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LIMIT PRESERVING SUMMABILITY OF SUBSEQUENCES

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ABSTRACT. The purpose of this paper is to characterize those matrices A with the property that if x is a sequence with limit point σ, then there exists a subsequence y of x such that $\lim Ay=\sigma$.

The direction for this paper has come from two basic sources. First, D. Gaier [3] has investigated necessary and sufficient conditions for a matrix A to have the property that whenever x is a sequence with limit point σ, then there exists a stretching y of x such that $\lim Ay=\sigma$. Second, R. P. Agnew [1], D. F. Dawson [2], and this author [4] have all investigated conditions for a matrix A to have the property that whenever x is a sequence, then there exists a subsequence y of x such that each limit point of x is a limit point of Ay. The investigations conducted by Dawson and this author have utilized certain subclasses of regular summability methods.

The matrix A is a regular summability method if whenever x is convergent, then so is Ax, and $\lim (Ax)_p=\lim x_n$. The familiar Silverman-Toeplitz characterization of regular matrices consists of the three conditions:

(1) $\{a_{pq}\}_{p=1}^{\infty}$ converges to 0, $q=1,2,3,\ldots,$

(2) $\{\sum_q a_{pq}\}_{p=1}^{\infty}$ converges to 1, and

(3) $\sup_p \sum_q |a_{pq}|$ is finite.

The subclass of regular summability methods which both Dawson and this author utilized when considering the preservation of finite limit points was the set of all those matrices that satisfy properties (1) and (2) of the Silverman-Toeplitz regularity conditions. When considering the preservation of infinite limit points, the additional requirement of row finiteness was added. This suggests the following two theorems.

THEOREM 1. *The matrix A has the property that whenever x is a sequence with finite limit point σ, then there exists a subsequence y of x such that* $\lim Ay=\sigma$ *if and only if A satisfies properties* (1) *and* (2) *of regularity.* ■

Proof. We first establish sufficiency by choosing a subsequence y of x satisfying $|y_q-\sigma|<1/(2^q \sup_p |a_{pq}|+1)$ for each q. Since

$$\left|\sum_q a_{pq}y_q-\sigma\right|\le \sum_q |a_{pq}|\,|y_q-\sigma|+|\sigma|\,\left|\sum_q a_{pq}-1\right|$$

and A satisfies properties (1) and (2) of regularity, $\lim Ay=\sigma$.

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The necessity of property (2) of regularity follows from the fact that $\lim Ay = 1$ for some subsequency y of $x = (1, 1, 1, \ldots)$.

To establish the necessity of property (1) of regularity, we assume the contrary and without loss of generality let $|a_{p1}| > 1$, $p = 1, 2, 3, \ldots$. Our desired contradiction will result by demonstrating the existence of a null sequence x of nonzero terms and an increasing sequence of rows $\{p(i)\}_{i=1}^{\infty}$ having the property that if y is any subsequence of x with $y_1 = x_n$, then $|\sum_q a_{p(m),q} y_q| > |x_n|/2$ whenever $m \geq n$.

Let $x_1 = 1$, $p(1) = 1$, and $p(1, n) = n$ for $n = 2, 3, 4, \ldots$. Let $\{x(1, n)\}_{n=2}^{\infty}$ be a null sequence of non-zero terms such that $|a_{p(1),1} x_1 + \sum_{q=2}^{\infty} a_{p(1),q} y_q| > 1$ whenever $\{y_q\}_{q=2}^{\infty}$ is a subsequence of $\{x(1, n)\}_{n=2}^{\infty}$.

Our next step is to establish a subsequence $\{p(2, i)\}_{i=2}^{\infty}$ of $\{p(1, i)\}_{i=2}^{\infty}$ having the property that for each $i \geq 2$

$$|a_{p(2,i),1} x_1| > |x_1|/2, \tag{2-1}$$

$$|a_{p(2,i),1} x_2| > |x_2|/2, \tag{2-2}$$

and

$$|a_{p(2,i),1} x_1 + a_{p(2,i),2} x_2| > |x_1|/2. \tag{2-3}$$

Regardless of our choice for x_2, forms (2-1) and (2-2) follow from the fact that $|a_{p1}| > 1$ for all p. Form (2-3) is dependent upon the character of $s(1) = \{a_{p(1,i),2}/a_{p(1,i),1} x_1\}_{i=2}^{\infty}$ as described in the following three cases:

(2-a) There exists a subsequence $\{a_{p(2,i),2}/a_{p(2,i),1} x_1\}_{i=2}^{\infty}$ of $s(1)$ that converges to zero. Let $x_2 = x(1, 2)$ and without loss of generality assume for all $i \geqslant 2$

$$|a_{p(2,i),1} x_1 + a_{p(2,i),2} x_2| \geq |a_{p(2,i),1} x_1| \, [1 - |x_2| \, |a_{p(2,i),2}/a_{p(2,i),1} x_1|] > |x_1|/2.$$

(2-b) Condition (2-a) fails, but $s(1)$ has a subsequence which converges to a limit $L \neq 0$. Let $x_2 = x(1, m)$ where m is the least positive integer such that $|Lx(1, m)| < \frac{1}{2}$. Without loss of generality we may assume $|a_{p(2,i),1} x_1 + a_{p(2,i),2} x_2| > x_1|/2$ whenever $i \geq 2$.

(2-c) Conditions (2-a) and (2-b) fail. Let $x_2 = x(1, 2)$ and without loss of generality assume for $i \geq 2$

$$|a_{p(2,i),1} x_1 + a_{p(2,i),2} x_2| \geq |a_{p(2,i),1} x_1| \, [|x_2| \, |a_{p(2,i),2}/a_{p(2,i),1} x_1| - 1] > |x_1|/2.$$

Regardless of which of the above three cases holds, let $p(2) = p(2, 2)$ and if $x_2 = x(1, m)$, then choose a subsequence $\{x(2, n)\}_{n=3}^{\infty}$ of $\{x(1, n)\}_{n=m+1}^{\infty}$ such that

$$\left| a_{p(2),1} x_1 + \sum_{q=2}^{\infty} a_{p(2),q} y_{q+1} \right| > |x_1|/2, \tag{2-4}$$

$$\left| a_{p(2),1} x_2 + \sum_{q=2}^{\infty} a_{p(2),q} y_{q+1} \right| > |x_2|/2, \tag{2-5}$$

and

$$\left|a_{p(2),1}x_1+a_{p(2),2}x_2+\sum_{q=3}^{\infty} a_{p(2),q}y_q\right|>|x_1|/2 \tag{2-6}$$

whenever $\{y_q\}_{q=3}^{\infty}$ is a subsequence of $\{x(2, n)\}_{n=3}^{\infty}$.

The next step in our inductive process involves choosing x_3, $p(3)$, and $\{x(3, n)\}_{n=4}^{\infty}$. To do this we must determine a subsequence $\{p(3, i)\}_{i=3}^{\infty}$ of $\{p(2, i)\}_{i=3}^{\infty}$ having the property that for each $i \geq 3$

$$|a_{p(3,i),1}x_1|>|x_1|/2, \tag{3-1}$$

$$|a_{p(3,i),1}x_2|>|x_2|/2, \tag{3-2}$$

$$|a_{p(3,i),1}x_3|>|x_3|/2, \tag{3-3}$$

$$|a_{p(3,i),1}x_1+a_{p(3,i),2}x_2|>|x_1|/2, \tag{3-4}$$

$$|a_{p(3,i),1}x_1+a_{p(3,i),2}x_3|>|x_1|/2, \tag{3-5}$$

$$|a_{p(3,i),1}x_2+a_{p(3,i),2}x_3|>|x_2|/2, \tag{3-6}$$

and

$$|a_{p(3,i),1}x_1+a_{p(3,i),2}x_2+a_{p(3,i),3}x_3|>|x_1|/2. \tag{3-7}$$

Regardless of our eventful choice for x_3, forms (3-1), (3-2), and (3-3) all follow from the fact that $|a_{p1}|>1$ for all p. Form (3-4) follows from the fact that $\{p(3, i)\}_{i=3}^{\infty}$ is to be a subsequence of $\{p(2, i)\}_{i=3}^{\infty}$. Forms (3-5) and (3-6) follow by placing appropriate restrictions on x_3 and $\{p(3, i)\}_{i=3}^{\infty}$ as determined by whichever of the above cases (2-a), (2-b), or (2-c) is applicable. Form (3-7) requires additional restrictions on x_3 and $\{p(3, i)\}_{i=3}^{\infty}$ as determined by considering the three analogous conditions to cases (2-a), (2-b), and (2-c) for

$$s(2)=\{a_{p(2,i),3}/[a_{p(2,i),1}x_1+a_{p(2,i),2}x_2]\}_{i=3}^{\infty}.$$

Let $p(3)=p(3, 3)$ and if $x_3=x(2, m)$, then choose $\{x(3, n)\}_{n=4}^{\infty}$ a subsequence of $\{x(2, n)\}_{n=m+1}^{\infty}$ such that

$$\left|a_{p(3),1}x_1+\sum_{q=2}^{\infty} a_{p(3),q}y_{q+2}\right|>|x_1|/2, \tag{3-8}$$

$$\left|a_{p(3),1}x_2+\sum_{q=2}^{\infty} a_{p(3),q}y_{q+2}\right|>|x_2|/2, \tag{3-9}$$

$$\left|a_{p(3),1}x_3+\sum_{q=2}^{\infty} a_{p(3),q}y_{q+2}\right|>|x_3|/2, \tag{3-10}$$

$$\left|a_{p(3),1}x_1+a_{p(3),2}x_2+\sum_{q=3}^{\infty} a_{p(3),q}y_{q+1}\right|>|x_1|/2, \tag{3-11}$$

$$\left|a_{p(3),1}x_1+a_{p(3),2}x_3+\sum_{q=3}^{\infty} a_{p(3),q}y_{q+1}\right|>|x_1|/2, \tag{3-12}$$

$$\left|a_{p(3),1}x_2+a_{p(3),2}x_3+\sum_{q=3}^{\infty} a_{p(3),q}y_{q+1}\right|>|x_2|/2, \tag{3-13}$$

and

$$(3\text{-}14)\qquad \left|a_{p(3),1}x_1+a_{p(3),2}x_2+a_{p(3),3}x_3+\sum_{q=4}^{\infty} a_{p(3),q}y_q\right|>|x_1|/2$$

whenever $\{y_q\}_{q=4}^{\infty}$ is a subsequence of $\{x(3, n)\}_{n=4}^{\infty}$.

This inductive pattern for defining x and $\{p(i)\}_{i=1}^{\infty}$ may be continued [the next case requires the consideration of fifteen separate forms], and the proof is complete. ∎

THEOREM 2. *The matrix A has the property that whenever x is a sequence with limit point σ (finite or infinite), then there exists a subsequence y of x with* $\lim Ay=\sigma$ *if and only if A is a row finite matrix that satisfies properties* (1) *and* (2) *of regularity.* ∎

Proof. Let $\{a_q\}_{q=1}^{\infty}$ be any null sequence not eventually equal to zero. Without loss of generality assume $a_1\neq 0$. Let $x_q=\max\{1/a_n : 1\leq n\leq q$ and $a_n\neq 0\}$ for $q=1,2,3,\ldots$. Clearly x diverges to infinity, $|x_{q+1}|\geq|x_q|$ for each q, and $|a_qx_q|=1$ whenever $a_q\neq 0$. Thus if y is any subsequence of x, then $\{a_qy_q\}_{q=1}^{\infty}$ has an infinite number of terms with absolute value greater than or equal to 1. Hence $\sum a_qy_q$ diverges, and it follows that A must be row finite. The remaining necessary conditions follow from Theorem 1.

The sufficiency case for a finite limit point σ follows from Theorem 1. To prove the sufficiency case for an infinite limit point σ, we first observe that since A satisfies property (1) of regularity, then for each column q of A there exist but a finite number of rows p such that $q=\max\{n : a_{pn}\neq 0\}$. Also since A satisfies property (2) of regularity, there exist but a finite number of zero rows. Thus it is easy to choose a subsequence y of x which diverges to infinity such that for each non-zero row p of A, $|(Ay)_p|\geq q$, where $q=\max\{n : a_{pn}\neq 0\}$. ∎

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