Canad. Math. Bull. Vol. **33** (1), 1990

# FACTORS OF FIELDS

BY

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ABSTRACT. Let $L$ be a finitely generated extension of a field $k$. $L$ is a $k$-rational factor if there is a field extension $K$ of $k$ such that the total quotient ring of $L \otimes_k K$ is a rational (pure transcendental) extension of $K$. We present examples of non-rational rational factors and explicitly determine both factors.

**Introduction.** Let $L$ be a finitely generated field extension of a field $k$. $L$ is called a $k$-rational factor if there exists a field extension $K$ of $k$ such that the quotient field of $L \otimes_k K$ is $k(x_1, \ldots, x_n)$, a pure transcendental extension of $k$. In view of the characterization of projective modules as direct summands of free modules, rational factors are in some sense the projectives in the category of fields. In a recent paper [3], Colliot-Thelene and Sansuc have shown the equivalence of being a rational factor and being retract rational in the special case of function fields of algebraic tori. We show that in general a rational factor must be retract rational (Theorem 1). In Section 2, after giving an elementary proof of the result of Colliot-Thelene and Sansuc, we explicitly determine both factors for a non-trivial family of rational factors.

§1. **Rational Factors.** Throughout this paper $k$ will denote a field. All homomorphisms will be $k$-homomorphisms. We will denote by $k^{(n)}$ the pure transcendental extension of $k$ of transcendence degree $n$.

DEFINITION 1. *If $L$ and $K$ are fields containing $k$ then the total quotient ring of $L \otimes_k K$ is called the local tensor product of $L$ and $K$ over $k$ and is denoted by $L \times_k K$.*

DEFINITION 2. *A field $L$ containing $k$ is said to be a $k$-rational factor (or simply a rational factor) if there is a field $K$ containing $k$ such that $L \times_k K \cong k^{(n)}$ for some $n$.*

A rational factor is, of course, regular and unirational over $k$. Any stably rational extension is a rational factor. ($L$ is *stably rational if* $L^{(m)} \cong k^{(n)}$ for some $m$ and $n$.) In [2] an example is given of a stably rational extension that is not rational so not all rational factors are rational. We will present other examples later.

A related concept is that of retract rationality.

Received by the editors May 13, 1988, in revised form, February 21, 1989.

AMS 1980 Subject Classification: 12F20, 14J26

DEFINITION 3. *A field $L$ containing $k$ is retract rational if there is a $k$-algebra $T$ whose quotient field is $L$ such that $T$ is a retract of $k[x_1,\ldots,x_n](1/w)$ for some variables $x_1,\ldots,x_n$ and $w \in k[x_1,\ldots,x_n]$ (i.e. there are $k$-algebra maps $i : T \longrightarrow k[x_1,\ldots,x_n](1/w)$ and $j : K[x_1,\ldots,x_n](1/w) \longrightarrow T$ such that $j \circ i = id$).*

Definition 3 is due to Saltman and some of the basic properties of retract rational extensions can be found in [6] and [7].

THEOREM 1. *If $L$ is a rational factor then $L$ is retract rational.*

PROOF. Suppose $L \times_k K \cong k^{(n)} = k(x_1,\ldots,x_n)$. Choose an affine $T_1$ so that $T_1$ has quotient field $L$ and $w_1 \in k[x_1,\ldots,x_n]$ so that $T_1 \subseteq k[x_1,\ldots,x_n](1/w_1)$. Now $L \times_k K \times_k L \cong L^{(n)} = L(y,\ldots,y_n)$. Choose $u \in T_1[y_1,\ldots,y_n]$ so that $T_1 \subseteq k[x_1,\ldots,x_n](1/w) \subseteq T_1[y_1,\ldots,y_n](1/u)$. Now choose $a_1,\ldots,a_n \in T_1$ so that $u(a_1,\ldots,a_n) = a \neq 0$ and define $j : T_1[y_1,\ldots,y_n](1/au) \longrightarrow T_1(1/a)$ by $j(y_i) = a_i$ for $i = 1,\ldots,n$. Then $j$ restricts to a retraction from $k[x_1,\ldots,x_n](1/w) \longrightarrow T$ where $w = aw_1$ and $T = T_1(1/a)$. $\square$

It is unclear at this point as to whether the converse of Theorem 1 holds although there is some evidence for it. We present this evidence in the next section.

§2. **The Fixed Field of a Group Action.** Let $G$ be a finite group and let $M$ be a finitely generated $\mathbb{Z}[G]$-module which is free as an abelian group. Such an $M$ will be called a *$G$-module*. A $G$-module $P$ is said to be a *permutation module* if the action of $G$ permutes the elements of some basis for $P$.

Considering $M$ as a free abelian group we can form the group algebra $k[M]$, $k(M)$ will denote its quotient field. $G$ acts naturally as a group of automorphisms on $k(M)$ and we are interested in the structure of the fixed field, $k(M)^G$.

A closely related construction is the function field of an algebraic torus. In this situation $G$ is the Galois group of a Galois extension $L$ of $k$ and our function field is $L(M)^G$, the fixed field of $G$ acting on $L(M)$. The difference here is that the elements of $G$ also act on the coefficients of the rational functions. A fundamental tool in the study of algebraic tori is the following:

THEOREM 2. *Suppose that $L$ is a Galois extension of $k$ with Galois group $G$ and $P$ is a permutation module. Then $L(P)^G \cong k^{(n)}$ for some $n$.*

PROOF. See [4], Proposition 1.4, p. 303. $\square$

The following result of [3] is the basic motivation for this paper. We present an elementary proof.

THEOREM 3. *Suppose that $M$ is a $G$-module and $L$ is a Galois extension of $k$ with group $G$. Then $L(M)^G$ is retract rational over $k$ if and only if $L(M)^G$ is a $k$ rational factor.*

PROOF. By theorem 1 we need only prove that $L(M)^G$ retract rational implies that

https://doi.org/10.4153/CMB-1990-014-6 Published online by Cambridge University Press

$L(M)^G$ is a $k$-rational factor. Assume that there is an affine $T$ with quotient field $L(M)^G$ and maps $i : T \longrightarrow k[x_1, \ldots, x_n](1/w)$ and $j : k[x_1, \ldots, x_n](1/w) \longrightarrow T$ with $j \circ i = id$. Since $L[M]^G$ also has quotient field $L(M)^G$, there is an $s \in L[M]^G$ and $t \in T$ such that $L[M]^G(1/s) = T(1/t)$. Thus we get

$$L[M]^G(1/s) \longrightarrow k[x_1, \ldots, x_n](1/tw) \longrightarrow L[M]^G(1/s)$$

where the composition is the identity. Tensoring with $L$ over $k$ we get

$$L[M](1/s) \longrightarrow L[x_1, \ldots, x_n](1/tw) \longrightarrow L[M](1/s)$$

where the composition is the identity and the maps respect the $G$-action.

Let $N = (L[M](1/s))^*/L^*$ and $P = (L[M](1/s))^*/L^*$ where $^*$ denotes the group of units. $N$ and $P$ are $G$-modules and the above sequence yields a retraction of $P$ onto $N$ so we get $P \cong N \oplus N'$ for some $G$-module $N'$. But $P$ is a permutation module. (The set of distinct prime factors for $tw$ forms a basis for $P$ and the action of $G$ permutes this basis because $tw$ is fixed by $G$.)

Now applying [8, Lemma 7, p. 151] to $L[M]$ and $L[M](1/s)$ we get an exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow Q \longrightarrow O$$

where $Q$ is a permutation module. Thus

$$L(N)^G \cong L(M \oplus Q)^G \text{ [4, Proposition 1.5 p. 304]}$$

But

$$L(P)^G \cong L(N \oplus N')^G \cong L(M \oplus Q \oplus N')^G = L(M)^G \times_R L(Q \oplus N')^G$$

(The last equality holds since tensoring either side with $L$ over $k$ yields $L(M \oplus Q \oplus N')$.)

Hence, $L(M)^G$ is a rational factor by Theorem 2. □

To get further examples of rational factors we turn to the fields of the form $k(Z[A])^A$ where $A$ is a finite abelian group. Saltman has shown in [6] that if the characteristic of $k \neq 2$, the exponent of $A$ is $2^r m$ with $m$ odd, $\xi_{2^r}$ is a primitive $2^r$ root of unity, and $k(\xi_{2^r})/k$ is a cyclic extension then $k(\mathbb{Z}[A])^A$ is retract rational (the conditions above are satisfied, for example, if $k = Q$ and $8 \mid |A|$). We will show that these fields are, in fact, rational factors and explicitly determine the other factor. Using Theorem 1 we get Saltman's result as a corollary.

LEMMA 1. *Let $H$ and $G$ be finite groups. Then $k(\mathbb{Z}[H \times G])^{H \times G}$ and $k(\mathbb{Z}[H])^H \times_k k(\mathbb{Z}[G])^G$ are stably isomorphic. (Note that the transcendence degree of $k(\mathbb{Z}[H \times G])^{H \times G}/k$ is $|H| \cdot |G|$ while that of $k(\mathbb{Z}[H])^H \times_k K(\mathbb{Z}[G]^G$ is $|H| + |G|$ so they are not, in general, isomorphic.)*

PROOF. $k(\mathbb{Z}[H \times G]) = k(x_{h,g} \mid (h, g) \in H \times G)$, $k(\mathbb{Z}[H]) = k(x_h \mid h \in H)$ and $k(\mathbb{Z}[G]) = k(x_g \mid g \in H)$ for some variables $x_{h,g}, x_h$ and $x_g$. Let $L = k(\{x_h \mid h \in H\}$,

$\{x_g \mid g \in G\}$, $\{x_{h,g} \mid (h,g) \in H \times G\}$) and $K = k(\{x_h \mid h \in H\}, \{x_g \mid g \in G\})$. Then we have Figure 1.

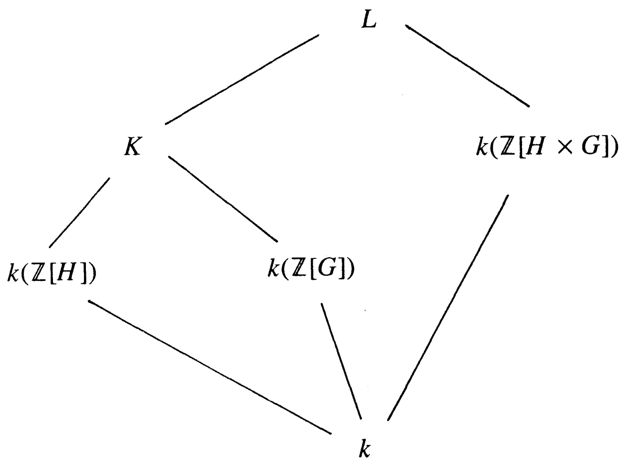


FIGURE 1

The natural projection $H \times G \longrightarrow H$ and $H \times G \longrightarrow G$ define a faithful action $H \times G$ on $K$. Clearly, $K^{H \times G}$ contains $k(\mathbb{Z}[H])^H \times_k k(\mathbb{Z}[G])^G$. But $K$ is of dimension $|H| \cdot |G|$ over both of these fields. Hence, they are equal. By Theorem 2, $L^{H \times G}$ is rational over both $K^{H \times G} = k(\mathbb{Z}[H])^H \times_k k(\mathbb{Z}[G])^H$ and $k(\mathbb{Z}[H \times G])^{H \times G}$. $\square$

NOTATION. By $L^n$ we will mean $L \times_k L \times_k L \times_k \cdots \times_k L$, the product taken $n$ times.

THEOREM 4. *Suppose that $A$ is a finite abelian group and that $t(A)$ is the highest power of 2 dividing the exponent of $A$. Assume $k(\xi_{r(A)})$ is a cyclic extension of $k$ where $\xi_{r(A)}$ is a primitive $r(A)$ root of unit and assume that the characteristic of $k \neq 2$. Then $k(\mathbb{Z}[A])^A$ is a $k$-rational factor. Moreover, there is an integer $w$ so that $(k(\mathbb{Z}[A]^A)^w \cong k^{(m)}$ for some $m$.*

PROOF. By the proof of [4, Corollary 7.5, p. 322] there exists an $n$ such that $k[\mathbb{Z}[A^n])^{A^n}$ is rational over $k$. Thus, by Lemma 1, $(k(\mathbb{Z}[A])^A)^n$ is a rational factor. If we choose $s$ so that $s \cdot n \cdot |A| \geqq |A|^n$ then $(k[\mathbb{Z}[A]^A)^{sn}$ will be stably rational and hence, by [4, theorem 6.4, p. 320] will actually be rational. $\square$

COROLLARY 1. *If $A = C_{47}$, and $k = Q$, the rationals, then $K = k(\mathbb{Z}[A])^A$ is a rational factor that is not stably rational. Furthermore, $K^w$ is rational for some $w$.*

PROOF. Follows from theorem 4 and [8]. $\square$

There are many other examples of groups $A$ with $Q(\mathbb{Z}[A])^A$ not rational and hence,

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many examples of rational factors that are not stably rational. In fact, if $A = C_{p^s}$ where $s \geqq 2$ and $p^s \notin \{2^2, 3^m, 5^2, 7^2 \mid m \in \mathbb{Z},\ m \geqq 2\}$ Lenstra has shown that $Q(\mathbb{Z}[A])^A$ is not rational.

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https://doi.org/10.4153/CMB-1990-014-6 Published online by Cambridge University Press