \theta$. Thus the inclusion ‘ \subseteq ’ is verified. The reverse inclusion is obvious since $\delta_0 \subseteq \mu \subseteq \mathcal{R}$.

The inverse operation ‘ \prime ’ involved in the definition of $\tau_{\mathbf{V}}^{\theta}$ will be chosen in such a way that δ_0 be compatible with it, that is, so that $s \delta_0 t$ implies $s' \delta_0 t'$ for every $s, t \in S$. Since θ is a group congruence, an inverse operation possesses this property if and only if μ is compatible with ‘ \prime ’. The next statement verifies that such an inverse operation exists.

LEMMA 2.4. *Let us fix an idempotent e_L in each \mathcal{L} -class L in E , and choose an inverse s' of s for every $s \in S$ such that $e_L \mathcal{R} s'$ provided $s \mathcal{L} e_L$. Then μ is compatible with the inverse operation ‘ \prime ’.*

PROOF. It is well known that such an inverse s' exists. Assume that $s, t \in S$ with $s \mu t$. Since $\mu \vee \gamma$ is an inverse semigroup congruence on S , we clearly have $s' (\mu \vee \gamma) t'$. Since $\mu \subseteq \mathcal{L}$, we have also $s' \mathcal{R} t'$ by definition. Hence, by Result 1.2(iii), we obtain that $s' \mu t'$, completing the proof.

From now on, we fix an inverse operation $'$ with which δ_0 is compatible. A crucial property of the relation δ_0 is the following:

LEMMA 2.5. *The relation θ/δ_0 is a group congruence on S/δ_0 and, if $(\rho(\mathbf{V}))_0 \subseteq \Delta$, then $\ker(\theta/\delta_0) \in \mathbf{V}_0$.*

PROOF. Since $\delta_0 \subseteq \theta$, the first statement is clear. Notice that $\ker(\theta/\delta_0) = \{s\delta_0 : s \in \ker\theta\}$. We should show that $\ker(\theta/\delta_0)$ satisfies all the bi-identities $u \hat{=} v$ where $(u, v) \in (\rho(\mathbf{V}, A))_0$ and A is an infinite alphabet. Let $(u, v) \in (\rho(\mathbf{V}, A))_0$ where

$$u = u(a_1, a_1^*, \dots, a_n, a_n^*) \quad \text{and} \quad v = v(a_1, a_1^*, \dots, a_n, a_n^*).$$

Consider $s_1, s_2, \dots, s_n \in \ker\theta$ and $s_1'' \in V(s_1), \dots, s_n'' \in V(s_n)$. We have to verify that $u(\underline{s}) \delta_0 v(\underline{s})$ where $u(\underline{s}) = u(s_1, s_1'', \dots, s_n, s_n'')$ and $v(\underline{s}) = v(s_1, s_1'', \dots, s_n, s_n'')$. Since $s_1, s_1'', \dots, s_n, s_n'' \in \ker\theta$, it is clear that they are θ -related. So it suffices to prove that

$$(2.1) \quad u(\underline{s}) \mu v(\underline{s}).$$

Since $(u, v) \in (\rho(\mathbf{V}, A))_0$, we have $(\bar{u}, \bar{v}) \in \rho(\mathbf{V}, A)$ such that $\bar{u} = ua_{n+1}\bar{u}$ and $\bar{v} = va_{n+1}\bar{v}$ where $A(\bar{u}), A(\bar{v}) \subseteq \{a_1, a_2, \dots, a_n, a_{n+1}\}$ and $a_{n+1} \notin \{a_1, a_2, \dots, a_n\}$. Then Result 1.9 ensures that $(ua_{n+1})\rho(\mathbf{V}, A) \mathcal{R} (va_{n+1})\rho(\mathbf{V}, A)$ in $A^\oplus/\rho(\mathbf{V}, A)$. This implies that, for any $e \in E$, we have $u(\underline{s})e \mathcal{R} v(\underline{s})e$ in S . Hence we see that $u(\underline{s})e(u(\underline{s}))'' \mathcal{R} v(\underline{s})e(v(\underline{s}))''$ for every $(u(\underline{s}))'' \in V(u(\underline{s}))$ and $(v(\underline{s}))'' \in V(v(\underline{s}))$. Since these elements are idempotent, it follows by Result 1.1 that $u(\underline{s})e(u(\underline{s}))'' = v(\underline{s})e(v(\underline{s}))''$. Thus (2.1) holds by Result 1.2(ii), which completes the proof.

Consider $C, A, \tilde{\vartheta}, \tau_{\mathbf{V}}^\emptyset$ and $\kappa_{\mathbf{V}}^\emptyset$ defined as in Section 2 by means of $S, \theta, ' and \mathbf{V}$. For simplicity, we omit the indices θ and \mathbf{V} in $\tau_{\mathbf{V}}^\emptyset$ and $\kappa_{\mathbf{V}}^\emptyset$. By Result 1.14 and Remark 1.13, we should prove the following assertion.

PROPOSITION 2.6. *Whenever $s, t \in S$ with $s \theta t$ and w_0, w_1, \dots, w_n is a sequence of words in A such that $w_0 = (1, s), w_n = (1, t)$ and, for any i ($0 \leq i < n$), the word w_{i+1} is obtained from w_i by one of the rules (S1)–(S3'), then $s = t$.*

In order to reduce the number of cases to be handled, later we will introduce rules containing (S2)–(S3').

From now on, we will fix a sequence of words w_0, w_1, \dots, w_n satisfying the previous assumptions. One can easily see that the subgraph in the graph \bar{C} with involution determined by $A(w_i)$ ($1 \leq i \leq n - 1$) need not be connected. Therefore we will assign a greater subgraph in \bar{C} to each word w in A^\oplus which will turn out to be connected for the words w_0, w_1, \dots, w_n . If there exists i ($0 \leq i < n$) such that w_{i+1} is obtained from w_i by means of rule (S3) or (S3') then we say that the triple $(a, b, c) \in A \times A \times A$ occurs in the sequence w_0, w_1, \dots, w_n . Denote the set of all triples occurring in w_0, w_1, \dots, w_n by \mathcal{T} . If we want to emphasize that rule (S3) or (S3') is applied with some $(a, b, c) \in \mathcal{T}$ then we will simply say that rule (S3) $^{\mathcal{T}}$ or (S3') $^{\mathcal{T}}$, respectively, is applied.

Given a subset $B \subseteq A$, we define $\langle B \rangle$ to be the subgraph in \bar{C} obtained from B by repeated application of \mathcal{T} . More precisely, put $C^0(B) = \{b, b' : b \in B\}$ and, if $k > 0$, then put

$$C^k(B) = C^{k-1}(B) \cup \{a, a', b, b' : (\exists c \in C^{k-1}(B)) (a, b, c) \in \mathcal{T}\} \\ \cup \{c, c' : (\exists a, b \in C^{k-1}(B)) (a, b, c) \in \mathcal{T}\}.$$

Define $\langle B \rangle$ to be the subgraph in \bar{C} determined by the set of arrows $\bigcup_{k=0}^\infty C^k(B)$. In particular, if $B = A(w)$ for some word $w \in A^\oplus$ then we will simply write $\langle w \rangle$ instead of $\langle A(w) \rangle$. Furthermore, if D is a subgraph in \bar{C} then by $\langle D \rangle$ we mean $\langle \text{Arr}(D) \rangle$. Note that if $\text{Obj}(D) = \{\alpha(a), \omega(a) : a \in \text{Arr}(D)\}$ then D is a subgraph in $\langle D \rangle$.

The following lemma formulates several simple but important properties of this construction.

LEMMA 2.7. *Let B be a subset in A .*

- (i) *If B is finite then $\langle B \rangle$ is a finite subgraph in \bar{C} . In particular, $\langle u \rangle$ is finite for every $u \in A^\oplus$.*
- (ii) *If B determines a connected subgraph in \bar{C} then $\langle B \rangle$ is also connected.*

PROOF. It is easily seen by induction that $C^k(B) \subseteq C^0(B) \cup \hat{\mathcal{T}}$ where $\hat{\mathcal{T}} = \{a, a', b, b', c, c' : (a, b, c) \in \mathcal{T}\}$. Thus we have

$$\text{Arr}(\langle B \rangle) \subseteq \{a, a^* : a \in C^0(B) \cup \hat{\mathcal{T}}\}.$$

Since $\hat{\mathcal{T}}$ is finite, statement (i) follows. Statement (ii) is clear because if B determines a connected subgraph in \bar{C} then $C^0(B)$ also determines a connected subgraph, and each extension of a connected subgraph by edges c, c', c^*, c'^* or $a, a', a^*, a'^*, b, b', b^*, b'^*$ appearing in the construction preserves connectedness.

Let $w \in A^\oplus$. We say that w is *connected* if $|A(w)| = 1$ or else $0(w)$ is connected and $\alpha(\bar{0}(w)) \in \text{Obj}(\langle 0(w) \rangle)$. Since $|A(0(w))| < |A(w)|$, this recursive definition is correct.

Now we verify several lemmas on connected words.

LEMMA 2.8. *If $w \in A^\oplus$ is connected then $\langle w \rangle$ is a finite connected subgraph in \bar{C} .*

PROOF. Lemma 2.7(i) implies that $\langle w \rangle$ is a finite subgraph in \bar{C} . In order to show that it is also connected, we proceed by induction on $|A(w)|$. If $|A(w)| = 1$ then the assertion is obvious. Assume that the lemma is valid provided $|A(w)| < N$ ($N > 1$), and let $w \in A^\oplus$ with $|A(w)| = N$. Then $|A(0(w))| < N$, and so the induction hypothesis ensures that $\langle 0(w) \rangle$ is connected. Since $\alpha(\bar{0}(w)) \in \text{Obj}(\langle 0(w) \rangle)$, the subgraph in \bar{C} determined by the set of arrows

$$B = \text{Arr}(\langle 0(w) \rangle) \cup \{a, a'\}, \quad \text{where } a = \begin{cases} \bar{0}(w) & \text{if } \bar{0}(w) \in A, \\ (\bar{0}(w))^* & \text{otherwise,} \end{cases}$$

is connected, so Lemma 2.7(ii) implies that $\langle w \rangle = \langle B \rangle$ is also connected. The former equality holds because $A(w) = A(0(w)\bar{0}(w))$.

LEMMA 2.9. *A word $w = a_1a_2 \cdots a_N$ ($a_1, a_2, \dots, a_N \in \bar{A}$) in A^\oplus is connected if and only if $\alpha(a_j) \in \text{Obj}(\langle a_1a_2 \cdots a_{j-1} \rangle)$ for every j ($1 \leq j \leq N$) provided $A(a_j) \not\subseteq A(a_1a_2 \cdots a_{j-1})$.*

PROOF. Firstly suppose that $w = a_1a_2 \cdots a_N \in A^\oplus$ is connected and $A(a_j) \not\subseteq A(a_1a_2 \cdots a_{j-1})$. Then $0^k(w) = a_1a_2 \cdots a_{j-1}$ and $\bar{0}^k(w) = a_j$ for some k . Applying the definition of connectedness several times, we infer that $\alpha(a_j) \in \text{Obj}(\langle a_1a_2 \cdots a_{j-1} \rangle)$.

Conversely, assume that the cword $w = a_1a_2 \cdots a_N \in A^\oplus$ satisfies the condition in the lemma. We show by induction on j that $a_1a_2 \cdots a_j$ ($1 \leq j \leq N$) is connected. For $j = 1$ the definition applies. Suppose that $u = a_1a_2 \cdots a_{j-1}$ ($j > 1$) is connected. If $A(a_j) \not\subseteq A(u)$ then $0(ua_j) = u$, $\bar{0}(ua_j) = a_j$ and, by assumption, we have $\alpha(a_j) \in \text{Obj}(\langle u \rangle)$. Thus the definition ensures that ua_j is connected. If $A(a_j) \subseteq A(u)$ then $0(ua_j) = 0(u)$ and $\bar{0}(ua_j) = \bar{0}(u)$. Since u is connected, $0(u)$ is connected and $\alpha(\bar{0}(u)) \in \text{Obj}(\langle 0(u) \rangle)$ by definition. This ensures that ua_j is also connected.

From now on, we use \mathbf{U} to denote an arbitrary subvariety in \mathbf{V} containing \mathbf{S} .

LEMMA 2.10. *If $(\rho(\mathbf{U}))_0 \subseteq \Delta$, $w \in A^\oplus$ is connected and $\bar{w} \in A^\oplus$ with $w\rho(\mathbf{U}, A)\bar{w}$ then \bar{w} is also connected.*

PROOF. We proceed by induction on $|A(w)|$. First of all, recall that the assumption $\mathbf{S} \subseteq \mathbf{U}$ implies $A(w) = A(\bar{w})$. Thus, if $|A(w)| = 1$ then the assertion is clearly valid. Suppose that the lemma holds provided $|A(w)| < N$ ($N > 1$), and let $w, \bar{w} \in A^\oplus$ satisfy the assumptions of the lemma with $|A(w)| = N$. Since w is connected, $0(w)$

is connected and $\alpha(\bar{0}(w)) \in \text{Obj}(\langle 0(w) \rangle)$. On the other hand, Result 1.6 implies that $\bar{0}(\bar{w}) = \bar{0}(w)$ and $0(\bar{w}) \rho(\mathbf{U}_0, A) 0(w)$. Since $|A(0(w))| < N$, the induction hypothesis implies that $0(\bar{w})$ is connected and so, by definition, we obtain that \bar{w} is also connected.

The following rule generalizes all of (S2), (S2'), (S3)^ℱ and (S3')^ℱ:

(S4)^ℱ $w_i = usv, w_{i+1} = utv$ for some $u, v \in (A^\oplus)^1$ and coterminal paths $s, t \in C^\oplus$ with $\langle s \rangle = \langle t \rangle$.

LEMMA 2.11. *If $w \in A^\oplus$ is connected and \bar{w} is obtained from w by rule (S4)^ℱ then \bar{w} is also connected.*

PROOF. Since s and t are paths and $\alpha(s) = \alpha(t), \omega(s) = \omega(t)$, Lemma 2.9 implies the lemma.

Now we introduce the property of words which is preserved by rules (S1) and (S4)^ℱ, and so which will play a crucial role in proving Proposition 2.6.

Let $g, h \in G$. We say that a word $w \in A^\oplus$ is $[g, h, \mathbf{U}]$ -connected (with respect to ℱ) if the following conditions are satisfied:

- (W1) $\langle w \rangle$ is a connected graph and $g, h \in \text{Obj}(\langle w \rangle)$;
- (W2) (i) if $(\rho(\mathbf{U}))_0 \subseteq h$ then $\alpha(h(w)) = g$,
- (ii) if $(\rho(\mathbf{U}))_0 \subseteq \Delta$ (and $|A(w)| > 1$) then w is connected.

In particular, it is immediate by Lemma 2.9 that each (g, h) -path is a $[g, h, \mathbf{U}]$ -connected word.

LEMMA 2.12. *If $w \in A^\oplus$ is $[g, h, \mathbf{U}]$ -connected and $\bar{w} \in A^\oplus$ is obtained from w by one of the rules (S1) and (S4)^ℱ then \bar{w} is also $[g, h, \mathbf{U}]$ -connected.*

PROOF. Since $\mathbf{S} \subseteq \mathbf{U}$, we have $A(w) = A(\bar{w})$ provided \bar{w} is obtained from w by rule (S1). Therefore $\langle w \rangle = \langle \bar{w} \rangle$. In the case of (S4)^ℱ this equality follows by definition. Thus \bar{w} satisfies condition (W1) because w does. If $(\rho(\mathbf{U}))_0 \subseteq h$ then $h(w) = h(\bar{w})$ provided \bar{w} is obtained from w either by rule (S1) or by rule (S4)^ℱ such that u is non-empty. Thus $\alpha(h(w)) = \alpha(h(\bar{w}))$ follows. In the remaining case this equality holds by assumption. Therefore property (W2)(i) for w also implies it for \bar{w} . Property (W2)(ii) for \bar{w} follows from that for w by means of Lemmas 2.10 and 2.11.

We will need later the following property of $[g, h, \mathbf{U}]$ -connected words.

LEMMA 2.13. *If $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $w \in A^\oplus$ is $[g, h, \mathbf{U}]$ -connected such that $w = uv$ with $|A(u)| \geq 1$, then u is $[g, \alpha(h(v)), \mathbf{U}_0]$ -connected.*

PROOF. The assumptions on U and w imply by definition that w is connected. Then, by Lemma 2.9, u is also connected and $\alpha(h(v)) \in \langle u \rangle$. This implies by Lemma 2.8 that $\langle u \rangle$ is connected. Furthermore, the condition $(\rho(U))_0 \subseteq \Delta$ implies $(\rho(U))_0 \subseteq h$ by Remark 1.4, and so $g = \alpha(h(w)) = \alpha(h(u))$. Thus we see that u is $[g, \alpha(h(v)), U_0]$ -connected.

The main idea in proving Proposition 2.6 is that we assign paths in $\langle w_i \rangle$ to w_i for every i ($0 \leq i \leq n$) such that these paths have the same label in the following sense.

Let p be a (g, h) -path in C^\oplus . The label $\ell(p)$ of p is defined to be $s \in S$ if $p\tilde{\vartheta} = (g, s)$ in C . Since V is bilocal by Result 1.11, we obtain the following statement from Proposition 1.16.

LEMMA 2.14. *If p, q are coterminal paths in C^\oplus such that $p \rho(V, A) q$ then $\ell(p) = \ell(q)$.*

Now we are able to give the rule which restricts $(S4)^\mathcal{F}$ but still generalizes $(S2)$, $(S2)'$, $(S3)^\mathcal{F}$ and $(S3)'^\mathcal{F}$. This rule will be frequently used later instead of treating $(S2)$ – $(S3)'^\mathcal{F}$ separately.

$(S2-3)^\mathcal{F}$ $w_i = usv, w_{i+1} = utv$ for some $u, v \in (A^\oplus)^1$ and for some coterminal paths $s, t \in C^\oplus$ with $\langle s \rangle = \langle t \rangle$ and $\ell(s) = \ell(t)$.

In the proof of Proposition 2.6 we will need a generalization of Lemma 2.14.

Let $\hat{V} \subseteq V$ and let ϵ be a congruence on S such that $\epsilon \subseteq \theta$ and $\ker(\theta/\epsilon) \in \hat{V}$. For brevity, put $\hat{S} = S/\epsilon$ and $\hat{\theta} = \theta/\epsilon$. Clearly, $\hat{\theta}$ is a group congruence on \hat{S} such that $\hat{S}/\hat{\theta}$ is isomorphic to S/θ . If ϵ is compatible with the inverse operation $'$, that is, $s' \epsilon t'$ provided $s \epsilon t$ in S , then we can define an inverse operation $\hat{'}$ on \hat{S} by $(s\epsilon)^\hat{'} = (s')\epsilon$. Notice that each inverse semigroup congruence is compatible with any inverse operation and $'$ is chosen to be compatible with δ_0 .

Let us construct \hat{C}, \hat{A} and $\hat{\tilde{\vartheta}}$ by means of $\hat{S}, \hat{\theta}$ and $\hat{'}$ as in Section 2. Moreover, define the label $\hat{\ell}(p)$ of a path in \hat{C}^\oplus by means of $\hat{\tilde{\vartheta}}$ as before. Similarly to Lemma 2.14, we obtain $\hat{\ell}(p) = \hat{\ell}(q)$ provided p, q are coterminal paths in \hat{C}^\oplus such that $p \rho(\hat{V}, \hat{A}) q$. Consider the morphism $\phi: C^\oplus \rightarrow \hat{C}^\oplus$ extending the graph function

$$f: \text{Obj}(C) \rightarrow \text{Obj}(\hat{C}), \quad (s\theta) \mapsto (s\epsilon)\hat{\theta},$$

$$f: \overline{C}(g, h) \rightarrow \overline{\hat{C}}(gf, hf), \quad (g, s) \mapsto (gf, s\epsilon) \quad \text{and} \quad (g, s)^* \mapsto (gf, s\epsilon)^*,$$

for $(g, s) \in C(g, h)$. Since $(s\theta) \mapsto (s\epsilon)\hat{\theta}$ is an isomorphism of S/θ onto $\hat{S}/\hat{\theta}$, the morphism ϕ is a quotient morphism, and the image of $\rho(\hat{V}, \hat{A})|C^\oplus$ under ϕ is $\rho(\hat{V}, \hat{A})|\hat{C}^\oplus$. Thus we infer that $\hat{\ell}(p\phi) = \hat{\ell}(q\phi)$ for every coterminal path $p, q \in C^\oplus$ with $p \rho(\hat{V}, \hat{A}) q$. Taking into consideration the definition of $\hat{'}$, we see that $\hat{\ell}(p\phi) =$

$(\ell(p))\epsilon$. Thus we have proved the following assertion. In connection with (ii), recall Lemma 2.5.

LEMMA 2.15. (i) *Let $\hat{V} \subseteq \mathbf{V} \cap \mathbf{SG}$ and let ϵ be an inverse semigroup congruence on S such that $\epsilon \subseteq \theta$ and $\ker(\theta/\epsilon) \in \hat{V}$. Then, for any coterminal paths $p, q \in C^\oplus$ with $p \rho(\hat{V}, A) q$, we have $\ell(p) \in \ell(q)$.*

(ii) *If $(\rho(\mathbf{V}))_0 \subseteq \Delta$ then, for any coterminal paths $p, q \in C^\oplus$ with $p \rho(\mathbf{V}_0, A) q$, we have $\ell(p) \delta_0 \ell(q)$.*

Now, consider the least inverse semigroup congruence γ on S and put $\bar{S} = S/\gamma$. Clearly, $\gamma \subseteq \theta$. Put also $\bar{\theta} = \theta/\gamma$. Since $\ker \theta \in \mathbf{V}$, we see that $\ker \bar{\theta} \in \mathbf{V} \cap \mathbf{SG}$. This implies that $\ker \bar{\theta}$ is included in the kernel of the greatest idempotent separating congruence $\bar{\mu}$ on \bar{S} , and hence $\bar{\theta} \cap \bar{\mu}$ is an idempotent separating congruence on \bar{S} with $\ker(\bar{\theta} \cap \bar{\mu}) = \ker \bar{\theta}$. The latter equality implies that $\bar{\theta} \cap \bar{\mu}$ is an E -unitary congruence on \bar{S} . Denote by ζ the congruence on S such that $\zeta/\gamma = \bar{\theta} \cap \bar{\mu}$. Then ζ is an E -unitary inverse semigroup congruence on S such that $\ker \zeta = \ker \theta$ and $\text{tr} \zeta = \text{tr} \gamma = \mathcal{D}_E$, the least semilattice congruence on E . Furthermore, we clearly have $\zeta \subseteq \theta$. Thus $\ker(\theta/\zeta) \in \mathbf{S}$. Applying Lemma 2.15(i) with $\hat{V} = \mathbf{S}$ and $\epsilon = \zeta$, we infer that $\ell(p) \zeta \ell(q)$ for any coterminal paths $p, q \in C^\oplus$ with $[p] = [q]$. In particular, if p is a loop then $(\ell(p))\zeta$ is an element in the semilattice E/\mathcal{D}_E . This allows us to assign an idempotent in E/\mathcal{D}_E to each pointed finite connected subgraph in \bar{C} . If D is a finite connected subgraph in \bar{C} and $g \in \text{Obj}(D)$ then put $e(D, g) = (\ell(p))\zeta$ where p is a (g, g) -path in C^\oplus spanning D . By means of $e(D, g)$, we will be able to compare the finite connected subgraphs in \bar{C} .

LEMMA 2.16. *Let D, \hat{D} be finite connected subgraphs in \bar{C} , and suppose that D is a subgraph in \hat{D} .*

- (i) *If $g \in \text{Obj}(D)$ then $e(D, g) \geq e(\hat{D}, g)$.*
- (ii) *If $g \in \text{Obj}(D)$ then $e(D, g) = e(\langle D \rangle, g)$.*
- (iii) *If $g, h \in \text{Obj}(D)$ then $e(D, g) > e(\hat{D}, g)$ if and only if $e(D, h) > e(\hat{D}, h)$.*

PROOF. (i) Since D, \hat{D} are finite, it suffices to verify the statement in the special case when

$$(2.2) \quad \text{Arr}(\hat{D}) = \text{Arr}(D) \cup \{a, a^*\} \quad \text{for some } a \in \text{Arr}(C).$$

The general statement follows by induction. Suppose that (2.2) holds and p is a (g, g) -path spanning D . Since \hat{D} is connected, we have $p = qr$ where $\omega(q) = \alpha(a)$ or $\omega(a)$. Thus $\hat{p} = qrqa^0r$, where $a^0 = aa^*$ or a^*a according to these possibilities, is a (g, g) -path spanning \hat{D} . Since $(\ell(p))\zeta \in E/\mathcal{D}_E$, we have $\ell(p)\ell(\hat{p}) = \ell((qr)^2qa^0r) = (\ell(qr))^2\ell(qa^0r)\zeta\ell(qrqa^0r) = \ell(\hat{p})$, and so $e(D, g) = (\ell(p))\zeta \geq (\ell(\hat{p}))\zeta = e(\hat{D}, g)$ in E/\mathcal{D}_E .

(ii) Taking into consideration the definition of $\langle D \rangle$, in the same way as before, we can restrict ourselves to the case when (2.2) holds such that one of the following is valid:

- (a) $a = x \circ y$ for some $x, y \in \text{Arr}(D)$,
- (b) $x = a \circ y$ for some $x, y \in \text{Arr}(D)$,
- (c) $x = y \circ a$ for some $x, y \in \text{Arr}(D)$,
- (d) $a = x'$ for some $x \in \text{Arr}(D)$.

We have a (g, g) -path spanning D of the form $p = qxyr$ in case (a) and of the form $p = qxr$ in the other cases. Clearly,

$$\hat{p} = \begin{cases} qa(xy)^*xyr & \text{in case (a),} \\ qayx^*xr & \text{in case (b),} \\ qyax^*xr & \text{in case (c),} \\ qxaxr & \text{in case (d),} \end{cases}$$

is a (g, g) -path spanning \hat{D} and $\ell(p) = \ell(\hat{p})$. Thus $e(D, g) = (\ell(p))\zeta = (\ell(\hat{p}))\zeta = e(\hat{D}, g)$ which was to be proved.

(iii) By (i), we have $e(D, g) \geq e(\hat{D}, g)$ and $e(D, h) \geq e(\hat{D}, h)$. Moreover, the statement is symmetric in g and h . Therefore it suffices to prove that if $e(D, g) = e(\hat{D}, g)$ then $e(D, h) = e(\hat{D}, h)$. Let us choose (g, g) -paths p and \hat{p} which span D and \hat{D} , respectively. By assumption, we have $\ell(p) \zeta \ell(\hat{p})$. Furthermore, consider a (h, g) -path q and a (g, h) -path r in D . Then qpr and $q\hat{p}r$ are (h, h) -paths spanning D and \hat{D} , respectively. Thus we have $e(D, h) = (\ell(qpr))\zeta = (\ell(q))\zeta(\ell(p))\zeta(\ell(r))\zeta = (\ell(q))\zeta(\ell(\hat{p}))\zeta(\ell(r))\zeta = (\ell(q\hat{p}r))\zeta = e(\hat{D}, h)$. The proof is complete.

Statement (iii) makes it possible to disregard the object g in the definition of $e(D, g)$. Therefore, if D, \hat{D} are subgraphs as in Lemma 2.16 then we will write $e(D) > e(\hat{D})$ or $e(D) = e(\hat{D})$ to mean that $e(D, g) > e(\hat{D}, g)$ or $e(D, g) = e(\hat{D}, g)$ for some/every $g \in \text{Obj}(D)$.

By analogy with the operations $\mathbf{0}$ and $\bar{\mathbf{0}}$ defined in Section 1, we will introduce operations on connected words in the following manner.

Let w be a connected word. Then Lemma 2.9 ensures that each initial segment u of w is connected, and so, by Lemma 2.8, $\langle u \rangle$ is a finite connected subgraph in \bar{C} . Define $\mathbf{0}(w)$ and $\bar{\mathbf{0}}(w)$ as follows:

$$\mathbf{0}(w) = w_1 \quad \text{and} \quad \bar{\mathbf{0}}(w) = a \quad \text{if} \quad w = w_1aw_2$$

$$\text{where} \quad a \in \bar{A} \quad \text{and} \quad e(\langle w_1 \rangle) > e(\langle w_1a \rangle) = e(\langle w \rangle).$$

In particular, if $e(\langle h(w) \rangle) = e(\langle w \rangle)$ then we define $\mathbf{0}(w)$ to be empty and $\bar{\mathbf{0}}(w)$ to be $h(w)$. It is obvious that $\mathbf{0}(w) = \mathbf{0}^k(w)$ and $\bar{\mathbf{0}}(w) = \bar{\mathbf{0}}^k(w)$ for some k .

We will iterate the operations $\mathbf{0}$ and $\overline{\mathbf{0}}$ in the same way as 0 and $\overline{0}$: if $k \geq 2$ and $\mathbf{0}^{k-1}(w)$ is non-empty then we define

$$\mathbf{0}^k(w) = \mathbf{0}(\mathbf{0}^{k-1}(w)) \quad \text{and} \quad \overline{\mathbf{0}}^k(w) = \overline{\mathbf{0}}(\mathbf{0}^{k-1}(w)).$$

In order to be able to control how a path corresponding to a $[g, h, \mathbf{U}]$ -connected word behaves itself modulo $\rho(\mathbf{U} \cap \mathbf{G}, A)$, we will assign a path $w_{(n)}$ to a $[g, h, \mathbf{U}]$ -connected word w in such a way that we connect the edges in w by means of a fixed family (n) of paths. Let D be a finite connected subgraph in \overline{C} . Let us choose an $e \in \text{Obj}(D)$ and an (e, g) -path n_g in D for every $g \in \text{Obj}(D)$. The family $(n) = \{n_g: g \in \text{Obj}(D)\}$ is called a *cone of paths in D* . Put

$$a_{(n)} = n_i a n_j^* \quad \text{for every } a \in D(i, j), \quad (i, j \in \text{Obj}(D)).$$

Notice that $(a^*)_{(n)} = (a_{(n)})^*$, and so it causes no confusion to write $a^*_{(n)}$. Now let $w = w(a_1, a_1^*, \dots, a_k, a_k^*)$ be a word in A^\oplus such that $a_1, a_2, \dots, a_k \in \text{Arr}(D)$. The (e, e) -path $w((a_1)_{(n)}, (a_1^*)_{(n)}, \dots, (a_k)_{(n)}, (a_k^*)_{(n)})$ will be denoted by $w_{(n)}$.

We intend to prove that, in a certain sense, the label of $(w_i)_{(n)}$ for every word w_i in Proposition 2.6 is independent of the choice of the cone of paths (n) . Firstly we consider $p_{(n)}$ for a path p .

LEMMA 2.17. *If p is a (g, h) -path in C^\oplus then $p \rho(\mathbf{G}, A) n_g^* p_{(n)} n_h$ for every cone of paths (n) in a finite connected subgraph D in \overline{C} containing $[p]$.*

PROOF. Let $p = a_1 a_2 \dots a_N$ where $i_k, j_k \in \text{Obj}(D)$ and $a_k \in \overline{A}(i_k, j_k)$ for every k ($1 \leq k \leq N$). Since p is a (g, h) -path, we have $g = i_1, j_1 = i_2, j_2 = i_3, \dots, j_{N-1} = i_N, j_N = h$. Thus

$$\begin{aligned} n_g^* p_{(n)} n_h &= n_g^* (n_{i_1} a_1 n_{j_1}^*) (n_{i_2} a_2 n_{j_2}^*) \dots (n_{i_N} a_N n_{j_N}^*) n_h \\ &= (n_g^* n_g) a_1 (n_{i_2}^* n_{i_2}) a_2 \dots a_{N-1} (n_{i_N}^* n_{i_N}) a_N (n_h^* n_h) \\ &\rho(\mathbf{G}, A) a_1 a_2 \dots a_N = p. \end{aligned}$$

The proof is complete.

Denote by ϵ a congruence on S such that $\epsilon \subseteq \theta$ and $\ker(\theta/\epsilon) \in \mathbf{U} \cap \mathbf{SG}$. Since $\ker(\theta/\epsilon)$ is a full inverse subsemigroup in S/ϵ , we see that $\gamma \subseteq \epsilon$. In particular, if $\mathbf{U} = \mathbf{V}$ then $\epsilon = \gamma$ satisfies the above requirements and if $(\rho(\mathbf{V}))_0 \subseteq \Delta$ and $\mathbf{U} = \mathbf{V}_0$ then γ_0 does.

Let p, D and (n) be as in the previous lemma. If (m) is another cone of paths in D then, clearly, we have $m_g^* p_{(m)} m_h \rho(\mathbf{G}, A) n_g^* p_{(n)} n_h$. Moreover, if q and r are (g, h) -paths spanning D such that $q \rho(\mathbf{U} \cap \mathbf{G}, A) m_g^* p_{(m)} m_h$ and $r \rho(\mathbf{U} \cap \mathbf{G}, A) n_g^* p_{(n)} n_h$

then $A(q) = A(r)$, and so $q \rho(\mathbf{U} \cap \mathbf{SG}, A) r$ follows by Remark 1.8. Hence, applying Lemma 2.15(i) with $\hat{\mathbf{V}} = \mathbf{U} \cap \mathbf{SG}$, we infer that $\ell(q) \in \ell(r)$.

Generalizing this property, we will call a $[g, h, \mathbf{U}, \epsilon]$ -connected word $w \in A^\oplus$ a $[g, h, \mathbf{U}, \epsilon]$ -prepath if $\ell(p) \in \ell(q)$ holds whenever D is a finite connected subgraph in \bar{C} containing $\langle w \rangle$, (m) and (n) are cones of paths in D , and p, q are (g, h) -paths spanning D such that $p \rho(\mathbf{U} \cap \mathbf{G}, A) m_g^* w_{(m)} m_h$ and $q \rho(\mathbf{U} \cap \mathbf{G}, A) n_g^* w_{(n)} n_h$.

As we have seen, an easy consequence of Lemma 2.17 is the following.

LEMMA 2.18. *Let ϵ be a congruence on S such that $\epsilon \subseteq \theta$ and $\ker(\theta/\epsilon) \in \mathbf{U} \cap \mathbf{SG}$. Then each (g, h) -path in C^\oplus is a $[g, h, \mathbf{U}, \epsilon]$ -prepath.*

Now we prove that the property of being a $[g, h, \mathbf{V}, \gamma]$ -prepath is preserved by rules (S1) and (S2-3)^ℱ.

LEMMA 2.19. *Let w, \bar{w} be $[g, h, \mathbf{V}]$ -connected words in A^\oplus such that \bar{w} is obtained from w by one of the rules (S1) and (S2-3)^ℱ. If w is a $[g, h, \mathbf{V}, \gamma]$ -prepath then \bar{w} is also a $[g, h, \mathbf{V}, \gamma]$ -prepath.*

PROOF. Recall firstly that \bar{w} is $[g, h, \mathbf{V}]$ -connected by Lemma 2.12, and we have $\langle w \rangle = \langle \bar{w} \rangle$. Let D be a finite connected subgraph in \bar{C} containing $\langle w \rangle$ and let (m) and (n) be two cones of paths in D . Suppose that p, q, \bar{p}, \bar{q} are (g, h) -paths spanning D such that $p \rho(\mathbf{V} \cap \mathbf{G}, A) m_g^* w_{(m)} m_h$, $q \rho(\mathbf{V} \cap \mathbf{G}, A) n_g^* w_{(n)} n_h$, $\bar{p} \rho(\mathbf{V} \cap \mathbf{G}, A) m_g^* \bar{w}_{(m)} m_h$ and $\bar{q} \rho(\mathbf{V} \cap \mathbf{G}, A) n_g^* \bar{w}_{(n)} n_h$. Then we have $\ell(p) \gamma \ell(q)$ since w is a $[g, h, \mathbf{V}, \gamma]$ -prepath. We should verify that $\ell(\bar{p}) \gamma \ell(\bar{q})$. Obviously, it suffices to show that $\ell(p) \gamma \ell(\bar{p})$ since, similarly, $\ell(q) \gamma \ell(\bar{q})$ also holds, whence the relation to be proved follows.

Assume firstly that \bar{w} is obtained from w by rule (S1). Since p and \bar{p} span D , we have $A(p) = A(\bar{p})$. Moreover, since $w \rho(\mathbf{V} \cap \mathbf{G}, A) \bar{w}$ and $\rho(\mathbf{V} \cap \mathbf{G}, A)$ is a bi-invariant congruence, we obtain that $w_{(m)} \rho(\mathbf{V} \cap \mathbf{G}, A) \bar{w}_{(m)}$. This implies by assumption that $p \rho(\mathbf{V} \cap \mathbf{G}, A) \bar{p}$. Thus we have $p \rho(\mathbf{V} \cap \mathbf{SG}, A) \bar{p}$ by Remark 1.8 and, by applying Lemma 2.15(i) with $\hat{\mathbf{V}} = \mathbf{V} \cap \mathbf{SG}$ and $\epsilon = \gamma$, we infer that $\ell(p) \gamma \ell(\bar{p})$. If \bar{w} is obtained from w by (S2-3)^ℱ then, by Lemma 2.17, we have $p \rho(\mathbf{V} \cap \mathbf{G}, A) m_g^* w_{(m)} m_h \rho(\mathbf{G}, A) (m_g^* u_{(m)} m_i) s (m_j^* v_{(m)} m_h)$ and, similarly, $\bar{p} \rho(\mathbf{V} \cap \mathbf{G}, A) (m_g^* u_{(m)} m_i) t (m_j^* v_{(m)} m_h)$ provided s, t are (i, j) -paths. Hence, if r is a (g, h) -path spanning D , we obtain in the same way as before that $\ell(p) \gamma \ell(rr^* (m_g^* u_{(m)} m_i) s (m_j^* v_{(m)} m_h))$ and $\ell(\bar{p}) \gamma \ell(rr^* (m_g^* u_{(m)} m_i) t (m_j^* v_{(m)} m_h))$. However, since $\ell(s) = \ell(t)$, we see that the right hand sides of these relations are equal, and so $\ell(p) \gamma \ell(\bar{p})$ which was to be proved.

We will need the analogue of this lemma for $\mathbf{0}^M(w)$ and $\mathbf{0}^M(\bar{w})$ ($M \geq 1$). Observe that if \bar{w} is obtained from w by rule (S2-3)^ℱ then $e(\langle us \rangle) = e(\langle ut \rangle)$, and so the following possibilities occur:

- (P1) $\mathbf{0}^M(w) = \mathbf{0}^k(u) = \mathbf{0}^M(\bar{w})$ and $\bar{\mathbf{0}}^M(w) = \bar{\mathbf{0}}^k(u) = \bar{\mathbf{0}}^M(\bar{w})$,
- (P2) $\mathbf{0}^M(w) = us_0$, $\mathbf{0}^M(\bar{w}) = ut_0$ and $s_0\bar{\mathbf{0}}^M(w)$, $t_0\bar{\mathbf{0}}^M(\bar{w})$ are initial segments of s and t , respectively, (s_0 or t_0 may be empty),
- (P3) $\mathbf{0}^M(w) = usv_0$, $\mathbf{0}^M(\bar{w}) = utv_0$, $\bar{\mathbf{0}}^M(w) = \bar{\mathbf{0}}^M(\bar{w})$ and $v_0\bar{\mathbf{0}}^M(w)$ is an initial segment of v .

In particular, if \bar{w} is obtained from w by rules (S2) or (S2') then (P2) is of the form

- (P2)(i) $\mathbf{0}^M(w) = \mathbf{0}^M(\bar{w}) = u$, $\bar{\mathbf{0}}^M(w) = a^*$, $\bar{\mathbf{0}}^M(\bar{w}) = a'$; and
- (P2)(i') $\mathbf{0}^M(w) = \mathbf{0}^M(\bar{w}) = u$, $\bar{\mathbf{0}}^M(w) = a'$, $\bar{\mathbf{0}}^M(\bar{w}) = a^*$,

respectively. If \bar{w} is obtained from w by rules (S3)^ℱ and (S3')^ℱ, respectively, then the following possibilities occur:

- (P2)(ii) $\mathbf{0}^M(w) = \mathbf{0}^M(\bar{w}) = u$, $\bar{\mathbf{0}}^M(w) = a$, $\bar{\mathbf{0}}^M(\bar{w}) = c$,
- (P2)(iii) $\mathbf{0}^M(w) = ua$, $\mathbf{0}^M(\bar{w}) = u$, $\bar{\mathbf{0}}^M(w) = b$, $\bar{\mathbf{0}}^M(\bar{w}) = c$; and
- (P2)(ii') $\mathbf{0}^M(w) = \mathbf{0}^M(\bar{w}) = u$, $\bar{\mathbf{0}}^M(w) = c$, $\bar{\mathbf{0}}^M(\bar{w}) = a$,
- (P2)(iii') $\mathbf{0}^M(w) = u$, $\mathbf{0}^M(\bar{w}) = ua$, $\bar{\mathbf{0}}^M(w) = c$, $\bar{\mathbf{0}}^M(\bar{w}) = b$,

respectively. Within (P2)(iii), we distinguish two subcases denoted by indices 1 and 2 according to $e(\langle u \rangle) > e(\langle ua \rangle)$ or $e(\langle u \rangle) = e(\langle ua \rangle)$. In the former case, we have $\mathbf{0}(ua) = u$ and $\bar{\mathbf{0}}(ua) = a$.

Notice that, in cases (P1) and (P3), we have $\langle \mathbf{0}^M(w) \rangle = \langle \mathbf{0}^M(\bar{w}) \rangle$.

LEMMA 2.20. *Let w, \bar{w} be $[g, h, \mathbf{V}]$ -connected words in A^\oplus such that \bar{w} is obtained from w by one of the rules (S1)–(S3')^ℱ. Assume that $(\rho(\mathbf{V}))_0 \subseteq \Delta$. If $\mathbf{0}^M(w)$ is a $[g, \alpha(\bar{\mathbf{0}}^M(w)), \mathbf{V}_0, \gamma_0]$ -prepath for every M such that $\mathbf{0}^M(w)$ is non-empty then $\mathbf{0}^M(\bar{w})$ is a $[g, \alpha(\bar{\mathbf{0}}^M(\bar{w})), \mathbf{V}_0, \gamma_0]$ -prepath for every M such that $\mathbf{0}^M(\bar{w})$ is non-empty.*

PROOF. By Lemma 2.13, $\mathbf{0}^M(w)$ is a $[g, \alpha(\bar{\mathbf{0}}^M(w)), \mathbf{V}_0]$ -connected word and $\mathbf{0}^M(\bar{w})$ is a $[g, \alpha(\bar{\mathbf{0}}^M(\bar{w})), \mathbf{V}_0]$ -connected word. If \bar{w} is obtained from w by rule (S1) or by rule (S2-3)^ℱ such that $\mathbf{0}^M(w)$ and $\mathbf{0}^M(\bar{w})$ are of the form (P3) then $\mathbf{0}^M(\bar{w})$ is obtained from $\mathbf{0}^M(w)$ by rule (S1) or (S2-3)^ℱ, respectively, with \mathbf{V}_0 instead of \mathbf{V} . Furthermore, we have $\bar{\mathbf{0}}^M(w) = \bar{\mathbf{0}}^M(\bar{w})$. Therefore the argument in the previous lemma with \mathbf{V}_0 and γ_0 instead of \mathbf{V} and γ , respectively, proves the assertion. If $\mathbf{0}^M(w)$ and $\mathbf{0}^M(\bar{w})$ are of the form (P1), (P2)(i), (i'), (ii) or (ii') then $\mathbf{0}^M(w) = \mathbf{0}^M(\bar{w})$, $\alpha(\bar{\mathbf{0}}^M(w)) = \alpha(\bar{\mathbf{0}}^M(\bar{w}))$, and so the assertion is straightforward. Similarly, in case (P2)(iii)₁, we have $\mathbf{0}^{M+1}(w) = u = \mathbf{0}^M(\bar{w})$ and $\alpha(\bar{\mathbf{0}}^{M+1}(w)) = \alpha(\bar{\mathbf{0}}^M(\bar{w}))$, and so the assertion immediately follows from the assumption.

Consider now the case (P2)(iii)₂. By assumption, ua is a $[g, j, \mathbf{V}_0, \gamma_0]$ -prepath where $a \in C(i, j)$. Since ua is $[g, j, \mathbf{V}_0]$ -connected, we see by Lemma 2.13 that u is $[g, i, \mathbf{V}_0]$ -connected. Let D be a finite connected subgraph in \bar{C} containing $\langle u \rangle$, let $(m), (n)$ be two cones of paths in D and let p, q be (g, i) -paths spanning D such

that $p \rho(\mathbf{V}_0 \cap \mathbf{G}, A) m_g^* u_{(m)} m_i$ and $q \rho(\mathbf{V}_0 \cap \mathbf{G}, A) n_g^* u_{(n)} n_i$. Consider the subgraph \hat{D} in \bar{C} obtained from D by joining the edges a and a^* . Clearly, \hat{D} is connected, and therefore $\langle \hat{D} \rangle$ is also connected. Moreover, since $e(\langle u \rangle) = e(\langle ua \rangle)$, we have $e(D) = e(\hat{D})$, and this implies by Lemma 2.16(ii) that $e(D) = e(\langle \hat{D} \rangle)$. Notice that

$$pa \rho(\mathbf{V}_0 \cap \mathbf{G}, A) m_g^* u_{(m)} m_i a \rho(\mathbf{G}, A) m_g^* u_{(m)} m_i a m_j^* m_j = m_g^* (ua)_{(m)} m_j$$

and, similarly, $qa \rho(\mathbf{V}_0 \cap \mathbf{G}, A) n_g^* (ua)_{(n)} n_j$. Thus, since ua is a $[g, j, \mathbf{V}_0, \gamma_0]$ -prepath, we have $\ell(rr^* pa) \gamma_0 \ell(rr^* qa)$ for every (g, i) -path r spanning $\langle \hat{D} \rangle$. Hence $\ell(rr^* paa^*) \gamma_0 \ell(rr^* qaa^*)$ follows. However, we have $e(\langle r \rangle) = e(\langle [p] \rangle) = e(\langle [pa] \rangle) \leq e(\langle [a] \rangle)$. Thus $\ell(rr^* paa^*) \gamma \ell(p)$ and $\ell(rr^* qaa^*) \gamma \ell(q)$. Since $\gamma \subseteq \gamma_0$, we obtain that $\ell(p) \gamma_0 \ell(q)$.

As far as the case (P2)(iii') is concerned, the argument is similar. Let D be a finite subgraph in \bar{C} containing $\langle ua \rangle$, let $(m), (n)$ be two cones of paths in D and let p, q be (g, j) -paths spanning D such that $p \rho(\mathbf{V}_0 \cap \mathbf{G}, A) m_g^* (ua)_{(m)} m_j$ and $q \rho(\mathbf{V}_0 \cap \mathbf{G}, A) n_g^* (ua)_{(n)} n_j$. Now we have $pa^* \rho(\mathbf{V}_0 \cap \mathbf{G}, A) m_g^* u_{(m)} m_i$ and $qa^* \rho(\mathbf{V}_0 \cap \mathbf{G}, A) n_g^* u_{(n)} n_i$ where pa^*, qa^* are (g, i) -paths spanning D . Since u is a $[g, i, \mathbf{V}_0, \gamma_0]$ -prepath by assumption, we see that $\ell(pa^*) \gamma_0 \ell(qa^*)$. Hence $\ell(pa^* a) \gamma_0 \ell(qa^* a)$ follows, whence we conclude that $\ell(p) \gamma_0 \ell(q)$.

Let us call a $[g, h, \mathbf{V}, \gamma]$ -prepath w an *almost* $[g, h, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -path if $\mathbf{0}^M(w)$ is a $[g, \alpha(\bar{\mathbf{0}}^M(w)), \mathbf{V}_0, \gamma_0]$ -prepath for every M ($M \geq 1$) provided $(\rho(\mathbf{V}))_0 \subseteq \Delta$ and $\mathbf{0}^M(w)$ is non-empty. It is immediate from the definition that if $(\rho(\mathbf{V}))_0 \subseteq \Delta$, w is an almost $[g, h, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -path and $\mathbf{0}^M(w)$ is non-empty, then $\mathbf{0}^M(w)$ is an almost $[g, \alpha(\bar{\mathbf{0}}^M(w)), \mathbf{V}_0, \gamma_0; \mathbf{V}_0, \gamma_0]$ -path.

Combining Lemmas 2.12, 2.19 and 2.20, we get the following assertion.

LEMMA 2.21. *Let $w, \bar{w} \in A^\oplus$ be such that \bar{w} is obtained from w by one of the rules (S1)–(S3')^ℱ. If w is an almost $[g, h, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -path, then \bar{w} is also an almost $[g, h, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -path.*

Since each (g, h) -path is an almost $[g, h, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -path by Lemma 2.18, we can immediately deduce the following lemma.

LEMMA 2.22. *Each word w_0, w_1, \dots, w_n in Proposition 2.6 is an almost $[1, g, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -path with $g = s\theta = t\theta$.*

Now we are able to define the paths which we intend to assign to the almost $[g, h, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -paths. In fact, we give our definition for any $[g, h, \mathbf{U}]$ -connected word.

Let us choose and fix a cone of paths in any finite connected subgraph in \bar{C} . Let w be a $[g, h, \mathbf{U}]$ -connected word in A^\oplus and p a path in C^\oplus . We say that p is a $[g, h, \mathbf{U}]$ -extension of w if

- (E1) p is a (g, h) -path which spans $\langle w \rangle$,
- (E2) $p \rho(\mathbf{U} \cap \mathbf{G}, A) n_g^* w_{(n)} n_h$ for the cone of paths (n) fixed in $\langle w \rangle$,
- (E3) (i) if $(\rho(\mathbf{U}))_0 \subseteq h$ then $h(p) = h(w)$,
- (ii) if $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$ is non-empty then $p = p_0 \bar{\mathbf{0}}(w) q$, where p_0 is a $[g, \alpha(\bar{\mathbf{0}}(w)), \mathbf{U}_0]$ -extension of $\mathbf{0}(w)$.

Let us recall that Lemma 2.13 ensures that $\mathbf{0}(w)$ is a $[g, \alpha(\bar{\mathbf{0}}(w)), \mathbf{U}_0]$ -connected word. Furthermore, we have $|A(\mathbf{0}(w))| < |A(w)|$, and so the definition is correct.

The set of all $[g, h, \mathbf{U}]$ -extensions of a $[g, h, \mathbf{U}]$ -connected word w will be denoted by $\tilde{w}[g, h, \mathbf{U}]$.

In the following lemmas we sum up several properties of the sets $\tilde{w}[g, h, \mathbf{V}]$. Firstly we prove that if p is a (g, h) -path and $q \in \tilde{p}[g, h, \mathbf{V}]$ then $\ell(p) = \ell(q)$. In order to be able to proceed by induction, we verify a slightly more general statement.

LEMMA 2.23. *Let \mathbf{U} be either the variety \mathbf{V} or the variety \mathbf{V}_0 provided $(\rho(\mathbf{V}))_0 \subseteq \Delta$. Denote by ϵ the equality relation and δ_0 , respectively. If $q \in \tilde{p}[g, h, \mathbf{U}]$ where p is a (g, h) -path then $\ell(p) \epsilon \ell(q)$.*

PROOF. Since $q \rho(\mathbf{U} \cap \mathbf{G}, A) p$ by Lemma 2.17, we can apply Lemma 2.15(i) with the variety $\mathbf{U} \cap \mathbf{SG}$ and the congruence $\epsilon \vee \gamma$, and we can deduce that $\ell(pr^*r) (\epsilon \vee \gamma) \ell(q)$ for every (g, h) -path r spanning $\langle p \rangle$. However, since

$$(2.3) \quad e([q]) = e(\langle p \rangle) = e([p])$$

by Lemma 2.16(ii), we obtain that

$$(2.4) \quad \ell(p) (\epsilon \vee \gamma) \ell(q).$$

If $(\rho(\mathbf{V}))_0 \not\subseteq \Delta$ then \mathbf{U} is necessarily \mathbf{V} and $q \in \tilde{p}[g, h, \mathbf{V}]$. Taking into account Lemma 2.17, we see by Result 1.6 and Remark 1.7 that $q \rho(\mathbf{V}, A) pr^*r$. Hence $\ell(q) = \ell(pr^*r)$ by Lemma 2.14. Again utilizing (2.3) and the fact that S is a left regular orthodox semigroup, we obtain that $\ell(q) = \ell(p)$.

Now assume that $(\rho(\mathbf{V}))_0 \subseteq \Delta$. If $\mathbf{0}(p)$ is empty then we have $e([p]) = e([h(p)])$ and $h(p) = h(q)$. Thus $\ell(q) \mathcal{R} \ell(h(q)) = \ell(h(p)) \mathcal{R} \ell(p)$. By Result 1.2(i) and Lemma 2.3, respectively, this relation together with (2.4) implies $\ell(p) \epsilon \ell(q)$. In particular, if $|A(p)| = 1$ then this is the case since $A(p) = A(h(p))$ implies $e([p]) = e([h(p)])$.

We proceed by induction on $|A(p)|$. Suppose that the statement of the lemma holds for every path p with $|A(p)| < N$ ($N > 1$), and consider a path p with $|A(p)| = N$. It

remains to consider only the case when $\mathbf{0}(p)$ is non-empty. Then $\mathbf{0}(p)$ is a (g, k) -path, $k = \alpha(\bar{\mathbf{0}}(p))$ and $\mathbf{U}_0 = \mathbf{V}_0$. So the induction hypothesis ensures that $\ell(q_0) \delta_0 \ell(\mathbf{0}(p))$ provided $q_0 \in \widetilde{\mathbf{0}}(p)[g, k, \mathbf{V}_0]$. This relation implies $\ell(q_0 \bar{\mathbf{0}}(p)) \delta_0 \ell(\mathbf{0}(p) \bar{\mathbf{0}}(p))$. Since q is of the form $q_0 \mathbf{0}(p) q_1$ for some $q_0 \in \widetilde{\mathbf{0}}(p)[g, k, \mathbf{V}_0]$ and we have $e([p]) = e([\mathbf{0}(p) \bar{\mathbf{0}}(p)])$, $e([q]) = e([q_0 \bar{\mathbf{0}}(p)])$, therefore we see that $\ell(p) \mathcal{R} \ell(\mathbf{0}(p) \bar{\mathbf{0}}(p))$ and $\ell(q) \mathcal{R} \ell(q_0 \bar{\mathbf{0}}(p))$. Hence we obtain $\ell(p) \mathcal{R} \ell(q)$ because $\delta_0 \subseteq \mathcal{R}$. Again applying Result 1.2(i) and Lemma 2.3, we obtain from (2.4) and the previous relation that $\ell(p) \in \ell(q)$. The proof is complete.

LEMMA 2.24. *Each $[g, h, \mathbf{U}]$ -connected word has a $[g, h, \mathbf{U}]$ -extension.*

PROOF. Let w be a $[g, h, \mathbf{U}]$ -connected word, r a (g, h) -path spanning $\langle w \rangle$, (n) the cone of paths fixed in $\langle w \rangle$ and $p_0 \in \widetilde{\mathbf{0}}(w)[g, \alpha(\bar{\mathbf{0}}(w)), \mathbf{U}_0]$, if exists, provided $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$ is non-empty. Denote the (g, h) -path $rr^*n_g^*w_{(n)}n_h$ by x , and consider the (g, h) -path

$$(2.5) \quad p = \begin{cases} x & \text{if } (\rho(\mathbf{U}))_0 \not\subseteq h, \\ h(w)(h(w))^*x & \text{if either } (\rho(\mathbf{U}))_0 \subseteq h \text{ and } (\rho(\mathbf{U}))_0 \not\subseteq \Delta, \\ & \text{or } (\rho(\mathbf{U}))_0 \subseteq \Delta \text{ and } \mathbf{0}(w) \text{ is empty,} \\ p_0 \bar{\mathbf{0}}(w)(p_0 \bar{\mathbf{0}}(w))^*x & \text{if } (\rho(\mathbf{U}))_0 \subseteq \Delta \text{ and } \mathbf{0}(w) \text{ is non-empty.} \end{cases}$$

The path p possesses the properties (E1) and (E2) by definition. If $(\rho(\mathbf{U}))_0 \not\subseteq \Delta$ or $\mathbf{0}(w)$ is empty then p satisfies (E3)(i) also by definition. In the opposite case, Result 1.3 and Remark 1.4 ensure that $\rho(\mathbf{U}_0) \subseteq h$. Therefore, if p_0 exists, then it fulfils (E3)(i), and so $h(p) = h(p_0) = h(\mathbf{0}(w)) = h(w)$. So p satisfies (E3)(i) also in this case. Thus we have proved the existence of a path in $\tilde{w}[g, h, \mathbf{U}]$ provided either $(\rho(\mathbf{U}))_0 \not\subseteq \Delta$, or $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$ is empty. In particular, this is the case if $|A(w)| = 1$, for, in this case, $A(w) = A(h(w))$, and so $e(\langle w \rangle) = e(\langle h(w) \rangle)$ follows.

We proceed by induction on $|A(w)|$. Suppose that every $[g, h, \mathbf{U}]$ -connected word w with $|A(w)| < N$ ($N > 1$) has a $[g, h, \mathbf{U}]$ -extension. Let w be a $[g, h, \mathbf{U}]$ -connected word with $|A(w)| = N$. It remains to be proved that $\tilde{w}[g, h, \mathbf{U}]$ is non-empty provided $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$ is non-empty. Since $|A(\mathbf{0}(w))| < N$, the induction hypothesis implies that a $p_0 \in \widetilde{\mathbf{0}}(w)[g, \alpha(\bar{\mathbf{0}}(w)), \mathbf{U}_0]$ exists. The above argument verifies that $p \in \tilde{w}[g, h, \mathbf{U}]$, and so the proof is complete.

LEMMA 2.25. *Let \mathbf{U} be either the variety \mathbf{V} or the variety \mathbf{V}_0 provided $(\rho(\mathbf{V}))_0 \subseteq \Delta$. Denote by ϵ the equality relation and δ_0 , respectively. If $w, \bar{w} \in A^\oplus$ are $[g, h, \mathbf{U}]$ -connected words with $w \rho(\mathbf{U}, A) \bar{w}$ and $p \in \tilde{w}[g, h, \mathbf{U}]$, $q \in \tilde{\bar{w}}[g, h, \mathbf{U}]$ then $\ell(p) \in \ell(q)$. In particular, we have $\ell(p) = \ell(q)$ for every $p, q \in \tilde{w}[g, h, \mathbf{V}]$.*

PROOF. Since $\mathbf{S} \subseteq \mathbf{U}$, the assumption $w \rho(\mathbf{U}, A) \bar{w}$ implies $A(w) = A(\bar{w})$, and hence $\langle w \rangle = \langle \bar{w} \rangle$. Thus the assumptions on p and q imply that $[p] = [q] = \langle w \rangle$ and $p \rho(\mathbf{U} \cap \mathbf{G}, A) n_g^* w_{(n)} n_h, q \rho(\mathbf{U} \cap \mathbf{G}, A) n_g^* \bar{w}_{(n)} n_h$ where (n) is the cone of paths fixed in $\langle w \rangle$. Since $\rho(\mathbf{U} \cap \mathbf{G}, A)$ is bi-invariant, we obtain that $w_{(n)} \rho(\mathbf{U} \cap \mathbf{G}, A) \bar{w}_{(n)}$ whence we see that $p \rho(\mathbf{U} \cap \mathbf{G}, A) q$. Therefore we have $p \rho(\mathbf{U} \cap \mathbf{S}\mathbf{G}, A) q$ by Remark 1.8. Since $\epsilon \vee \gamma \subseteq \theta$ and $\ker(\theta/(\epsilon \vee \gamma)) \in \mathbf{U} \cap \mathbf{S}\mathbf{G}$, it follows by Lemma 2.15(i) that

$$(2.6) \quad \ell(p) (\epsilon \vee \gamma) \ell(q).$$

Notice by Result 1.3 that if $(\rho(\mathbf{V}))_0 \not\subseteq \Delta$ then necessarily $\mathbf{U} = \mathbf{V}$. So, as we have seen formerly, $p \rho(\mathbf{V} \cap \mathbf{G}, A) q$ holds. If, moreover, $(\rho(\mathbf{V}))_0 \subseteq h$ then, by definition, we have also $h(p) = h(w) = h(\bar{w}) = h(q)$. Thus Result 1.6 and Remark 1.7 imply that $p \rho(\mathbf{V}, A) q$, and so Lemma 2.14 ensures that $\ell(p) = \ell(q)$.

Now assume that $(\rho(\mathbf{V}))_0 \subseteq \Delta$. By Result 1.2(i) and Lemma 2.3, it suffices to prove that $\ell(p) \mathcal{R} \ell(q)$ since (2.6) is already verified. Taking into consideration the remark after the definition of $\mathbf{0}(w)$ and $\bar{\mathbf{0}}(w)$, the relation $w \rho(\mathbf{U}, A) \bar{w}$ implies

$$(2.7) \quad \bar{\mathbf{0}}(w) = \bar{\mathbf{0}}(\bar{w}) \quad \text{and} \quad \mathbf{0}(w) \rho(\mathbf{V}_0, A) \mathbf{0}(\bar{w})$$

by Result 1.6 since $\mathbf{U}_0 = \mathbf{V}_0$. Moreover, we have $h(w) = h(\bar{w})$ by Remark 1.4. Therefore, if $\mathbf{0}(w)$ is empty then $\mathbf{0}(\bar{w})$ is also empty, and this is the case if and only if $e(\langle w \rangle) = e(\langle h(w) \rangle)$. Then it follows by Lemma 2.16(ii) that $e([p]) = e([h(p)]) = e([q])$. Furthermore, by definition, we have $h(p) = h(w) = h(q)$. Thus we obtain

$$\ell(pp^*) = \ell(h(w)(h(w))^* pp^*) \mathcal{R} \ell(h(w)(h(w))^* qq^*) \mathcal{R} \ell(qq^*)$$

whence $\ell(p) \mathcal{R} \ell(q)$ follows. In particular, this is the case if $|A(w)| = 1$, for $A(w) = A(h(w))$ implies $e(\langle w \rangle) = e(\langle h(w) \rangle)$.

The remaining case is treated by induction. Suppose that the assertion of the lemma holds for $\mathbf{U}, \epsilon, g, h, w, \bar{w}, p, q$ satisfying the assumptions of the lemma and the condition that $|A(w)| < N$ ($N > 1$). Let $\mathbf{U}, \epsilon, g, h, w, \bar{w}, p, q$ be as in the lemma, and assume that $|A(w)| = N$. As we have seen, we need consider only the case when $(\rho(\mathbf{V}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$ is non-empty. By definition, we have $p = p_0 \bar{\mathbf{0}}(w) r$ and $q = q_0 \bar{\mathbf{0}}(\bar{w}) s$ where $p_0 \in \mathbf{0}(w)[g, \alpha(\bar{\mathbf{0}}(w)), \mathbf{V}_0]$ and $q_0 \in \bar{\mathbf{0}}(\bar{w})[g, \alpha(\bar{\mathbf{0}}(\bar{w})), \mathbf{V}_0]$. Since $|A(\mathbf{0}(w))| < N$, (2.7) shows that the induction hypothesis can be applied. We infer $\ell(p_0) \delta_0 \ell(q_0)$, and so $\ell(p_0 \bar{\mathbf{0}}(w)) \delta_0 \ell(q_0 \bar{\mathbf{0}}(\bar{w}))$ follows. However, we have $e([p]) = e(\langle p \rangle) = e(\langle \mathbf{0}(w) \bar{\mathbf{0}}(w) \rangle) = e(\langle p_0 \bar{\mathbf{0}}(w) \rangle) = e([p_0 \bar{\mathbf{0}}(w)])$ by definition and Lemma 2.16(ii). Therefore we see that $\ell(p) \mathcal{R} \ell(pp^*) = \ell(p_0 \bar{\mathbf{0}}(w) r (p_0 \bar{\mathbf{0}}(w) r)^*) \mathcal{R} \ell(p_0 \bar{\mathbf{0}}(w) (p_0 \bar{\mathbf{0}}(w))^*) \mathcal{R} \ell(p_0 \bar{\mathbf{0}}(w))$. Similarly, we have also $\ell(q) \mathcal{R} \ell(q_0 \bar{\mathbf{0}}(\bar{w}))$. Hence we obtain $\ell(p) \mathcal{R} \ell(q)$, for $\delta_0 \subseteq \mathcal{R}$. The proof is complete.

For later use, we formulate a consequence of the previous lemma.

LEMMA 2.26. *Assume that $(\rho(\mathbf{V}))_0 \subseteq \Delta$. Let $u, v \in A^\oplus$ be $[g, h, \mathbf{V}]$ -connected and $[g, i, \mathbf{V}]$ -connected words, respectively, such that $\mathbf{0}(u) = \mathbf{0}(v)$ and $\bar{\mathbf{0}}(u) = \bar{\mathbf{0}}(v)$. Then $\ell(p) \mathcal{R} \ell(q)$ for every $p \in \tilde{u}[g, h, \mathbf{V}]$ and $q \in \tilde{v}[g, i, \mathbf{V}]$.*

PROOF. For brevity, denote $\mathbf{0}(u) = \mathbf{0}(v)$ by w and $\bar{\mathbf{0}}(u) = \bar{\mathbf{0}}(v)$ by a . Put $k = \alpha(a)$. By Lemma 2.13, w is $[g, k, \mathbf{V}_0]$ -connected, and, by definition, we have $p = p_0ap_1$ and $q = q_0aq_1$ where $p_0, q_0 \in \tilde{w}[g, k, \mathbf{V}_0]$. Lemma 2.25 implies that $\ell(p_0) \delta_0 \ell(q_0)$. Thus $\ell(p_0a) \delta_0 \ell(q_0a)$ follows. Since $e(\langle u \rangle) = e(\langle wa \rangle) = e(\langle p_0a \rangle) = e(\langle q_0a \rangle) = e(\langle v \rangle)$, we infer that $\ell(p) \mathcal{R} \ell(pp^*) \mathcal{R} \ell(p_0a(p_0a)^*) \mathcal{R} \ell(p_0a)$ and, similarly, $\ell(q) \mathcal{R} \ell(q_0a)$. Hence we obtain that $\ell(p) \mathcal{R} \ell(q)$ since $\delta_0 \subseteq \mathcal{R}$.

Before proving that the equality $\ell(p) = \ell(q)$ holds provided $w, \bar{w} \in A^\oplus$ are almost $[g, h, \mathbf{V}, \gamma; \mathbf{V}_0, \gamma_0]$ -paths such that \bar{w} is obtained from w by one of the rules (S2)–(S3)^g and $p \in \tilde{w}[g, h, \mathbf{V}]$, $q \in \tilde{\bar{w}}[g, h, \mathbf{V}]$, we verify an auxiliary lemma.

LEMMA 2.27. *Assume that $(\rho(\mathbf{V}))_0 \subseteq \Delta$. If $u \in A^\oplus$ is an almost $[g, i, \mathbf{V}_0, \gamma_0; \mathbf{V}_0, \gamma_0]$ -path and $s \in C^\oplus(i, j)$ then us is an almost $[g, j, \mathbf{V}_0, \gamma_0; \mathbf{V}_0, \gamma_0]$ -path and $\ell(p) \delta_0 \ell(zs)$ for every $p \in \tilde{u}s[g, j, \mathbf{V}_0]$ and $z \in \tilde{u}[g, i, \mathbf{V}_0]$.*

PROOF. Recall first that the sets $\tilde{u}s[g, j, \mathbf{V}_0]$ and $\tilde{u}[g, i, \mathbf{V}_0]$ are non-empty by Lemma 2.24. By Lemma 2.25, it suffices to prove the statement for some p and z . In order to show that us is an almost $[g, j, \mathbf{V}_0, \gamma_0; \mathbf{V}_0, \gamma_0]$ -path, it is enough to verify that us_0 is a $[g, h, \mathbf{V}_0, \gamma_0]$ -prepath for every initial segment $s_0 \in C^\oplus(i, h)$ of s . For, u is supposed to be an almost $[g, i, \mathbf{V}_0, \gamma_0; \mathbf{V}_0, \gamma_0]$ -path, and $\mathbf{0}^M(us)$ is either of the form $\mathbf{0}^k(u)$ or of the form $u\mathbf{0}^k(s)$ for some k ($k \geq 1$). Let D be a finite connected subgraph in \bar{C} containing $\langle us_0 \rangle$, (m) a cone of paths in D and (n) a cone of paths in $\langle u \rangle$. Furthermore, let q be a (g, h) -path spanning D and r a (g, i) -path spanning $\langle u \rangle$ such that $q \rho(\mathbf{V}_0 \cap \mathbf{G}, A) m_g^*(us_0)_{(m)} m_h$ and $r \rho(\mathbf{V}_0 \cap \mathbf{G}, A) n_g^*u_{(n)} n_i$. Since $s_0 \in C^\oplus(i, h)$, Lemma 2.17 implies $qs_0^* \rho(\mathbf{V}_0 \cap \mathbf{G}, A) m_g^*u_{(m)} m_i$. Considering any (i, i) -path y spanning D , we obtain that ryy^* also spans D and $ryy^* \rho(\mathbf{V}_0 \cap \mathbf{G}, A) n_g^*u_{(n)} n_i$. Since u is a $[g, i, \mathbf{V}_0, \gamma_0]$ -prepath by assumption, we infer that $\ell(qs_0^*) \gamma_0 \ell(ryy^*)$. However, we have $e(\langle s_0 \rangle) \geq e(\langle q \rangle) = e(\langle y \rangle)$, therefore $\ell(qs_0^*s_0) \gamma \ell(q)$ and $\ell(ryy^*s_0) \gamma \ell(rs_0s_0^*yy^*s_0) \gamma \ell(rs_0)$. Since $\gamma \subseteq \gamma_0$, we obtain that $\ell(q) \gamma_0 \ell(rs_0)$. Since (m) is an arbitrary cone of paths in D , we have verified on the one hand, that us_0 is a $[g, h, \mathbf{V}_0, \gamma_0]$ -prepath. On the other hand, if we choose $s_0 = s$, $D = \langle us \rangle$, (m) and (n) to be the cones of paths fixed in $\langle us \rangle$ and $\langle u \rangle$, respectively, and $q = p$, $r = z$, then we obtain that $\ell(p) \gamma_0 \ell(zs)$ for every $p \in \tilde{u}s[g, j, \mathbf{V}_0]$ and $z \in \tilde{u}[g, i, \mathbf{V}_0]$.

By Lemma 2.3, it remains to prove that $\ell(p) \mathcal{R} \ell(zs)$ for some $p \in \tilde{u}s[g, j, \mathbf{V}_0]$ and $z \in \tilde{u}[g, i, \mathbf{V}_0]$. We proceed by induction on $|A(s)|$. If $|A(s)| = 0$ then we can

choose $p = z$, and the relation $\ell(p) \delta_0 \ell(zs)$ is obviously valid. Suppose that the relation $\ell(p) \delta_0 \ell(zs)$ holds for some $p \in \tilde{u}s[g, j, \mathbf{V}_0]$ and $z \in \tilde{u}[g, i, \mathbf{V}_0]$ provided $|A(s)| < N$ ($N > 0$), and consider u and s with $|A(s)| = N$ which fulfil the assumptions of the lemma.

If $e(\langle us \rangle) = e(\langle u \rangle)$ then we have $\mathbf{0}(us) = \mathbf{0}(u)$ and $\bar{\mathbf{0}}(us) = \bar{\mathbf{0}}(u)$. Thus $e(\langle p \rangle) = e(\langle zs \rangle) = e(\langle z \rangle)$ follows, and so

$$\ell(p) \mathcal{R} \ell(pp^*) \mathcal{R} \ell(y\bar{\mathbf{0}}(u)(y\bar{\mathbf{0}}(u))^*) \mathcal{R} \ell(zz^*) \mathcal{R} \ell(zs(zs)^*) \mathcal{R} \ell(zs)$$

where $y \in \widetilde{\mathbf{0}}(u)[g, \alpha(\bar{\mathbf{0}}(u)), \mathbf{V}_0]$ provided $\mathbf{0}(u)$ is non-empty and y is empty otherwise.

If $e(\langle us \rangle) < e(\langle u \rangle)$ then $\mathbf{0}(us) = us_0$ (s_0 may be empty) and $s_0\bar{\mathbf{0}}(us)$ is an initial segment of s . Assume that $s_0 \in C^\oplus(i, h)$ provided it is non-empty, and put $h = i$ if s_0 is empty. Since $|A(s_0)| < N$, the induction hypothesis implies that $\ell(p_0) \delta_0 \ell(zs_0)$ is valid for some $p_0 \in \tilde{u}s_0[g, h, \mathbf{V}_0]$ and $z \in \tilde{u}[g, i, \mathbf{V}_0]$. Since δ_0 is a congruence, we infer $\ell(p_0\bar{\mathbf{0}}(us)) \delta_0 \ell(zs_0\bar{\mathbf{0}}(us))$. However, $e(\langle p_0\bar{\mathbf{0}}(us) \rangle) = e(\langle us \rangle) = e(\langle p \rangle) = e(\langle zs \rangle) = e(\langle zs_0\bar{\mathbf{0}}(us) \rangle)$ by Lemma 2.16(ii). So, on the one hand, we obtain for p defined by means of p_0 as in (2.5) that $\ell(p) \mathcal{R} \ell(p_0\bar{\mathbf{0}}(us))$. On the other hand, we see that $\ell(zs) \mathcal{R} \ell(zs_0\bar{\mathbf{0}}(us))$. Since $\delta_0 \subseteq \mathcal{R}$, we infer $\ell(p) \mathcal{R} \ell(zs)$, completing the proof.

LEMMA 2.28. *Let \mathbf{U} be either the variety \mathbf{V} or the variety \mathbf{V}_0 provided $(\rho(\mathbf{V}))_0 \subseteq \Delta$. Denote by ϵ the equality relation and δ_0 , respectively. Let $w, \bar{w} \in A^\oplus$ be almost $[g, h, \mathbf{U}, \epsilon \vee \gamma; \mathbf{U}_0, \gamma_0]$ -paths such that \bar{w} is obtained from w by one of the rules (S2)–(S3)^g. Then we have $\ell(p) \epsilon \ell(q)$ for every $p \in \tilde{w}[g, h, \mathbf{U}]$ and $q \in \bar{w}[g, h, \mathbf{U}]$.*

PROOF. By Lemma 2.25, it suffices to show that $\ell(p) \epsilon \ell(q)$ for some $p \in \tilde{w}[g, h, \mathbf{U}]$ and $q \in \bar{w}[g, h, \mathbf{U}]$. Clearly, we have $\langle w \rangle = \langle \bar{w} \rangle$. Let us choose a (g, h) -path r spanning $\langle w \rangle$, and suppose by Lemma 2.24 that $p_0 \in \mathbf{0}(\tilde{w})[g, \alpha(\bar{\mathbf{0}}(w)), \mathbf{U}_0]$ and $q_0 \in \mathbf{0}(\bar{w})[g, \alpha(\bar{\mathbf{0}}(\bar{w})), \mathbf{U}_0]$ provided $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$ and $\mathbf{0}(\bar{w})$ are non-empty. Later we will choose p_0 and q_0 in an appropriate way. Denote the cone of paths fixed in $\langle w \rangle$ by (n) . By means of w, p_0 and \bar{w}, q_0 , respectively, let us define p and q as in (2.5). By Result 1.2(i) and Lemma 2.3, we should verify that $\ell(p) ((\epsilon \vee \gamma) \cap \mathcal{R}) \ell(q)$.

Firstly we show that $\ell(p) (\epsilon \vee \gamma) \ell(q)$. Since \bar{w} is obtained from w by rule (S2-3)^g, by definition, we have $w_{(n)} = u_{(n)}s_{(n)}v_{(n)}$ and $\bar{w}_{(n)} = u_{(n)}t_{(n)}v_{(n)}$. Here, by Lemma 2.17, we have $s_{(n)} \rho(\mathbf{G}, A) n_i s n_j^*$ and $t_{(n)} \rho(\mathbf{G}, A) n_i t n_j^*$ provided $s, t \in C^\oplus(i, j)$. Thus we see from the definition of p and q that $p \rho(\mathbf{G}, A) P s Q$ and $q \rho(\mathbf{G}, A) P t Q$ for some paths $P \in C^\oplus(g, i), Q \in C^\oplus(j, h)$ with $[p] = [P s Q], [q] = [P t Q]$. Thus $p \rho(\mathbf{SG}, A) P s Q$ and $q \rho(\mathbf{SG}, A) P t Q$ where $\rho(\mathbf{SG}, A) \subseteq \rho(\mathbf{U} \cup \mathbf{SG}, A)$. Applying Lemma 2.15(i) with the variety $\mathbf{U} \cap \mathbf{SG}$ and the congruence $\epsilon \vee \gamma$, we deduce that $\ell(p) (\epsilon \vee \gamma) \ell(P s Q)$ and $\ell(q) (\epsilon \vee \gamma) \ell(P t Q)$. Since $\ell(s) = \ell(t)$, it immediately follows that $\ell(p) (\epsilon \vee \gamma) \ell(q)$.

Turning to the proof of $\ell(p) \mathcal{R} \ell(q)$, observe that $e([p]) = e([r]) = e([q])$ and $e([p]) = e([p_0\bar{\mathbf{0}}(w)])$, $e([q]) = e([q_0\bar{\mathbf{0}}(\bar{w})])$, respectively, provided $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$, $\mathbf{0}(\bar{w})$ are non-empty. So we have $\ell(p) \mathcal{R} \ell(rr^*) \mathcal{R} \ell(q)$ if $(\rho(\mathbf{U}))_0 \not\subseteq h$, and

$$(2.8) \quad \ell(p) \mathcal{R} \ell(h(w)(h(w))^*rr^*), \quad \ell(q) \mathcal{R} \ell(h(\bar{w})(h(\bar{w}))^*rr^*),$$

respectively, if either $(\rho(\mathbf{U}))_0 \subseteq h$ and $(\rho(\mathbf{U}))_0 \not\subseteq \Delta$, or $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$, $\mathbf{0}(\bar{w})$ are empty, and

$$(2.9) \quad \ell(p) \mathcal{R} \ell(p_0\bar{\mathbf{0}}(w)), \quad \ell(q) \mathcal{R} \ell(q_0\bar{\mathbf{0}}(\bar{w})),$$

respectively, if $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$, $\mathbf{0}(\bar{w})$ are non-empty. Hence it is obvious that $\ell(p) \mathcal{R} \ell(q)$ in the cases when $(\rho(\mathbf{U}))_0 \not\subseteq h$, and when u is non-empty and either $(\rho(\mathbf{U}))_0 \subseteq h$ and $(\rho(\mathbf{U}))_0 \not\subseteq \Delta$ or $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and both $\mathbf{0}(w)$ and $\mathbf{0}(\bar{w})$ are empty. In connection with the latter case, let us notice that $\mathbf{0}(w)$ is empty if and only if $\mathbf{0}(\bar{w})$ is empty. For, both $\mathbf{0}(w)$ and $\mathbf{0}(\bar{w})$ are empty by definition if and only if $e([h(u)]) = e(\langle w \rangle)$.

Now assume that u is empty. If $(\rho(\mathbf{U}))_0 \subseteq h$ and $(\rho(\mathbf{U}))_0 \not\subseteq \Delta$ then we have $\mathbf{U} = \mathbf{V}$, and Result 1.6 and Remark 1.7 ensure that $h(w)(h(w))^*rr^* \rho(\mathbf{V}, A) ss^*rr^*$. Hence it follows by Lemma 2.14 that $\ell(h(w)(h(w))^*rr^*) = \ell(ss^*rr^*)$. Similarly, we see that $\ell(h(\bar{w})(h(\bar{w}))^*rr^*) = \ell(tt^*rr^*)$. Since $\ell(ss^*) \mathcal{R} \ell(s) = \ell(t) \mathcal{R} \ell(tt^*)$ and E is left regular, we obtain that $\ell(ss^*) = \ell(tt^*)$. This implies $\ell(ss^*rr^*) = \ell(tt^*rr^*)$ whence we infer by (2.8) that $\ell(p) \mathcal{R} \ell(q)$. If $(\rho(\mathbf{U}))_0 \subseteq \Delta$ and $\mathbf{0}(w)$ is empty then $e([h(s)]) = e(\langle w \rangle)$. Thus, by Lemma 2.16(i) and (ii), we obtain that $e(\langle w \rangle) = e([s]) = e([h(s)])$, and so

$$(2.10) \quad \ell(p) \mathcal{R} \ell(h(s)(h(s))^*) \mathcal{R} \ell(h(s)) \mathcal{R} \ell(s).$$

If $e([h(t)]) = e([t])$ also holds then the equality $e([s]) = e([t])$ implies $e(\langle \bar{w} \rangle) = e([h(t)])$, and so $\mathbf{0}(\bar{w})$ is also empty. Similarly to (2.10), we have $\ell(q) \mathcal{R} \ell(t)$ whence $\ell(p) \mathcal{R} \ell(q)$ follows. In the opposite case, we have $\bar{\mathbf{0}}(\bar{w}) = \bar{\mathbf{0}}(t)$ and $\mathbf{0}(\bar{w}) = \mathbf{0}(t)$ where $\mathbf{0}(t)$ is non-empty. Then, by (2.9), we see that $\ell(q) \mathcal{R} \ell(q_0\bar{\mathbf{0}}(t))$ where $q_0 \in \bar{\mathbf{0}}(t)[g, \alpha(\bar{\mathbf{0}}(t)), \mathbf{U}_0]$. Here $\mathbf{U}_0 = \mathbf{V}_0$ and $\mathbf{0}(t)$ is a path, and so Lemma 2.23 implies $\ell(q_0) \delta_0 \ell(\mathbf{0}(t))$. Hence we infer $\ell(q_0\bar{\mathbf{0}}(t)) \delta_0 \ell(\mathbf{0}(t)\bar{\mathbf{0}}(t)) \mathcal{R} \ell(t)$. Since $\delta_0 \subseteq \mathcal{R}$ we see that $\ell(q) \mathcal{R} \ell(t)$, whence, again, $\ell(p) \mathcal{R} \ell(q)$ follows. The case when $\mathbf{0}(w)$ is non-empty and $\mathbf{0}(\bar{w})$ is empty is treated similarly.

In particular, if $\min(|A(w)|, |A(\bar{w})|) = 1$ then either $A(w) = A(h(w))$ or $A(\bar{w}) = A(h(\bar{w}))$ which implies in case $(\rho(\mathbf{U}))_0 \subseteq \Delta$ that $\mathbf{0}(w)$ or $\mathbf{0}(\bar{w})$, respectively, is empty. Thus the argument in the previous paragraphs shows that the statement in the lemma is valid in this case.

We proceed by induction on $\min(|A(w)|, |A(\bar{w})|)$. Let us suppose that the lemma holds provided $\min(|A(w)|, |A(\bar{w})|) < N$ ($N > 1$), and consider words w, \bar{w} with $\min(|A(w)|, |A(\bar{w})|) = N$ which satisfy the requirements in the lemma. It remains to

handle the case when $(\rho(U))_0 \subseteq \Delta$ and both $\mathbf{0}(w)$ and $\mathbf{0}(\bar{w})$ are non-empty. Then we clearly have $U_0 = V_0$. Moreover, $\mathbf{0}(w)$ is an almost $[g, \alpha(\bar{\mathbf{0}}(w)), V_0, \gamma_0; V_0, \gamma_0]$ -path and $\mathbf{0}(\bar{w})$ is an almost $[g, \alpha(\bar{\mathbf{0}}(\bar{w})), V_0, \gamma_0; V_0, \gamma_0]$ -path. We distinguish several cases according to the forms of $\mathbf{0}(w)$ and $\mathbf{0}(\bar{w})$ listed in (P1)–(P3).

(P1) We can choose p_0 and q_0 to be equal, and so $\ell(p) \mathcal{R} \ell(q)$ is straightforward from (2.9).

(P2) By symmetry, it suffices to consider only the cases when \bar{w} is obtained from w by rules (S2) and (S3)^ℱ, that is, we should consider only the subcases (i), (ii) and (iii). Since $\mathbf{0}(\bar{w}) = u$, u is an almost $[g, i, V_0, \gamma_0; V_0, \gamma_0]$ -path where $i = \alpha(\bar{\mathbf{0}}(\bar{w}))$. Put

$$s = \begin{cases} a^* & \text{in subcase (i),} \\ ab & \text{in subcases (ii) and (iii),} \end{cases} \quad \text{and} \quad t = \begin{cases} a' & \text{in subcase (i),} \\ c & \text{in subcases (ii) and (iii),} \end{cases}$$

respectively. Applying Lemma 2.27, we see that if $\hat{p} \in \tilde{u}s[g, \omega(s), V_0]$, $\hat{q} \in \tilde{u}t[g, \omega(t), V_0]$ and $z \in \tilde{u}[g, i, V_0]$ then

$$(2.11) \quad \ell(\hat{p}) \delta_0 \ell(zs) \quad \text{and} \quad \ell(\hat{q}) \delta_0 \ell(zt).$$

Since $\mathbf{0}(us) = \mathbf{0}(w)$, $\bar{\mathbf{0}}(us) = \bar{\mathbf{0}}(w)$ and $\mathbf{0}(ut) = \mathbf{0}(\bar{w})$, $\bar{\mathbf{0}}(ut) = \bar{\mathbf{0}}(\bar{w})$ in all subcases, Lemma 2.26 ensures that $\ell(p) \mathcal{R} \ell(\hat{p})$ and $\ell(q) \mathcal{R} \ell(\hat{q})$, respectively. Thus (2.11) implies that $\ell(p) \mathcal{R} \ell(zs)$ and $\ell(q) \mathcal{R} \ell(zt)$. However, we have $\ell(s) = \ell(t)$, and so $\ell(p) \mathcal{R} \ell(q)$ immediately follows.

(P3) Now $\mathbf{0}(\bar{w})$ is obtained from $\mathbf{0}(w)$ by one of the rules (S2)–(S3')^ℱ, and we have $\min(|A(\mathbf{0}(w))|, |A(\mathbf{0}(\bar{w}))|)$. Therefore, the induction hypothesis ensures that $\ell(p_0) \delta_0 \ell(q_0)$ for every $p_0 \in \mathbf{0}(w)[g, i, V_0]$ and $q_0 \in \mathbf{0}(\bar{w})[g, i, V_0]$ where $i = \alpha(\bar{\mathbf{0}}(w)) = \alpha(\bar{\mathbf{0}}(\bar{w}))$. This implies $\ell(p_0 \bar{\mathbf{0}}(w)) \delta_0 \ell(q_0 \bar{\mathbf{0}}(\bar{w}))$ and so, by (2.9), we have $\ell(p) \mathcal{R} \ell(q)$. The proof is complete.

PROOF OF PROPOSITION 2.6. By Lemma 2.22, the words w_0, w_1, \dots, w_n in Proposition 2.6 are almost $[1, g, V, \gamma; V_0, \gamma_0]$ -paths where $g = s\theta = t\theta$. Obviously, we have $\ell(w_0) = \ell((1, s)) = s$ and $\ell(w_n) = \ell((1, t)) = t$. On the other hand, there exists $p_i \in \tilde{w}_i[1, g, V]$ for $i = 0, 1, \dots, n$ by Lemma 2.24. According to whether w_{i+1} is obtained from w_i by rule (S1) or by one of the rules (S2)–(S3')^ℱ, the equality $\ell(p_i) = \ell(p_{i+1})$ follows from Lemmas 2.25 and 2.28, respectively. Since w_0 and w_n are edges, Lemma 2.23 implies $\ell(w_0) = \ell(p_0)$ and $\ell(w_n) = \ell(p_n)$. Thus $s = \ell(w_0) = \ell(p_0) = \ell(p_1) = \dots = \ell(p_n) = \ell(w_n) = t$, proving Proposition 2.6 and, consequently, Theorem 2.1.

Acknowledgement

This paper was written while the author was a Humboldt fellow at the FB Mathematik, AG 1, Technische Hochschule Darmstadt. The excellent circumstances provided by the fellowship and by the THD are gratefully acknowledged. Research partially supported by Hungarian National Foundation for Scientific Research grant no. 1903. The result of the paper was announced at the Meeting on Semigroup Theory, Oberwolfach, 21-27 July, 1991.

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