Can. J. Math., Vol. XXXI, No. 2, 1979, pp. 347–353

ON BERNSTEIN'S INEQUALITY

A. GIROUX, Q. I. RAHMAN AND G. SCHMEISSER

1. Introduction and statement of results. If $p_n(z)$ is a polynomial of degree at most n, then according to a famous result known as Bernstein's inequality (for references see [**4**])

$$(1)\qquad \max_{|z|=1} |p_n'(z)| \leqq n \max_{|z|=1} |p_n(z)|.$$

Here equality holds if and only if $p_n(z)$ has all its zeros at the origin and so it is natural to seek for improvements under appropriate assumptions on the zeros of $p_n(z)$. Thus, for example, it was conjectured by P. Erdös and later proved by Lax [**2**] that if $p_n(z)$ does not vanish in $|z| < 1$, then (1) can be replaced by

$$(2)\qquad \max_{|z|=1} |p_n'(z)| \leqq (n/2) \max_{|z|=1} |p_n(z)|.$$

On the other hand, Turán [**5**] showed that if $p_n(z)$ is a polynomial of degree n having all its zeros in $|z| \leqq 1$, then

$$(3)\qquad \max_{|z|=1} |p_n'(z)| \geqq (n/2) \max_{|z|=1} |p_n(z)|.$$

Thus in (2) as well as in (3) equality holds for those polynomials of degree n which have all their zeros on $|z| = 1$. These results were extended by Malik [**3**] who proved that if $p_n(z)$ does not vanish in $|z| < K$, where $K \geqq 1$, then

$$(4)\qquad \max_{|z|=1} |p_n'(z)| \leqq \frac{n}{1+K} \max_{|z|=1} |p_n(z)|,$$

whereas

$$(5)\qquad \max_{|z|=1} |p_n'(z)| \geqq \frac{n}{1+k} \max_{|z|=1} |p_n(z)|$$

provided $p_n(z)$ is a polynomial of degree n having all its zeros in $|z| \leqq k \leqq 1$. In this connection E. B. Saff mentioned to us the following:

Problem. Let

$$(6)\qquad p_n(z) = \prod_{\nu=1}^{n} (z - z_\nu)$$

be a polynomial having all its zeros in $\operatorname{Re} z \geqq 1$. Is it true that

$$(7)\qquad \max_{|z|=1}|p_n'(z)| \leqq \sum_{\nu=1}^{n} \frac{1}{1+\operatorname{Re} z_\nu} \max_{|z|=1} |p_n(z)| \;?$$

Here equality must hold if in addition the zeros are all real.

Received November 21, 1977.

https://doi.org/10.4153/CJM-1979-039-3 Published online by Cambridge University Press

The interesting fact about this problem is that in (7) each zero is supposed to make a contribution which is independent of the other zeros as well as of the degree of the polynomial. This is rather surprising in view of (2), (3), and a result of Giroux and Rahman [**1**] according to which there exists a polynomial $p_n(z)$ of degree n, satisfying $p_n(1) = 0$ and

$$(8) \quad \max_{|z|=1} |p_n'(z)| \geqq n(1 - c/n^2) \max_{|z|=1} |p_n(z)|,$$

where c is a constant independent of n. Thus, for large n the influence of all the zeros together may be much stronger than n times the contribution of a single zero. This indicates that the rôle of each individual zero cannot be independent of the others unless their location is somehow restricted.

Although the above problem appears to have been around for some time no contribution to it has appeared in print so far. Here we prove

THEOREM 1. *For $1 \leqq n \leqq 2$ the answer in the above problem is affirmative.*

We have no idea whether (7) is true or not for $n \geqq 3$ but we can prove considerably more if $p_n(z)$ happens to be real for real z.

THEOREM 2. *If the polynomial $p_n(z)$ in (6) is real for real z, then*

$$(9) \quad \max_{|z|=1} |p_n'(z)| \leqq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \max_{|z|=1} |p_n(z)|$$

provided all the zeros lie in

$$D = \{z \in \mathbf{C} : \operatorname{Re} z \geqq 0, \quad |z| \geqq 1\}.$$

The example $p_2(z) = (z + 1)(z - 3)$ shows that (9) may not hold if the zeros are not required to lie in D.

We also prove

THEOREM 3. *Provided $p_n'(z)$ is real for real z the answer in the above problem is affirmative if only* $\operatorname{Re} z_\nu \geqq 0, \quad \nu = 1, 2, \ldots, n$.

Next we present an inequality in the opposite direction.

THEOREM 4. *Let the polynomial* (6) *have all its zeros in* $\operatorname{Re} z \geqq 1$. *Then*

$$(10) \quad \max_{|z|=1} |p_n'(z)| \geqq n \prod_{\nu=1}^{n} (1 + \operatorname{Re} z_\nu)^{1/n} \max_{|z|=1} |p_n(z)|^{1-(2/n)}.$$

Equality holds if in addition $p_n(z)$ is real for real z and there is a $\lambda \geqq 1$ such that all the zeros lie in

$$(11) \quad \{z : |z - \lambda + 1| \leqq \lambda\} \cap \{z : \operatorname{Re} z \geqq 1\}.$$

Applying Theorem 2 to the polynomial $z^n p_n(1/z)$ we can easily deduce that if the polynomial (6) is real for real z then

$$(12) \quad \max_{|z|=1} |p_n'(z)| \geqq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \max_{|z|=1} |p_n(z)|$$

provided all the zeros lie in

$$E = \{z \in \mathbf{C} : \operatorname{Re} z \geqq 0, \quad |z| \leqq 1\}.$$

However, we observe that (12) holds under a considerably weaker hypothesis.

THEOREM 5. *Inequality* (12) *holds if all the zeros of* $p_n(z)$ *lie in* $|z| \leqq 1$. *There is equality if the zeros are all positive.*

From Theorem 5 we deduce the following refinement of Malik's result (5).

COROLLARY. *If all the zeros of the polynomial* (6) *lie in* $|z| \leqq 1$ *and* $\rho = (1/n) \sum_1^n |z_\nu|$, *then*

$$\text{(13)} \quad \max_{|z|=1} |p_n'(z)| \geqq \frac{n}{1+\rho} \max_{|z|=1} |p_n(z)|.$$

2. Proofs. *Proof of Theorem* 1. The case $n = 1$ presents no difficulties. Now let

$$\text{(14)} \quad p(z) = (z - z_1)(z - z_2)$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $x_1 \geqq 1$, $x_2 \geqq 1$, y_1, $y_2 \in \mathbf{R}$. Then $p'(z)$ vanishes at $\zeta + i\eta$, where

$$\zeta = (x_1 + x_2)/2, \ \eta = (y_1 + y_2)/2.$$

Here we may assume that $\eta \geqq 0$ and $x_1 \leqq x_2$. In order to prove (7) it is clearly enough to show that

$$\text{(15)} \quad \max_{|z|=1} |p'(z)|/|p(-1)| \leqq 1/(1 + x_1) + 1/(1 + x_2).$$

For fixed x_1, x_2 and η consider the family

$$\mathscr{P}_{x_1,x_2,\eta} = \{f_\lambda(z) = (z - x_1 - i(\eta - \lambda))(z - x_2 - i(\eta + \lambda)) : \lambda \in \mathbf{R}\}.$$

Note that $p(z)$ belongs to the family $\mathscr{P}_{x_1,x_2,\eta}$ and $\max_{|z|=1} |f_\lambda'(z)|$ is the same for each member $f_\lambda(z)$ of the family. It is therefore sufficient to prove (15) for the polynomial $f_\lambda(z)$ in $\mathscr{P}_{x_1,x_2,\eta}$ for which $|f_\lambda(-1)|$ is smallest. Setting

$$A(\lambda) = |\sqrt{(1 + x_1)^2 + (\eta - \lambda)^2}|, \quad B(\lambda) = |\sqrt{(1 + x_2)^2 + (\eta + \lambda)^2}|$$

we see that $|f_\lambda(-1)|$ is smallest when

$$B(\lambda)(\eta - \lambda)/A(\lambda) = A(\lambda)(\eta + \lambda)/B(\lambda),$$

i.e. (see Fig. 1)

$$B(\lambda) \sin \varphi_1 = A(\lambda) \sin \varphi_2 = \mu \quad \text{(say)}.$$

In other words, the line passing through the points $\Omega_1 = A(\lambda)e^{i\varphi_2} - 1$, $\Omega_2 = B(\lambda)e^{i\varphi_1} - 1$ should be parallel to the real axis. Let the points Z_1, Z_2, M, A, O, X_1, N and X_2 of the complex plane correspond to $x_1 + i(\eta - \lambda)$,

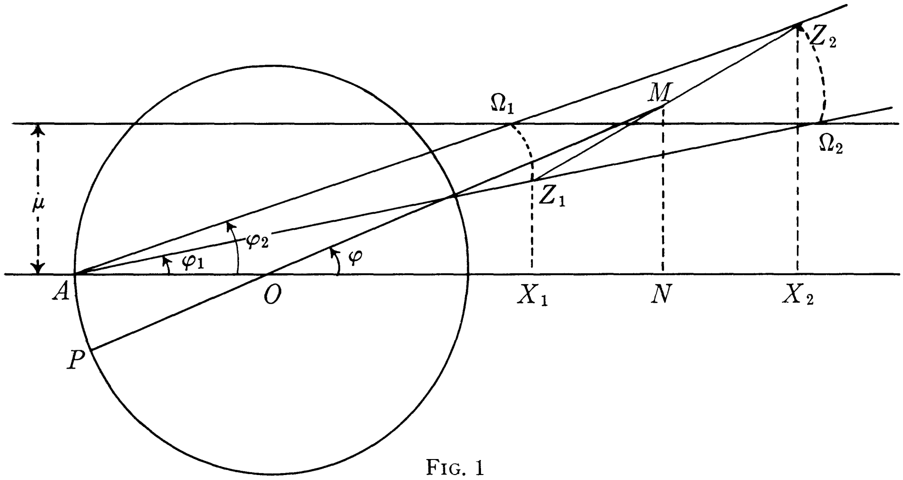

FIG. 1

$x_2 + i(\eta + \lambda)$, $(x_1 + x_2)/2 + i\eta$, -1, 0, x_1, $(x_1 + x_2)/2$ and x_2 respectively. If we denote by $|C, D|$ the distance between two points C, D, then we have to prove that

$$(|A, O| + |O, M|)/|A, Z_1| \cdot |A, Z_2| \leqq (|A, O| + |O, X_1|/2 + |O, X_2|/2) \\ /|A, X_1| \cdot |A, X_2|$$

or equivalently

$$\cos\varphi_1 \cos\varphi_2(|A, O| + |O, M|)/(|A, O| + |O, N|) \leqq 1. \tag{16}$$

Since $|A, X_1| \geqq 1$ we may write $|A, O| = \sigma|A, X_1|$ for some $\sigma \leqq \frac{1}{2}$ and (16) becomes

$$\cos\varphi_1 \cos\varphi_2(\sigma|A, X_1| + \sqrt{|M, N|^2 + (|A, N| - \sigma|A, X_1|)^2}) \\ /|A, N| \leqq 1. \tag{17}$$

But clearly

$$|A, \Omega_2| = \mu/\sin\varphi_1,\ |Z_2, X_2| = \mu \sin\varphi_2/\sin\varphi_1,\ |A, X_2| = \mu\cos\varphi_2/\sin\varphi_1,$$
$$|A, \Omega_1| = \mu/\sin\varphi_2,\ |Z_1, X_1| = \mu \sin\varphi_1/\sin\varphi_2,\ |A, X_1| = \mu\cos\varphi_1/\sin\varphi_2,$$
$$|M, N| = (\mu/2)(\sin\varphi_2/\sin\varphi_1 + \sin\varphi_1/\sin\varphi_2),\ |A, N| \\ = (\mu/2)(\cos\varphi_2/\sin\varphi_1 + \cos\varphi_1/\sin\varphi_2).$$

Hence (17) is equivalent to

$$F(\varphi_1, \varphi_2, \sigma) := \cos\varphi_1 \cos\varphi_2 \\ \times (2\sigma \sin 2\varphi_1 + \sqrt{4(\sin^2\varphi_1 + \sin^2\varphi_2)^2 + (\sin 2\varphi_1 + \sin 2\varphi_2 - 2\sigma \sin 2\varphi_1)^2}) \\ /(\sin 2\varphi_1 + \sin 2\varphi_2) \leqq 1. \tag{18}$$

Calculating $\partial F/\partial\sigma$ we see that the left-hand side of (18) is increased if we replace σ by $\frac{1}{2}$. Hence, it will be enough to prove the inequality

$$\cos\varphi_1 \cos\varphi_2(\sin 2\varphi_1 + 2\sqrt{\sin^2\varphi_2 + 2\sin^2\varphi_1 \sin^2\varphi_2 + \sin^4\varphi_1}) \\ /(\sin 2\varphi_1 + \sin 2\varphi_2) \leqq 1$$

which is equivalent to

$$(19)\quad \tfrac{1}{2}(\sin 2\varphi_1 \sin 2\varphi_2)^2 + (\sin 2\varphi_1 \sin \varphi_1 \cos \varphi_2)^2 \leqq \sin^2 2\varphi_1 \times (1 - \cos \varphi_1 \cos \varphi_2)^2 + 2 \sin 2\varphi_1 \sin 2\varphi_2 (1 - \cos \varphi_1 \cos \varphi_2) + (\sin 2\varphi_2 \sin \varphi_1)^2.$$

This latter inequality will be proved if we show that

$$(20)\quad (\sin 2\varphi_1 \sin \varphi_1 \cos \varphi_2)^2 \leqq (\sin 2\varphi_2 \sin \varphi_1)^2$$

and

$$(21)\quad \tfrac{1}{2}(\sin 2\varphi_1 \sin 2\varphi_2)^2 \leqq 2 \sin 2\varphi_1 \sin 2\varphi_2 (1 - \cos \varphi_1 \cos \varphi_2).$$

In fact, the sum of the left-hand sides of (20), (21) is equal to the left-hand side of (19) whereas the sum of the right-hand sides of (20), (21) is smaller than the right-hand side of (19). Now, as far as inequality (20) is concerned it is obvious. As for inequality (21) it is equivalent to

$$\cos \varphi_1 \cos \varphi_2 (1 + \sin \varphi_1 \sin \varphi_2) \leqq 1$$

or in turn to

$$\{\cos (\varphi_1 - \varphi_2) + 1\}^2 - \{\cos (\varphi_1 + \varphi_2) - 1\}^2 \leqq 4$$

which is certainly true. With this the proof of Theorem 1 is complete.

Proof of Theorem 2. The polynomial $p_n'(z)$ is also real for real z, so that its complex zeros occur in conjugate pairs. Besides, they (the zeros of $p_n'(z)$) all lie in the right half-plane. Hence $\max_{|z|=1} |p_n'(z)|$ is attained at the point -1. To every factor $z - z_\nu$ in (6) where z_ν is non-real there corresponds a factor $z - \bar{z}_\nu$, and therefore we may rearrange the factors to write

$$p_n(z) = \prod_{\nu=1}^{m} \{(z - z_\nu)(z - \bar{z}_\nu)\} \prod_{\nu=2m+1}^{n} (z - z_\nu)$$

where $z_{2m+1}, z_{2m+2}, \ldots, z_n$ are real and $\geqq 1$. Thus

$$\max_{|z|=1} |p_n'(z)| = |p_n'(-1)| = |p_n(-1)| \left| \sum_1^m \{(-1 - z_\nu)^{-1} + (-1 - \bar{z}_\nu)^{-1}\} + \sum_{2m+1}^n (-1 - z_\nu)^{-1} \right|$$
$$\leqq \max_{|z|=1} |p_n(z)| \sum_1^m |(1 + z_\nu)^{-1} + (1 + \bar{z}_\nu)^{-1}| + \sum_{2m+1}^n (1 + z_\nu)^{-1}.$$

Now we note that

$$|(1 + z_\nu)^{-1} + (1 + \bar{z}_\nu)^{-1}| \leqq 2/(1 + |z_\nu|)$$

if $z_\nu \in D$ and hence the desired result follows.

Proof of Theorem 3. Under the assumptions of Theorem 3 we obviously have

$$\max_{|z|=1} |p_n'(z)| = |p_n'(-1)|.$$

Hence

$$\max_{|z|=1} |p_n'(z)| / \max_{|z|=1} |p_n(z)| \leqq |p_n'(-1)|/|p_n(-1)| = \left|\sum_1^n (-1 - x_\nu - iy_\nu)^{-1}\right| \leqq \sum_1^n (1 + x_\nu)^{-1}.$$

Proof of Theorem 4. Without loss of generality we may assume that $\max_{|z|=1} |p_n(z)|$ is attained at a point $-e^{i\varphi}$ where $0 \leqq \varphi < \pi/2$. Putting

$$t(\theta) = |p_n(-e^{i(\varphi+\theta)})|^2$$

we must have $t'(0) = 0$ or equivalently

$$\text{(22)} \quad \operatorname{Im} e^{i\varphi} p_n'(-e^{i\varphi})/p_n(-e^{i\varphi}) = 0.$$

Using the abbreviation

$$A_\nu = (\cos \varphi + x_\nu)^2 + (\sin \varphi + y_\nu)^2$$

we deduce from (22)

$$\text{(23)} \quad -e^{i\varphi} p_n'(-e^{i\varphi})/p_n(-e^{i\varphi}) = \sum_1^n (1 + x_\nu \cos \varphi + y_\nu \sin \varphi)/A_\nu$$

and

$$\text{(24)} \quad 0 = \sum_1^n (x_\nu \sin \varphi - y_\nu \cos \varphi)/A_\nu.$$

These relations can also be written as

$$\text{(23}'\text{)} \quad -e^{i\varphi} p_n'(-e^{i\varphi})/p_n(-e^{i\varphi}) = \sum_1^n (1 + x_\nu)/A_\nu \\ - 2 \sin (\varphi/2) \sum_1^n (x_\nu \sin (\varphi/2) - y_\nu \cos (\varphi/2))/A_\nu$$

and

$$\text{(24}'\text{)} \quad \tan \varphi = \left(\sum_1^n y_\nu/A_\nu\right)/\left(\sum_1^n x_\nu/A_\nu\right),$$

respectively. Since $\tan (\varphi/2) \leqq \tan \varphi$ for $0 \leqq \varphi < \pi/2$ we deduce from (24′)

$$\sum_1^n (x_\nu \sin (\varphi/2) - y_\nu \cos (\varphi/2))/A_\nu \leqq 0.$$

Hence it follows from (23′) that

$$|p_n'(-e^{i\varphi})/p_n(-e^{i\varphi})| \geqq \sum_1^n (1 + x_\nu)/A_\nu.$$

Now using the well-known inequality between the harmonic and the geometric mean, namely

$$(a_1 a_2 \ldots a_n)^{1/n} \geqq n/(a_1^{-1} + a_2^{-1} + \ldots + a_n^{-1})$$

valid for positive numbers a_ν $(\nu = 1, 2, \ldots, n)$, we finally obtain

$$|n\, p_n(-e^{i\varphi})/p_n'(-e^{i\varphi})| \leqq \left(\prod_1^n A_\nu/(1 + x_\nu)\right)^{1/n} \\ = |p_n(-e^{i\varphi})|^{2/n}/\prod_1^n (1 + x_\nu)^{1/n};$$

that is,

$$\max_{|z|=1} |p_n'(z)| \geqq |p_n'(-e^{i\varphi})| \\ \geqq n \prod_1^n (1 + x_\nu)^{1/n} \cdot \max_{|z|=1} |p_n(z)|^{1-(2/n)}.$$

Case of equality. If $p_n(z)$ is real for real z, then

$$\max_{|z|=1} |p_n(z)| = |p_n(-1)| \quad \text{and} \quad \max_{|z|=1} |p_n'(z)| = |p_n'(-1)|.$$

https://doi.org/10.4153/CJM-1979-039-3 Published online by Cambridge University Press

Moreover, if all the zeros of $p_n(z)$ belong to the set (11), then

$$(1 + x_\nu)/A_\nu = 1/2\lambda \quad \text{for all } \nu = 1, 2, \ldots, n.$$

Hence under these conditions there is equality throughout the whole proof.

Proof of Theorem 5. For all $\theta \in \mathbf{R}$ we have

$$\operatorname{Re} e^{i\theta} p_n'(e^{i\theta})/p_n(e^{i\theta}) = \operatorname{Re} \sum_1^n e^{i\theta}/(e^{i\theta} - z_\nu) \geqq \sum_1^n (1 + |z_\nu|)^{-1},$$

so that

$$|p_n'(e^{i\theta})| \geqq \sum_1^n (1 + |z_\nu|)^{-1} |p_n(e^{i\theta})|.$$

From this we readily obtain the desired result.

Proof of the corollary. Since the function $f : x \to (1 + x)^{-1}$ is convex for $x \geqq 0$ we have

$$n^{-1} \sum_1^n (1 + |z_\nu|)^{-1} = n^{-1} \sum_1^n f(|z_\nu|) \geqq f(n^{-1} \sum_1^n |z_\nu|) = (1 + \rho)^{-1}$$

and hence (13) follows from Theorem 5.

REFERENCES

1. A. Giroux and Q. I. Rahman, *Inequalities for polynomials with a prescribed zero*, Trans. Amer. Math. Soc. *193* (1974), 67–98.
2. P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc. *50* (1944), 509–513.
3. M. A. Malik, *On the derivative of a polynomial*, J. London Math. Soc. *1* (1969), 57–60.
4. A. C. Schaeffer, *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*, Bull. Amer. Math. Soc. *47* (1941), 565–579.
5. P. Turán, *Über die ableitung von polynomen*, Compositio Math. *7* (1939–40), 89–95.

Université de Montréal,
Montréal, Québec;
Universität Erlangen Nürnberg,
Erlangen, West Germany