Solutions of Euclid's Problems, with a rule and one fixed aperture of the compasses, by the Italian geometers of the sixteenth century.

By J. S. Mackay.

Haec ac similia ad ostentationem ingenii, utilitatem vero pene nullam, inventa sunt.

Hieronymus Cardanus, *De Subtilitate*, Liber XV.

Chasles in his *Aperçu Historique sur l'origine et le développement des Méthodes en Géométrie* (seconde édition, 1875, pp. 214-215) makes the following statement:

"Essays of the same kind as the geometry of the rule and that of the compasses, and which hold, so to speak, the mean between the two, had long previously engaged the attention of famous mathematicians. Cardan first of all in his book *De Subtilitate* had resolved several of Euclid's problems by the straight line and a single aperture of the compasses, as if one had in practice only a rule and invariable compasses. Tartalea was not long in following his rival on this field, and extended this mode of treatment to some new problems. (*General trattato di numeri et misure; 5ta parte, libro terzo;* in-fol. Venise, 1560). Finally, a learned Piedmontese geometer, J.-B. de Benedictis, made it the object of a treatise entitled: *Resolutio omnium Euclidis problematum, aliorumque ad hoc necessario inventorum, una tantummodo circini data apertura;* in-4°. Venise, 1553."

As very little is known of these essays, and as the treatises in which they occur are far from being readily accessible, I have thought it worth while to make an abstract of them. This abstract, if its interest should prove to be historical rather than scientific, may afford some amusement to those who are fond of geometrical curiosities. I have confined myself to the solutions of the problems in Euclid's first six books; those of the problems in the remaining books are not given at all by Cardano, are very briefly dismissed by Benedetti, and in any case are now not worth resuscitation. With reference to the form in which the solutions are presented, it ought perhaps to be said that the sequence of the problems has in all the three cases been rigidly adhered to. The lettering of the figures has frequently been changed, most of the demonstrations have been omitted as unnecessary, such as have been inserted are not always those given by the authors, but the constructions have not been tampered with. When

https://doi.org/10.1017/S0013091500001334 Published online by Cambridge University Press

the constructions are the same or substantially the same as Euclid's, they have not been reproduced. Before, however, proceeding to the solutions, a word or two must be said in rectification of the statement of Chasles.

In the early part of 1547 Lodovico Ferraro, who was then a public lecturer on mathematics in Milan, and had been a dependant of the celebrated physician and algebraist Hieronimo Cardano (Tartaglia is never tired of calling him "suo creato"), sent an impertinent challenge to Tartaglia, and a brisk war of pamphlets ensued between the two. Thirty-one questions were proposed by each to be answered by the other within a given time, and it was with Tartaglia's questions that the problem arose which forms the subject of the present paper. The questions were first published in the *Seconda Riposta data da Nicolo Tartalea Brisciano a Messer Lodovico Ferraro* (Venice, 21 April, 1547). They are published again, along with their solutions and with the solutions of twenty-two of the thirty-one questions set by Ferraro, in Tartaglia's *General Trattato di Numeri et Misure* (Venice, 1560), Fifth Part, Third Book. Here Tartaglia tells us how the research originated. "I set myself one day (in order not to be unemployed) to try if it was possible to resolve, with any aperture of the compasses proposed by an adversary, the 26th* proposition of the sixth book of Euclid, namely, that where it is proposed to describe a superficies similar to a given rectilineal superficies and equal to another, a thing which not only I soon assured myself was possible, but even found it to be possible to resolve (with such a condition) all his other geometrical problems to be worked on a plane, excepting those where it falls to describe a terminated circle (as is the case in the 4th, 5th, 8th, 9th, 13th, and 14th propositions of his fourth book, and likewise in the 25th and 33rd of the third)."

Ferraro's answers to Tartaglia's questions were first printed in a pamphlet, the heading of which begins, *Quinto Cartello di Lodovico Ferraro contra Messer Nicolo Tartaglia*, and the last part of which is entitled *Risolutione fatto per Lodovico Ferraro à i trentaun' quesiti mandatagli da risolvere per Messer Nicolo Tartaglia* (Milan, October, 1547).

---

* This is the 25th proposition of Euclid's sixth book. Tartaglia in his translation of Euclid's *Elements* into Italian (Venice, 1543) now and then changes the order of Euclid's propositions.

Cardano, in his work *De Subtilitate Libri XXI.* (Norimbergae, 1550) entitles the fifteenth book *De incerti generis aut inutilibus subtilitatibus.* He there, without naming Tartaglia, mentions that both Ferraro and himself found out in a few days how all that Euclid proves, (with the exception of the inscription and circumscription of circles), when the aperture of the compasses may be varied, could be shown by them when the aperture was any invariable one proposed by an adversary. He says that Ferraro printed the demonstration of all this, but as it was for a controversial purpose, he did not think it would survive, and in order that so rare an example of subtilty might not perish, he gives an account of it.

Last of all, Giovanni Battista Benedetti (whose name is Latinised into Joannes Baptista de Benedictis) took the matter up, and published the treatise mentioned by Chasles. In his verbose preface or dedication, Benedetti says nothing about the origin of the problem, or about the published solutions of Ferraro and Cardano.

Tartaglia's Solutions.

1. *On a given straight line to describe an equilateral triangle.*

Euclid I. 1. See figures 1, 2.

Let AB be the given straight line.

Make AC and BD each equal to the given aperture. On AC describe the equilateral triangle ACE, and on BD the equilateral triangle BDF. Let AE and BF intersect at G.

GAB is the equilateral triangle required.

2. *Through a given point to draw a straight line parallel to a given straight line.*

Euclid I. 31. See figures 3, 4.

Let A be the given point, BC the given straight line.

With A as centre cut BC at D. Join AD and produce it, making DE equal to AD. With E as centre cut BC at F. Join EF and produce it, making FG equal to EF. Join AG.

AG is the parallel required.

If the circle with A as centre and the given aperture as radius do not cut BC, choose a point H near enough to BC, and through it draw the parallel HK, as before. Then through A draw AG parallel to HK.

https://doi.org/10.1017/S0013091500001334 Published online by Cambridge University Press

3. *To bisect a given angle.*

Euclid I. 9. See figure 5.

Let AOB be the given angle.

With O as centre cut OA and OB at D and E. With D and E as centres describe circles intersecting in F. Join OF.

OF is the bisector required.

4. *To bisect a given straight line.*

Euclid I. 10. See figure 6.

Let AB be the given straight line.

With A and B as centres describe circles cutting AB at C and D. If these circles do not intersect, with C and D as centres describe circles; and so on. Finally, let E and F be the points of intersection. Join EF cutting AB at G.

G is the point of bisection.

5. *To draw a perpendicular to a given straight line from a given point in it.*

Euclid I. 11. See figure 7.

Let AB be the given straight line, O the given point.

With O as centre cut OA and OB at D and E. Bisect OD and OE at G and H. With G and H as centres describe circles intersecting in F. Join OF.

OF is the perpendicular required.

6. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12. See figure 8.

Let AB be the given straight line, C the given point.

With C as centre cut AB at D and E. With D and E as centres describe circles intersecting in F. Join CF.

CF is the perpendicular required.

If the circle with C as centre and the given aperture as radius do not cut AB, draw a straight line near enough to C and parallel to AB To that straight line draw a perpendicular from C, and produce the perpendicular to meet AB.

7. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2. See figures 9, 10.

(*a*) Let A be the given point, BC the given straight line. (Fig. 9)

Join AB, and through C draw CD parallel to AB. Through A draw AE parallel to BC meeting CD at E.

AE is the straight line required.

(*b*) Let A be the given point, BC the given straight line. (Fig. 10)

Join AB, and on it describe the equilateral triangle DAB. Produce DA and DB to E and F. With B as centre cut BC and BF at G and H. Join GH, and through C draw CK parallel to GH and meeting BF at K. Through K draw KL parallel to BA and meeting AE at L.

AL is the straight line required.

8. *From the greater of two given straight lines to cut off a part equal to the less.*

Euclid I. 3. See figures 11, 12, 13, 14, 15, 16.

(*a*) Let the given straight lines AB and CD, of which CD is the greater, be parallel. (Fig. 11)

Join AC, and through B draw BE parallel to AC, and meeting CD at E.

CE is the part required.

(*b*) Let the given straight lines AB and AD, of which AD is the greater, be joined at A. (Fig. 12)

With A as centre cut AD and AB at E and F. Join EF, and through B draw BG parallel to FE, and meeting AD at G.

AG is the part required.

(*c*) Let the given straight lines AB and AD, of which AD is the greater, be in the same straight line. (Fig. 13)

From A draw AE perpendicular to AB. From AE cut off AF equal to AB; and from AD cut off AG equal to AF.

AG is the part required.

(*d*) Let the given straight lines AB and CD, of which CD is the greater, be neither parallel, joined at a point, nor in the same straight line. (Fig. 14)

Through A draw AE parallel to CD. From AE cut off AF equal to AB; and from CD cut off CG equal to AF.

CG is the part required.

(*e*) Let the given straight lines AB and CD, of which CD is the greater, be in the same straight line but not contiguous. (Fig. 15)

From B and C draw BE and CF respectively perpendicular to AB and CD. Cut off BE equal to BA; through E draw EF parallel to AD and meeting CF at F. From CD cut off CG equal to CF.

CG is the part required.

(*f*) Let the given straight lines AB and AC, of which AC is the greater, be joined at A; and let it be required to cut off a part equal to AB from the end C. (Fig. 16)

Join BC, and on it describe the equilateral triangle DBC. Produce DB and DC to E and F. Cut off BE equal to BA; and through E draw EF parallel to BC and meeting DF at F. Cut off CG equal to CF.

CG is the part required.

9. *At a given point in a given straight line to make an angle equal to a given angle.*

Euclid I. 23. See figure 17.

Let A be the given point, AB the given straight line, and CDE the given angle.

With D as centre cut DE at F; with F as centre cut DC at G, and join FG. From AB cut off AH equal to DG; with A and H as centres describe circles intersecting in K. Join KA, KH.

HAK is the angle required.

When angle CDE is obtuse, make, as before, angle HAK equal to its supplement; and find the supplement of angle HAK.

10–14. Euclid I. 42, 44, 45, 46; II. 11.

15. *Given two squares, to apply to one of them a gnomon equal to the other.*

This is not one of Euclid's problems, but is given in Tartaglia's edition of *The Elements* as II. 14. See figure 18.

Let ABCD, EFGH be the two squares.

https://doi.org/10.1017/S0013091500001334 Published online by Cambridge University Press

Produce AB, and cut off BK equal to EF. Join CK, and from AK cut off AL equal to CK. On AL describe the square ALMN.

BCDNML is the gnomon required.

16. Euclid VI. 10.

17. *To find a mean proportional between two given straight lines.*

Euclid VI. 13. See figure 19.

Let AB and BC be the two given straight lines.

Draw any straight line AD making an angle with AC, and cut off AE equal to the given aperture of the compasses. With E as centre describe a circle cutting AD again at F. Join CF, and through B draw BG parallel to CF and meeting AF at G. From G draw GH perpendicular to AF and meeting the circumference at H. Cut off GK equal to GH, and through K draw KL parallel to FC and meeting AC produced at L.

BL is the mean proportional required.

The following demonstration may be given:
On account of the parallels BG, LK, CF,

$$\begin{aligned} AB : BL : BC &= AG : GK : GF, \\ &= AG : GH : GF. \end{aligned}$$

Now GH is a mean proportional between AG and GF;
therefore BL is a mean proportional between AB and BC.

18. *To describe a square equal to a given triangle.*

This is a particular case of Euclid II. 14. It is given as II. 15 in Tartaglia's edition of *The Elements.*

Construct a rectangle equal to the given triangle, and find a mean proportional between its sides. The square on this mean proportional is the square required.

19. *To describe a square equal to a given rectilineal figure.*

Euclid II. 14. It is given by Tartaglia as the last proposition of the second book.

(*a*) The method of solution is the same as that of the preceding problem.

(*b*) Resolve the rectilineal figure into triangles. Find the side of a square equal to the first triangle. At one end of it raise a perpendicular equal to the side of a square which is equal to the second triangle; and draw the hypotenuse of this right-angled triangle. At one end of the hypotenuse raise a perpendicular equal to the side of a square which is equal to the third triangle; and draw the hypotenuse of this right-angled triangle. Continue this process till all the triangles which make up the rectilineal figure are exhausted. The last hypotenuse is a side of the required square.

20. Euclid III. 1.

21. *From a given point to draw a tangent to a given circle.*

Euclid III. 17. See figure 20.

Let ABG be the given circle, and D the given point.

Find C the centre of the given circle, and draw the straight line DACB. Produce BD to E, and make DE equal to DA. Divide BA at F, so that BF is to FA as BD to DE; and from F draw FG perpendicular to AB and meeting the circumference at G. Join DG.

DG is the tangent required.

The following demonstration may be given:

Because $$BF : FA = BD : DE,$$
$$= BD : DA;$$

therefore BA is divided harmonically at F and D.
Hence FG is the polar of D, and DG a tangent to the circle.

Tartaglia gives no proof of his construction, but refers to Apollonius's *Conics*, Book I., Prop. 34, and adds that in the same manner a tangent may be drawn to the three other conic sections.* He mentions that this problem was the first of the thirty-one questions proposed by him to Cardano and Ferraro in their public dispute.

22–26. Euclid III. 30, 34; VI. 11, 12, 9.

---

* Apollonius proves that this construction will give a tangent to the hyperbola, the ellipse, and the circle; but Prop. 35 shows that he knew how to modify it to obtain a tangent to the parabola.

27. *In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle.*

Euclid IV. 1. See figure 21.

Let CDF be the given circle, and AB the given straight line.

Draw any diameter CD, and from it cut off CE equal to the third proportional to CD and AB. From E draw EF perpendicular to CD and meeting the circumference at F. Join CF.

CF is the chord required.

28–38. Euclid IV. 2, 3, 6, 7, 10, 11, 12, 15, 16; VI. 18, 25.

The last of these problems was the second of Tartaglia's questions.

39. *To find a square equal to the difference of two given squares.*

This is not one of Euclid's problems, but is inserted as subsidiary to the problem which follows. See figure 22.

Let ABCD, EFGH be the given squares, the latter being the greater.

Take a straight line LM equal to twice the given aperture, and on it describe a semicircle. Find a fourth proportional to EF, AB, and LM; in the semicircle place a chord LK equal to this fourth proportional; and join KM. Lastly, find PQ a fourth proportional to LM, KM, and EF.

PQ is a side of the square required.

40. *To make a triangle the sides of which shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22. See figure 23.

Let AB, CD, EF be the three given straight lines.

Take a straight line HK equal to CD. Find a square equal to the sum of the squares on AB and CD; then find a second square equal to the difference between this square and the square on EF. On HK describe a rectangle equal to the half of the second square; and cut off HL equal to the breadth of the rectangle. Find a third square equal to the difference between the squares on AB and HL; from L draw LG perpendicular to HK and equal to a side of this third square; and join GH, GK.

GHK is the triangle required.

Tartaglia adds that to ensure the perpendicular always falling inside the triangle, the longest side should be chosen for base.

The demonstration follows from Euclid II. 12 or 13.

41. Euclid VI. 28.

This was the third of Tartaglia's questions.

42. *To describe a parallelogram which shall be similar to a given parallelogram and equal to another parallelogram and a triangle.*

This is not one of Euclid's problems, but is inserted as subsidiary to the problem which follows.

43. Euclid VI. 29.

This was the fourth of Tartaglia's questions.

44. Euclid VI. 30.

Ferraro's and Cardano's Solutions.

1. Euclid I. 9. Tartaglia's solution.

2. Euclid I. 10. Tartaglia's solution.

3. Euclid I. 11. Tartaglia's solution.

4. *From the greater of two unequal straight lines drawn from the same point to cut off a part equal to the less.*

Particular case of Euclid I. 3. See figure 24.

Let AB and AC be the given straight lines, AC being the greater.

Bisect the angle BAC by AD. With B as centre describe a circle cutting AD at E; with E as centre describe a circle cutting AC at F.

AF is the part required.

If the circle with B as centre does not meet AD, bisect the angles BAD, CAD, and repeat the preceding construction.

5. *On a given straight line to describe an isosceles triangle.*

This is not one of Euclid's problems, but is inserted as subsidiary to the problem which follows. The construction is effected by the 2nd and 3rd problems.

6. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2. See figure 25.

Let A be the given point, BC the given straight line.

Join AB, and on it describe the isosceles triangle ABD. Produce DB and DA; cut off BE equal to BC, and DF equal to DE.

AF is the straight line required.

7. Euclid I. 3.

8. *At a given point in a given straight line to make an angle equal to a given angle.*

Euclid I. 23. Tartaglia's solution.

If the aperture is too small, which will be the case when the given angle is obtuse, bisect the given angle; at the given point make an angle equal to half the given angle; and repeat the construction.

9. *On a given straight line to describe an equilateral triangle.*

Euclid I. 1.

Let AB be the given straight line.

With the given aperture describe an equilateral triangle; and at A and B make angles equal to two of the angles of this triangle.

10. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12.

Let AB be the given straight line, C the given point.

Through C draw CD parallel to AB; and from C draw CE perpendicular to CD.

11. *To find a mean proportional between two given straight lines.*

Euclid VI. 13. See figure 26.

Let AC and CB be the given straight lines.

Place AC and CB in the same straight line. At A draw AF perpendicular to AB and equal to twice the given aperture. Join BF; and through C draw CE parallel to BF. Make CG equal to EF, and CK equal to AE. On KG, which is equal to twice the given aperture, describe the semicircle KLG; Draw CL perpendicular to KG, and meeting the circumference at L. Join GL; and through B draw BM parallel to GL.

CM is the mean proportional required.

The following demonstration may be given:

Because	AE : AC = EF : CB,
	= CG : CB,
	= CL : CM;
therefore	AE : CL = AC : CM;
therefore	KC : CL = AC : CM.
Again	CL : CG = CM : CB;
and	KC : CL = CL : CG;
therefore	AC : CM = CM : CB.

12. Euclid II. 14. Tartaglia's solution.

13. *To draw a tangent to a circle from a given external point.*

Euclid III. 17. See figure 27.

Let BGC be the given circle, A the external point.

Through O, the centre of the circle BGC, draw the secant ABC; and find DE a mean proportional between AB and AC. Draw EF perpendicular to DE, and equal to the radius OB; and join DF. At O make the angle AOG equal to the angle DFE; and join AG.

AG is the tangent required.

14–15. Euclid III. 25, 34.

16. *On a given straight line to describe a segment of a circle which shall contain an angle equal to a given angle.*

Euclid III. 33. See figure 28.

Let AB be the given straight line, C the given angle.

With the given aperture describe a circle DEF; and from it cut off a segment DFE containing an angle equal to C. Take any point F in the arc of the segment, and join FD, FE. At A and B make angles BAG, ABG respectively equal to the angles EDF, DEF.

Then G is a point on the arc of the segment described on AB, and as many other points as may be necessary can be found in a similar manner.

17. *In a given circle to place a chord equal to a given straight line which is not greater than the diameter of the circle*

Euclid IV. 1. See figure 29.

Let O be the centre of the given circle, CD the given straight line.

Draw any radius OA; and find FG a fourth proportional to OA, CD, and the given aperture. On FG describe an isosceles triangle EFG, whose sides EF, EG are each equal to the given aperture. At O make angle AOB equal to angle FEG; and join AB.

AB is the chord required.

18. *To construct a triangle whose sides shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22. See figure 30.

Let A, B, C be the three given straight lines, and let A be greater than B, and B greater than C.

With the given aperture describe a circle, and through O its centre draw the diameter DG. Take OE such that A : B = DO : OE; and EF such that B : C = OE : EF.
Again take HK such that FE : EG = DE : HK; and produce HK to L, so that KL may be equal to FE. In the circle place the chord MN equal to HL; from MN cut off MP equal to KL; and join OP. Lastly, at the extremities of A make angles respectively equal to angles OMP and MOP.

The following demonstration may be given:

(*a*) To prove that the point F lies outside the circle.

A, B, and C are proportional to DO, OE, and EF. But B and C are greater than A; therefore OE and EF are greater than DO.

(*b*) To prove that DG is greater than HL.

DE : HK = FE : EG, by construction.
Now DE is greater than EG;
and DE is greater than FE, because DE : FE = A + B : C;
and DE is greater than HK, because FE is greater than EG.

https://doi.org/10.1017/S0013091500001334 Published online by Cambridge University Press

Hence of the four proportionals DE, HK, FE, EG, the greatest is DE, and consequently the least is EG; therefore DE + EG is greater than HK + FE, that is DG is greater than HL.

(*c*) To prove that MO, OP, PM are respectively equal to DO, OE, EF, and therefore proportional to A, B, C.

MO and DO are radii of the same circle; and PM was made equal to KL or EF. It remains therefore to prove OP equal to OE.

Because $\quad DE : HK = FE : EG$;

therefore $\quad DE \cdot EG = HK \cdot FE$;

$$= HK \cdot KL,$$
$$= NP \cdot PM,$$
$$= RP \cdot PQ.$$

Now DG = RQ; therefore DE = RP, and OE = OP.

Hence the triangle MOP is similar to the triangle whose sides are A, B, C

19. *To describe an isosceles triangle having each of the base angles double of the vertical angle.*

Euclid IV. 10.

Take any straight line AB, and divide it at C so that AB · BC is equal to AC². Construct a triangle whose sides shall be equal or proportional to AB, AB, and AC.

Cardano adds the two following problems:

20. *A diameter of a circle is given, and a point in it. To raise from the given point a perpendicular of such a length that it shall just meet the circumference.*

The length of the perpendicular is equal to a mean proportional between the segments of the diameter.

21. *On a given hypotenuse to construct a right-angled triangle having one of its sides equal to a given straight line.*

See figure 31.

Let AB be the hypotenuse, CD the side.

With the given aperture describe the circle EFG, and draw a diameter EF. To AB, CD, EE find a fourth proportional; in the circle EFG place FG equal to this fourth proportional; and join EG.

At B make angle ABH equal to angle EFG; cut off BH equal to CD; and join AH.

ABH is the triangle required.

BENEDETTI'S SOLUTIONS.

1. *To draw a perpendicular to a given straight line from a given point in it.*

Euclid I. 11. See figure 32.

Let AB be the given straight line, C the given point.

With C as centre describe the semicircle DFE. On CD and CE construct the equilateral triangles CDF, CGE; and join FG. On FG construct the equilateral triangle FHG; and join CH.

CH is the perpendicular required.

2. *To produce a given straight line, which is less than the given aperture, so that the part produced may be equal to the given straight line.*

See figure 33.

Let AB be the given straight line.

At B draw BD perpendicular to AB. With A as centre cut BD at E; and with E as centre cut AB produced at F.

BF is the part required.

3. *To do the same thing when the given straight line is greater than the given aperture.*

See figure 34.

Let AB be the given straight line.

From AB cut off consecutively distances equal to the given aperture, till DB the part that remains is less than the given aperture. Produce DB to E so that BE may be equal to DB. From AE produced, beginning at E, cut off as many distances equal to the given aperture as have been cut off from AB.

4. Euclid I. 10. Tartaglia's solution.

5. *To draw a perpendicular to a given straight line from a given point outside it.*

Euclid I. 12. See figure 35.

Let AB be the given straight line, C the given point.

Join BC, and produce it so that CD may be equal to BC. Join DA, and bisect it at E. Join CE, and from C draw CF perpendicular to CE.

CF is the perpendicular required.

6. *Through a given point to draw a straight line parallel to a given straight line.*

Euclid I. 31.

Let AB be the given straight line, C the given point.

From B draw BD perpendicular to AB; and from C draw CE perpendicular to BD.

CE is the parallel required.

7. *From a given point to draw a straight line equal and parallel to a given straight line.*

Let AB be the given straight line, C the given point.

Join CB. Through C draw CD parallel to AB, and through A draw AD parallel to BC.

CD is the straight line required.

8. *To cut off from the greater of two given straight lines a part equal to the less, or to produce the less till it is equal to the greater.*

Euclid I. 3. See figure 36.

Let AB and CD be the two given straight lines.

From C draw CE equal and parallel to AB. With C as centre describe a circle cutting CE and CD or these lines produced at F and G. Join FG; and through E draw EH parallel to FG, and meeting CD or CD produced at H.

CH is the straight line required.

9. *To bisect a given angle.*

Euclid I. 9. See figure 37.

Let BAC be the given angle.

https://doi.org/10.1017/S0013091500001334 Published online by Cambridge University Press

With A as centre describe a circle cutting AB and AC at D and E. Join DE; bisect DE at F; and join AF.

AF is the bisector required.

10. Euclid I. 23. Tartaglia's solution.

11. Euclid I. 42.

12. *On a given [indefinite] straight line to construct a triangle equal and similar to a given triangle.*

See figure 38.

Let AB be the given straight line, CDE the given triangle.

Cut off BF equal to DC; and make angle FBG equal to angle CDE. Cut off BG equal to DE; and join FG.

FBG is the triangle required.

13–14. Euclid I. 44, 46.

15. *Given two squares, to apply to one of them a gnomon equal to the other.*

Tartaglia's solution.

16–17. Euclid II. 11; VI. 10.

18. Euclid VI. 13. Tartaglia's solution.

19. *To describe a square equal to a given triangle.*

Tartaglia's solution.

20. Euclid III. 1.

21. *In a given circle to place a chord equal to one given straight line, which is less than the diameter, and parallel to another given straight line.*

See figure 39.

Let EKF be the given circle, AB the straight line to which the chord is to be equal, CD the straight line to which the chord is to be parallel.

Through O the centre of the given circle draw the diameter EF parallel to CD ; cut off OG and OH each equal to half of AB. Draw GK and HL perpendicular to EF ; and join KL.

KL is the chord required.

22. *With three given straight lines, two of which are equal to each other, and are together greater than the third, to construct a triangle.*

See figure 40.

Let A, B, C be the three given straight lines, of which A and B are equal.

Take any straight line DEF, making DE equal to A, and EF equal to C ; from D draw DG equal to the given aperture. Join EG, and through F draw FH parallel to EG. In a circle whose centre is O, and which is described with the given aperture as radius, place the chord KL equal to GH ; and join OK, OL. Make OM, ON equal to A, B ; and join MN.

OMN is the triangle required.

23. *To construct a triangle equal to a given triangle, and such that it may have an angle equal to a given angle, and a side equal to a given straight line.*

The construction is effected by the 13th problem.

24. *From a given point to draw a tangent to a given circle.*

Euclid III. 17. See figure 41.

Let DBE be the given circle, A the given point.

Find O the centre of the circle ; and join AO cutting the circumference at B. Divide BO at C so that BC : CO = AB : BO ; from C draw CD perpendicular to AO and cutting the circumference at D ; and join AD.

AD is the tangent required.

The following demonstration may be given :

Because $\quad BC : CO = AB : BO$,

therefore $\quad BO : CO = AO : BO$;

therefore $\quad AO \cdot CO = BO^2$.

Hence CD is the polar of A, and AD a tangent to the circle.

25–26. Euclid III. 30, 34.

https://doi.org/10.1017/S0013091500001334 Published online by Cambridge University Press

27. *To make a triangle the sides of which shall be equal to three given straight lines, any two of which are greater than the third.*

Euclid I. 22.

Benedetti discusses (in six pages) five cases of this problem.

(1) When two of the sides are equal.

(2) When the three sides are equal.

(3) When the square on one side is equal to the sum of the squares on the two other sides.

(4) When the square on one side is less than the sum of the squares on the two other sides.

(5) When the square on one side is greater than the sum of the squares on the two other sides.

In the first case the problem becomes the 22nd; in the second case the 36th; in the third case it reduces to that of constructing a right-angled triangle, having given the sides containing the right angle. The solutions of the fourth and fifth cases, depending on Euclid II. 13 and 12, are substantially the same as Tartaglia's. Benedetti concludes with the remark: "Et contra illos omnes excellentissimos Mathematicos priscos modernosque qui dixerunt impossibile esse hoc problema alio modo posse concludi quam ut docet xxii primi Euclidis, ego vero deo dante labente Anno diuinae incarnationis MDLII Die xv Octobris illud inueni."

28–34. Euclid IV. 2, 3, 6, 7, 10, 11, 12.

35. *In a given circle to inscribe a regular hexagon.*

Euclid IV. 15.

Find O the centre of the circle, and draw the diameter AOD. In the circle place the chord BC equal to the radius and parallel to AD. Join BO, CO, and produce them to meet the circumference in E, F. Join AB, CD, DE, EF, FA.

ABCDEF is the hexagon required.

36. Euclid I. 1. Ferraro's solution.

37. *From a given point in the circumference of a given circle to draw a chord which shall be equal to a given straight line.*

Euclid IV. 1. See figure 42.

Let AB be the given straight line. C the given point.

Find O the centre of the given circle, and draw the diameter COD. In the circle place the chord EF equal to AB, and parallel to CD. Draw the diameter FOG; and through C draw CH parallel to FG.

CH is the chord required.

38–45. Euclid IV. 26; VI. 11, 12, 9, 18, 25, 28, 29, 30.

46. *On a given hypotenuse to construct a right-angled triangle having one of its sides equal to a given straight line.*

Cardano's solution.

47. *Given two straight lines, one of which is less than half of the other, to divide the greater into two segments such that the smaller straight line shall be a mean proportional between them.*

See figure 43.

Let AB and BC be the two straight lines, BC being less than half of AB.

Place BC at right angles to AB, and join C with O the middle point of AB. With O as centre describe the semicircle DGE; from E draw EF perpendicular to AB and meeting CO at F. Through F draw FG parallel to AB and meeting the semicircle at G; from G draw GH perpendicular to AB. Lastly, divide AB at K so that AK : KB = DH : HE.

AK and KB are the required segments.

The following demonstration may be given:

$$\begin{aligned} EF : BC &= EO : BO, \\ &= 2EO : 2BO, \\ &= DE : AB. \end{aligned}$$

Now EF, which is equal to HG, is a mean proportional between DH and HE, the segments of DE;

therefore BC is a mean proportional between the segments of AB, when AB is divided similarly to DE;

that is, AB is a mean proportional between AK and KB.

48. Euclid IV. 5.

49. *To construct a triangle similar to a given triangle, and such that the centre of its circumscribed circle may be at a given point and the radius of it equal to a given straight line.*

See figure 44.

Let ABC be the given triangle, D the given point, and E [illegible] given radius.

Find F the centre of the circle circumscribed about triangle ABC; and join FA, FB, FC. From D draw DK equal to E; on DK construct triangle DKG similar to triangle FCA, and triangle DKH similar to triangle FCB; and join GH.

GHK is the triangle required.

50-54. Euclid IV. 4, 8, 9, 13, 14.

55. *About a given centre to describe a regular figure which shall be similar to a given regular figure and shall have a given circumscribed radius.*

The method of solution is similar to that of 49.

56. Euclid III. 33.

57. *From a given point to draw a straight line equal to a given straight line.*

Euclid I. 2.

Let A be the given point, BC the given straight line.

From A draw an indefinite straight line AD, and from AD cut off AE equal to BC.

58. *To find the centre of a circle, an arc of which is given.*

Euclid III. 25.

Draw any two chords not parallel to each other. The straight lines which bisect these chords perpendicularly will meet at the required centre.

59. *To find the diameter of a circle which shall be equivalent to two given circles.*

Make the diameters of the two given circles the base and perpendicular of a right-angled triangle. The hypotenuse of this triangle will be the diameter of the required circle.