ON UNITARY AND SYMMETRIC MATRICES WITH REAL QUATERNION ELEMENTS

N. A. WIEGMANN

1. Introduction. In general when symmetric matrices are considered, the elements of the matrix are taken at least in a principal ideal ring. It is interesting to determine what can be attained when the elements are not, in general, commutative and, to this end, the following is concerned with symmetric matrices with elements in the non-commutative field of real quaternions. At the same time some properties of real quaternion unitary matrices are obtained which involve symmetric matrices.

If A has complex elements, a necessary and sufficient condition that A be symmetric is that there exists a unitary matrix U such that $UAU^{\mathrm{T}} = D$ is a real diagonal matrix where U^{T} denotes the transpose of U. (See the principal result of **(2)**.) One of the properties of complex matrices importantly involved here is that the transpose of a product of two matrices is the product of their transposes taken in reverse order. For real quaternion matrices this property does not hold in general. The following topics are considered: first, if U is unitary and quaternion, necessary and sufficient conditions that U^{T} be unitary are determined; next, another proof of the above-mentioned theorem for the complex case is given; then, by paralleling this proof, necessary and sufficient conditions are determined that a quaternion matrix have the form UDU^{T} where D is quaternion diagonal and U is real orthogonal (and UDU^{T} is, of course, symmetric); finally, another canonical form for another set of quaternion symmetric matrices is found. (For relevant material on quaternion matrices see **(1)** and **(4)**.)

2. The transpose of a unitary matrix. If U is a unitary matrix (i.e., $UU^{\mathrm{CT}} = I$ where U^{CT} denotes the conjugate transpose of U), it does not follow that U^{T} is unitary (as in the complex case). Theorems 1 and 2 supply necessary and sufficient conditions for this, the latter being expressed in terms of symmetric matrices.

THEOREM 1. *If V is a unitary quaternion matrix, a necessary and sufficient condition that V^{T} be unitary is that there exist real orthogonal matrices U and W such that $UVW = D$ is a diagonal quaternion matrix.*

Proof. Let $V = V_1 + jV_2$, where V_1 and V_2 are complex matrices, and let $V_1 = T_1 + iT_2$ and $V_2 = W_1 + iW_2$ where T_1, T_2, W_1, and W_2 are real matrices.

Received April 25, 1955.

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press

Let V^T be unitary. Since $VV^{CT} = V^{CT}V = I$, the following relations result:

$$V_1^{CT}V_1 + V_2^{CT}V_2 = I, \quad V_1V_1^{CT} + V_2^CV_2^T = I,$$
$$V_1^TV_2 = V_2^TV_1, \quad V_2V_1^{CT} = V_1^CV_2^T.$$

Similarly, since $V^TV^C = V^CV^T = I$,

$$V_1^{CT}V_1 + V_2^TV_2^C = I, \quad V_1V_1^{CT} + V_2V_2^{CT} = I,$$
$$V_2^TV_1^C = V_1^{CT}V_2, \quad V_2V_1^T = V_1V_2^T.$$

Since $V_1^{CT}V_1 + V_2^{CT}V_2 = V_1^{CT}V_1 + V_2^TV_2^C$ and $V_1V_1^{CT} + V_2^CV_2^T = V_1V_1^{CT} + V_2V_2^{CT}$, it follows that $V_2^{CT}V_2 = V_2^TV_2^C$ and $V_2^CV_2^T = V_2V_2^{CT}$. In a similar manner, $V_1^{CT}V_1 = V_1^TV_1^C$ and $V_1V_1^{CT} = V_1^CV_1^T$ since $I = I^T$. Consider the following relations:

$$V_2^TV_1 = V_1^TV_2, \quad V_2^TV_1^C = V_1^{CT}V_2, \quad V_2V_1^T = V_1V_2^T, \quad V_2V_1^{CT} = V_1^CV_2^T,$$
$$V_2^{CT}V_2 = V_2^TV_2^C, \quad V_2V_2^{CT} = V_2^CV_2^T, \quad V_1^{CT}V_1 = V_1^TV_1^C, \quad V_1V_1^{CT} = V_1^CV_1^T.$$

Since $V_1 = T_1 + iT_2$ and $V_2 = W_1 + iW_2$, it follows from the first pair of relations that $W_i^TT_j = T_j^TW_i$, for $i = 1, 2$; $j = 1, 2$. From the next pair it follows that $W_iT_j^T = T_jW_i^T$, for $i = 1, 2$; $j = 1, 2$. From the next pair it follows that $W_1^TW_2 = W_2^TW_1$ and $W_2W_1^T = W_1W_2^T$, and from the last pair that $T_1^TT_2 = T_2^TT_1$ and $T_2T_1^T = T_1T_2^T$. Therefore the set of real matrices $\{T_1, T_2, W_1, W_2\}$ is such that if X and Y are any two matrices of the set, then XY^T and Y^TX are real symmetric. Now the following holds by a known theorem **(3)**: if A_i is an arbitrary set of non-zero complex matrices, there exist unitary matrices U and W such that $UA_iW = D_i$ where D_i is diagonal and real if and only if $A_iA_j^{CT} = A_jA_i^{CT}$ and $A_j^{CT}A_i = A_i^{CT}A_j$ for all i and j. In our case the matrices are all real and it is easily seen that the U and W will be real orthogonal matrices. Then since $V = T_1 + iT_2 + j(W_1 + iW_2)$, therefore $UVW = UT_1W + iUT_2W + j(UW_1W + iUW_2W) = D_1 + iD_2 + j(D_3 + iD_4)$ is a diagonal quaternion matrix.

Conversely, let V be unitary such that real orthogonal matrices U and W exist so that $UVW = D$ is quaternion and diagonal. Then $V = U^TDW^T$, and since U^T and W^T are real, it is true that $V^T = (U^TDW^T)^T = WDU$ and so $V^T\cdot(V^T)^{CT} = (WDU)(WDU)^{CT} = WDU\cdot U^TD^CW^T = I$, since $D\cdot D^{CT} = UVW\cdot W^TV^{CT}U^T = I$, and so V^T is unitary.

COROLLARY 1.1. *If V is a complex unitary matrix, there exist real orthogonal matrices U and W such that $UVW = D$ is a diagonal matrix with complex elements.*

This follows since in this case V^T is always unitary.

Let us define a unitary quaternion matrix U to be T-unitary if U^T is unitary.

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press

COROLLARY 1.2. *A* T*-unitary matrix V is symmetric if and only if there exists a real orthogonal matrix U such that $UVU^{\mathrm{T}} = D$ is a quaternion diagonal matrix.*

For if V is T-unitary and symmetric, then in the above proof $T_i = T_i^{\mathrm{T}}$, $W_i = W_i^{\mathrm{T}}$ for $i = 1, 2$, and the set of matrices $\{T_1, T_2, W_1, W_2\}$ are real, symmetric, and commutative in pairs and consequently can be diagonalized by a single real orthogonal similarity transformation. The converse is evident.

The following may also be noted:

THEOREM 2. *If V is unitary, then V^{T} is unitary if and only if VV^{T} and $V^{\mathrm{T}}V$ are symmetric.*

If V and V^{T} are unitary, by the preceding theorem there exist real orthogonal matrices U and W such that $V = UDW$ where D is quaternion and diagonal. Then $V^{\mathrm{T}} = W^{\mathrm{T}}DU^{\mathrm{T}}$ and $VV^{\mathrm{T}} = UD^2U^{\mathrm{T}}$ and $V^{\mathrm{T}}V = W^{\mathrm{T}}D^2W$ are symmetric.

Conversely, let V be unitary and let $V^{\mathrm{T}}V$ and VV^{T} be symmetric. Let $V = V_1 + jV_2$ where V_1 and V_2 are complex; then

$$\begin{aligned} VV^{\mathrm{T}} &= V_1V_1^{\mathrm{T}} - V_2^{\mathrm{C}}V_2^{\mathrm{T}} + j(V_2V_1^{\mathrm{T}} + V_1^{\mathrm{C}}V_2^{\mathrm{T}}) \\ &= V_1V_1^{\mathrm{T}} - V_2V_2^{\mathrm{CT}} + j(V_1V_2^{\mathrm{T}} + V_2V_1^{\mathrm{CT}}) \end{aligned}$$

$$\begin{aligned} V^{\mathrm{T}}V &= V_1^{\mathrm{T}}V_1 - V_2^{\mathrm{CT}}V_2 + j(V_2^{\mathrm{T}}V_1 + V_1^{\mathrm{CT}}V_2) \\ &= V_1^{\mathrm{T}}V_1 - V_2^{\mathrm{T}}V_2^{\mathrm{C}} + j(V_1^{\mathrm{T}}V_2 + V_2^{\mathrm{T}}V_1^{\mathrm{C}}). \end{aligned}$$

From this it follows that

$$V_2^{\mathrm{C}}V_2^{\mathrm{T}} = V_2V_2^{\mathrm{CT}},\ V_2^{\mathrm{T}}V_2^{\mathrm{C}} = V_2^{\mathrm{CT}}V_2,\ V_2V_1^{\mathrm{T}} + V_1^{\mathrm{C}}V_2^{\mathrm{T}} = V_1V_2^{\mathrm{T}} + V_2V_1^{\mathrm{CT}},$$
$$V_2^{\mathrm{T}}V_1 + V_1^{\mathrm{CT}}V_2 = V_1^{\mathrm{T}}V_2 + V_2^{\mathrm{T}}V_1^{\mathrm{C}}.$$

The latter two relations may be rewritten as

$$\begin{aligned} V_1^{\mathrm{C}}V_2^{\mathrm{T}} - V_2V_1^{\mathrm{CT}} &= V_1V_2^{\mathrm{T}} - V_2V_1^{\mathrm{T}}, \\ V_2^{\mathrm{T}}V_1 - V_1^{\mathrm{T}}V_2 &= V_2^{\mathrm{T}}V_1^{\mathrm{C}} - V_1^{\mathrm{CT}}V_2. \end{aligned}$$

Now $VV^{\mathrm{CT}} = I = V^{\mathrm{CT}}V$ and so:

$$\begin{aligned} I &= V_1V_1^{\mathrm{CT}} + V_2^{\mathrm{C}}V_2^{\mathrm{T}} + j(V_2V_1^{\mathrm{CT}} - V_1^{\mathrm{C}}V_2^{\mathrm{T}}) \\ &= V_1^{\mathrm{CT}}V_1 + V_2^{\mathrm{CT}}V_2 + j(-V_2^{\mathrm{T}}V_1 + V_1^{\mathrm{T}}V_2), \end{aligned}$$

$$\begin{aligned} I^{\mathrm{T}} &= V_1^{\mathrm{C}}V_1^{\mathrm{T}} + V_2V_2^{\mathrm{CT}} + j(V_1^{\mathrm{C}}V_2^{\mathrm{T}} - V_2V_1^{\mathrm{CT}}) \\ &= V_1^{\mathrm{T}}V_1^{\mathrm{C}} + V_2^{\mathrm{T}}V_2^{\mathrm{C}} + j(-V_1^{\mathrm{T}}V_2 + V_2^{\mathrm{T}}V_1). \end{aligned}$$

This means that

$$\begin{aligned} V_1^{\mathrm{C}}V_1^{\mathrm{T}} + V_2V_2^{\mathrm{CT}} &= I = V_1^{\mathrm{T}}V_1^{\mathrm{C}} + V_2^{\mathrm{T}}V_2^{\mathrm{C}}, \\ V_1^{\mathrm{C}}V_2^{\mathrm{T}} - V_2V_1^{\mathrm{CT}} &= 0 = -V_1^{\mathrm{T}}V_2 + V_2^{\mathrm{T}}V_1. \end{aligned}$$

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press

Substituting the relations obtained above in these expressions, it follows that

$$V_1{}^{\mathrm{C}} V_1{}^{\mathrm{T}} + V_2{}^{\mathrm{C}} V_2{}^{\mathrm{T}} = I = V_1{}^{\mathrm{T}} V_1{}^{\mathrm{C}} + V_2{}^{\mathrm{CT}} V_2,$$
$$V_1 V_2{}^{\mathrm{T}} - V_2 V_1{}^{\mathrm{T}} = 0 = V_2{}^{\mathrm{T}} V_1{}^{\mathrm{C}} - V_1{}^{\mathrm{CT}} V_2.$$

But this means that $V^{\mathrm{C}} V^{\mathrm{T}} = I = V^{\mathrm{T}} V^{\mathrm{C}}$ and so V^{T} is unitary.

It may be noted that if the matrices in the above Theorem are all complex, the result holds since then V^{T} is always unitary and VV^{T} and $V^{\mathrm{T}}V$ are always symmetric.

It is known **(1)** that a matrix A is normal if and only if there exists a unitary matrix U such that $UAU^{\mathrm{CT}} = D$ is a complex diagonal matrix. It is of interest to determine what characterizes normal matrices which can be brought into diagonal form by T-unitary matrices.

THEOREM 3. *A normal quaternion matrix A can be brought into complex diagonal form by means of a* T*-unitary similarity transformation if and only if A is unitarily similar to a complex symmetric matrix S (i.e., $U_1 A U_1{}^{\mathrm{CT}} = S$) under a matrix of the form $U_1 = DW_1$ where W_1 is real orthogonal and D is a unitary quaternion diagonal matrix.*

If $A = U^{\mathrm{CT}} D_1 U$ where U is T-unitary and D_1 is diagonal and complex, there exist real orthogonal matrices V and W such that $VUW = D$ is a quaternion diagonal unitary matrix. Then

$$W^{\mathrm{T}} A W = W^{\mathrm{T}} U^{\mathrm{CT}} V^{\mathrm{T}} V D_1 V^{\mathrm{T}} V U W = D^{\mathrm{CT}} S D$$

where $S = VD_1V^{\mathrm{T}}$ is complex symmetric and normal. Therefore $DW^{\mathrm{T}} A W D^{\mathrm{CT}} = S$ where $U_1 = DW^{\mathrm{T}}$ is unitary and of the above form.

If there exists a $U_1 = DW^{\mathrm{T}}$ of the type described such that $U_1 A U_1{}^{\mathrm{CT}} = S$ is complex symmetric and normal, then there exists a real orthogonal matrix V such that $V^{\mathrm{T}} S V = D_1$ and so

$$A = W D^{\mathrm{CT}} V D_1 V^{\mathrm{T}} D W^{\mathrm{T}} = U^{\mathrm{CT}} D_1 U,$$

where U is T-unitary.

COROLLARY 3.1. *If $A = A^{\mathrm{CT}}$, S is real symmetric.*

COROLLARY 3.2. *If $A = -A^{\mathrm{CT}}$, $S = iT$, where T is real symmetric.*

COROLLARY 3.3. *If $A \cdot A^{\mathrm{CT}} = I$, S is unitary symmetric.*

3. Matrices of the form UDU^{T}, U real orthogonal and D quaternion and diagonal. Let us consider first the following proof of the above-mentioned theorem:

THEOREM 4. *If $A = A^{\mathrm{T}}$ has complex elements, there exists a complex unitary matrix U such that $UAU^{\mathrm{T}} = D$ is a real diagonal matrix.* (The converse is obvious.)

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press

Let $A = HV = VK$ (where H, K are hermitian and V is unitary) be the polar form of A. (If A is non-singular, H, V, and K are uniquely determined **(6)**; if A is singular **(5)**, some arbitrariness is involved in V.) Then $A = HV = VK = V^{\mathrm{T}}H^{\mathrm{T}} = K^{\mathrm{T}}V^{\mathrm{T}}$ and so $H = K^{\mathrm{T}}$. If A is non-singular, $V = V^{\mathrm{T}}$; if A is singular, this can also be attained by means of a proper choice of V as the following will show.

Let $UHU^{\mathrm{CT}} = D = D_1 \dotplus 0$ where D_1 is diagonal and real with like roots arranged together along the diagonal. Let $UVU^{\mathrm{CT}} = W$, $UV^{\mathrm{T}}U^{\mathrm{CT}} = W_1$, and $UKU^{\mathrm{CT}} = UH^{\mathrm{T}}U^{\mathrm{CT}} = M$. Then $H^{\mathrm{T}} = U^{\mathrm{T}}DU^{\mathrm{C}}$ and $UH^{\mathrm{T}}U^{\mathrm{CT}} = UU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}} = M$. Let D_1 be of order r, i.e., $r < n$ where n is the order of D. Then

$$UAU^{\mathrm{CT}} = DW = DW_1 = WM = W_1M$$

and so $\quad WUU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}} = W_1UU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}}, \qquad WUU^{\mathrm{T}}D = W_1UU^{\mathrm{T}}D.$

From $DW = DW_1$, it follows that W and W_1 have like first r rows. From $WUU^{\mathrm{T}}D = W_1UU^{\mathrm{T}}D$, WUU^{T} and W_1UU^{T} have like first r columns. Since W and W_1 have like first r rows, WUU^{T} and W_1UU^{T} have like first r rows also. Since $DW = W_1M = W_1UU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}}$, then $DWUU^{\mathrm{T}} = W_1UU^{\mathrm{T}}D$ Then

$$WUU^{\mathrm{T}} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & X \end{bmatrix}$$

where B_{11} is an $r \times r$ matrix and W_1UU^{T} has the same form except for the matrix X. Let the elements common to both matrices (i.e., the elements of B_{11}, B_{12} and B_{21}) be denoted by w_{ij} (according to their row and column location in WUU^{T}). From $DWUU^{\mathrm{T}} = W_1UU^{\mathrm{T}}D$, it follows that if d_j denotes the jth diagonal element of D,

$$\begin{aligned} d_iw_{ij} &= 0, \quad (i = 1, 2, \ldots, r;\ j = r+1, \ldots, n), \\ w_{ij}d_j &= 0, \quad (j = 1, 2, \ldots, r;\ i = r+1, \ldots, n). \end{aligned}$$

For this range of subscripts, then, $w_{ij} = 0$, i.e., B_{12} and B_{21} are zero matrices. Therefore

$$\begin{aligned} WUU^{\mathrm{T}} &= B_{11} \dotplus X, & W_1UU^{\mathrm{T}} &= B_{11} \dotplus Y, \\ W &= (B_{11} \dotplus X)U^{\mathrm{C}}U^{\mathrm{CT}} & W_1 &= (B_{11} \dotplus Y)U^{\mathrm{C}}U^{\mathrm{CT}} \\ &= UVU^{\mathrm{CT}}, & &= UV^{\mathrm{T}}U^{\mathrm{CT}}. \end{aligned}$$

Therefore

$$V = U^{\mathrm{CT}}\begin{bmatrix} B_{11} & 0 \\ 0 & X \end{bmatrix}U^{\mathrm{C}}, \quad V^{\mathrm{T}} = U^{\mathrm{CT}}\begin{bmatrix} B_{11} & 0 \\ 0 & Y \end{bmatrix}U^{\mathrm{C}}.$$

But

$$V^{\mathrm{T}} = U^{\mathrm{CT}}\begin{bmatrix} B_{11}^{\mathrm{T}} & 0 \\ 0 & X^{\mathrm{T}} \end{bmatrix}U^{\mathrm{C}} = U^{\mathrm{CT}}\begin{bmatrix} B_{11} & 0 \\ 0 & Y \end{bmatrix}U^{\mathrm{C}},$$

and so $B_{11}{}^{\mathrm{T}} = B_{11}$, $Y = X^{\mathrm{T}}$. Now if X is chosen to be any unitary and symmetric matrix of dimension $(n - r) \times (n - r)$ then $X = X^{\mathrm{T}}$ and $V^{\mathrm{T}} = V$.

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press

Consequently, it is always possible in the singular case to find a $V = V^{\mathrm{T}}$ which is suitable for the polar unitary matrix of A.

Now $A = HV = VH^{\mathrm{T}}$ and so

$$UAU^{\mathrm{T}} = UHU^{\mathrm{CT}}UVU^{\mathrm{T}} = UVU^{\mathrm{T}}U^{\mathrm{C}}H^{\mathrm{T}}U^{\mathrm{T}} = DW = WD$$

where $W = UVU^{\mathrm{T}} = W^{\mathrm{T}}$ is unitary. If $W = W_3 + iW_4$ where W_3 and W_4 are real, it is then true that D, W_3, and W_4 are real symmetric matrices which commute in pairs and so there exists a real orthogonal matrix U_1 that diagonalizes all of them. Therefore $U_1UAU^{\mathrm{T}}U_1^{\mathrm{T}} = U_1(DW)U_1^{\mathrm{T}} = D_2$ where D_2 is diagonal with complex elements along the diagonal. Let the diagonal elements of D_2 be

$$\rho_k e^{-i\theta_k} \qquad (k = 1, 2, \ldots, r),$$

and form the diagonal unitary matrix D_u with

$$e^{\frac{1}{2}i\theta_k} \qquad (k = 1, 2, \ldots, r),$$

in the first r diagonal positions followed by ones in the remaining. Then $(D_uU_1U)\,A\,(D_uU_1U)^{\mathrm{T}} = D_3$ is real and diagonal and the Theorem is true.

Next consider the case where $A = A^{\mathrm{T}}$ is a quaternion matrix and the possibility of transforming such a matrix by means of a real orthogonal matrix into a diagonal quaternion matrix. It may be noted first that a quaternion matrix has a polar form **(4)** $A = HV = VK$ with the same properties as mentioned above in connection with the singular and non-singular case. V may, of course, be merely unitary and not necessarily T-unitary.

THEOREM 5. *If A is a quaternion matrix, there exists a real orthogonal matrix U such that $UAU^{\mathrm{T}} = D$ is diagonal and quaternion if and only if A is symmetric with a real hermitian polar matrix.*

Let $A = HV = VK$ where H has real elements and A is symmetric. Then $A^{\mathrm{T}} = (HV)^{\mathrm{T}} = V^{\mathrm{T}}H^{\mathrm{T}} = V^{\mathrm{T}}H$ and so $VK = V^{\mathrm{T}}H$. Whether A is singular or non-singular, the preceding proof may be followed; the matrix U is real in this case, and the relations $DW = DW_1 = WD = W_1D$ result so that $W = B_{11} \dotplus X$ and $W_1 = B_{11} \dotplus Y$ from which $V = U^{\mathrm{T}}(B_{11} \dotplus X)U$ and $V^{\mathrm{T}} = U^{\mathrm{T}}(B_{11} \dotplus Y)U$. Then $V^{\mathrm{T}} = U^{\mathrm{T}}(B_{11}{}^{\mathrm{T}} \dotplus X^{\mathrm{T}})U = U^{\mathrm{T}}(B_{11} \dotplus Y)U$, which is permissible since U is real. Therefore $B_{11}{}^{\mathrm{T}} = B_{11}$ and $X^{\mathrm{T}} = Y$. If A is singular and if X is chosen to be unitary and symmetric, then $V = V^{\mathrm{T}}$; if A is non-singular, then X does not appear above and $V = V^{\mathrm{T}}$ holds automatically. Then $HV = VH = A$ where V is T-unitary and symmetric. By Corollary 1.2, there exists a real orthogonal U_1 such that $U_1VU_1{}^{\mathrm{T}} = D_1$ is diagonal and quaternion with like diagonal elements grouped together. Then

$$U_1AU_1{}^{\mathrm{T}} = U_1VU_1{}^{\mathrm{T}}U_1HU_1{}^{\mathrm{T}} = U_1HU_1{}^{\mathrm{T}}U_1VU_1{}^{\mathrm{T}} = D_1U_1HU_1{}^{\mathrm{T}} = U_1HU_1{}^{\mathrm{T}}D_1.$$

Then $U_1HU_1{}^{\mathrm{T}}$ falls into a direct sum of real symmetric block matrices as

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press

determined by D_1 and a real orthogonal U_2 can then be determined so that $U_2U_1AU_1^{\mathrm{T}}U_2^{\mathrm{T}} = D$ is quaternion diagonal.

Conversely, if $A = U^{\mathrm{T}}DU$ for U real orthogonal, let $D = D_r \cdot D_q$ where D_r consists of the real parts of each diagonal element of D and D_q is composed of the corresponding quaternion part of absolute value one. (If A is singular, D_q is arbitrary to some extent but not necessarily diagonal.) Then

$$A = U^{\mathrm{T}}D_rU \cdot U^{\mathrm{T}}D_qU = U^{\mathrm{T}}D_qU \cdot U^{\mathrm{T}}D_rU$$

is symmetric and the hermitian polar matrix, $U^{\mathrm{T}}D_rU$, has the required property.

THEOREM 6. *If A is a symmetric quaternion matrix with a real polar unitary matrix, there exists a real orthogonal matrix U such that*

$$UAU^{\mathrm{T}} = \begin{bmatrix} D_1 & M \\ M^{\mathrm{T}} & D_2 \end{bmatrix},$$

where D_1 and D_2 are real diagonal and $M = -M^{\mathrm{C}}$; and conversely.

Let $A = HV = VK = V^{\mathrm{T}}H^{\mathrm{T}} = K^{\mathrm{T}}V^{\mathrm{T}}$ since V is real orthogonal. As before $H = K^{\mathrm{T}}$ and $V = V^{\mathrm{T}}$. (In the singular case, let U be a complex unitary matrix such that $UVU^{\mathrm{CT}} = D$; then $UV^{\mathrm{T}}U^{\mathrm{CT}} = D^{\mathrm{CT}}$ and applying this transformation to $V^{\mathrm{T}}H^{\mathrm{T}} = VH^{\mathrm{T}} = HV = HV^{\mathrm{T}}$, there results

$$D^{\mathrm{CT}}M = DM = M_1D = M_1D^{\mathrm{CT}}$$

where $M = UH^{\mathrm{T}}U^{\mathrm{CT}}$ and $M_1 = UHU^{\mathrm{CT}}$ are hermitian, and D has along the diagonal elements of absolute value one.) If in a given row, say k, of M one element $m_{kj} \neq 0$, then $\bar{d}_k m_{kj} = d_k m_{kj}$ (where d_k is the kth diagonal element of D) and so d_k is $+1$ or -1; if all elements of the kth row are zero, d_k is arbitrary except that it must have absolute value one. If the kth row of M is zero, so is the corresponding row of M_1, and conversely. By choosing the arbitrary d_k to be either $+1$ or -1, it is evident that $D = D^{\mathrm{CT}}$ and therefore $V = V^{\mathrm{T}}$.

If $V = I$, (or can be taken to be I), then $A = H = H^{\mathrm{CT}} = H^{\mathrm{T}}$ and so A is real and symmetric and consequently can be brought into diagonal form by a real orthogonal transformation. For the case where A is quaternion symmetric (as considered here), there exists a real orthogonal matrix W so that $WVW^{\mathrm{T}} = I_1 \dotplus I_2 = D$ where I_1 and I_2 are diagonal with $+1$ and -1, respectively, along the diagonal. Then

$$WAW^{\mathrm{T}} = WHW^{\mathrm{T}}WVW^{\mathrm{T}} = WVW^{\mathrm{T}}WH^{\mathrm{T}}W^{\mathrm{T}} = H_1D = DH_1^{\mathrm{T}}$$

where $H_1 = WHW^{\mathrm{T}}$. Let

$$H_1 = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press

be subdivided to correspond to $I_1 \dotplus I_2$ (where $H_1^{\mathrm{T}} = H_1^{\mathrm{C}}$). Then $K_1 = K_1^{\mathrm{T}}$ and $K_4 = K_4^{\mathrm{T}}$ are real symmetric and $K_2 = - K_2^{\mathrm{C}}$ and $K_3 = - K_3^{\mathrm{C}}$ while $K_3 = - K_2^{\mathrm{T}}$. Therefore,

$$WAW^{\mathrm{T}} = \begin{bmatrix} K_1 & -K_2 \\ -K_2^{\mathrm{T}} & -K_4 \end{bmatrix}.$$

Let $W_1K_1W_1^{\mathrm{T}} = D_1$ and $W_2K_4W_2^{\mathrm{T}} = D_2$ be real and diagonal where W_1 and W_2 are real orthogonal, and $U_1 = W_1 \dotplus W_2$. Then

$$U_1WAW^{\mathrm{T}}U_1^{\mathrm{T}} = \begin{bmatrix} D_1 & M \\ M^{\mathrm{T}} & D_2 \end{bmatrix},$$

where $M = - W_1K_2W_2^{\mathrm{T}} = - M$. The converse is evident.

REFERENCES

1. H. C. Lee, *Eigenvalues and canonical forms of matrices with quaternion coefficients*, Proc. Royal Irish Academy, *52*, Section A, No. 117 (1949), 253–260.
2. I. Schur, *Ein Satz über quadratische Formen mit komplexen Koeffizienten*, Amer. J. Math., *67* (1945), 472.
3. N. A. Wiegmann, *Some analogs of the generalized principal axis transformation*, Bull. Amer. Math. Soc., *54* (1948), 905.
4. ———, *Some theorems on matrices with real quaternion elements*, Can. J. Math., *7* (1955), 191–201.
5. J. Williamson, *A polar representation of singular matrices*, Bull. Amer. Math. Soc., *41* (1935), 118–123.
6. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular square matrices*, Proc. Nat. Acad. Sci., U.S.A., *17* (1931), 676–678.

Catholic University
Washington, D.C.

https://doi.org/10.4153/CJM-1956-006-1 Published online by Cambridge University Press