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Classical limit of the scaled elliptic algebra $\mathcal{A}_{\hbar,\eta}(\widetilde{\mathfrak{sl}_2})$

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Abstract. The classical limit of the scaled elliptic algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$is investigated. The limiting Lie algebra is described in two equivalent ways: as a central extension of the algebra of generalized automorphic sl_2 valued functions on a strip and as an extended algebra of decreasing automorphic sl_2 valued functions on the real line. A bialgebra structure and an infinite-dimensional representation in the Fock space are studied. The classical limit of elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ is also briefly presented.

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1. Introduction

The elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ has been introduced in the paper [FIJKMY]. Its definition was induced mostly by the bosonization formulas for massive integrable field theories, proposed in [L]. Then it was shown in [JKM] that within the framework of the representation theory of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$, which is a scaling limit of the algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$, one can obtain integral representations for the correlation functions in the XXZ-model in massless regime.

The structure of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$ is rather unusual. For its precise definition one should introduce a continuous family of generators being Fourier harmonics of the Gauss coordinates of the L-operator. The elements of the algebra are formal integrals over the generators with certain conditions on analyticity and on asymptotics of the coefficients. Next, both algebras $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ and $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$ are not Hopf algebras but form a Hopf family of algebras (see Section 4) and even at level $c = 0$ when these algebras become usual Hopf algebras, they do not have the structure of a double which can be reconstructed either in the Yangian limit $\eta \to 0$ from

$\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$ or in the quantum affine limit $(p \to 0)$ of $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$. We are interested in corresponding properties of the limiting (with respect to $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$) classical algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ which we consider in details. We observe also the elliptic case and the rational degeneration of Lie algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$.

The limiting algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ can be described in two ways. First, it can be realized as a central extension of the algebra of automorphic sl_2-valued generalized functions on a strip. The cocycle is given by an integral of a form which includes derivatives over the period (over the elliptic nome in case of $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$) instead of derivatives over spectral parameter. These algebras are isomorphic for different strips. In contrast to [RS] and [U], there are natural isotropic subalgebras only in the limit $\eta \to 0$ ($p \to 0$ in case of $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$). Nevertheless, at $c = 0$ we have Lie bialgebra which includes according to Sokhotsky formulas the usual currents on the line (on the circle in case of $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$).

On the other hand, we can describe the Lie algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ in terms of Fourier harmonics of the generating functions for $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. This language is a variant of the usual description of a current algebra in terms of Fourier modes for the case of vanishing at infinity currents on the real line. The structural constants of the algebras $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$ or $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ become standard (do not depend on p or η) in terms of Fourier components of the generating functions, but the cobracket has a nontrivial form.

We would like to emphasize the ideology for the description of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ which we keep throughout the paper. The algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ cannot be described in terms of a discrete basis like Taylor coefficients of the analytical functions. In order to define an analog of a basis we are forced to use the language of Fourier harmonics on an open line. Since we continuously get in this way many generators we should specify which integrals over them belong to the algebra. Equivalently, we fix a region for a spectral parameter where the generating functions are meromorphic. In representation theory it means that their matrix coefficients remain meromorphic in this region. Then we define an analytical continuation of the generating functions to larger domains of the spectral parameter. For the definition of this analytical continuation, as well as for the definition of the central extension, we need the language of a generalized function on a strip. Our construction of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ includes essentially all this stuff.

The ideology has been borrowed from [KLP] and the main motivation of writing this paper was to clarify this ideology. We also present an example of an infinite-dimensional representation of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ in the Fock space. One can see that this representation requires a description which differs from the usual construction of the integrable representations of $\widehat{sl_2}$.

2. The Lie algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ of the generalized automorphic functions on a strip

2.1. The algebra $\mathfrak{a}_\eta^0(\widehat{\mathfrak{sl}}_2)$ and its central extension

Let e, f, h be standard generators of Lie algebra $\mathfrak{sl}_2$:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h,$$

and $\eta = 1/\xi \in \mathbb{R}$ be a positive real parameter. Consider the meromorphic $\mathfrak{sl}_2$-valued functions of $z \in \mathbb{C}$ with the period $2i/\eta$ which satisfy the following asymptotical and automorphic conditions:

$$(\mathfrak{i}) \quad e(z) = -e(z + i/\eta), \quad f(z) = -f(z + i/\eta), \quad h(z) = h(z + i/\eta),$$

$$(\mathfrak{ii}) \quad |e(x + iy)| \underset{x\to\pm\infty}{<} C\,\mathrm{e}^{-\pi\eta|x|},$$

$$|f(x + iy)| \underset{x\to\pm\infty}{<} C\,\mathrm{e}^{-\pi\eta|x|}, \quad |h(x + iy)| \underset{x\to\pm\infty}{<} C.$$

One can verify that the functions

$$e(z) = e \otimes \frac{\mathrm{cth}^n \pi\eta(z - a)}{\mathrm{sh}\,\pi\eta(z - a)}, \quad n \geqslant 0, \quad a \in \mathbb{C}, \tag{2.1}$$

$$f(z) = f \otimes \frac{\mathrm{cth}^n \pi\eta(z - b)}{\mathrm{sh}\,\pi\eta(z - b)}, \quad n \geqslant 0, \quad b \in \mathbb{C}, \tag{2.2}$$

$$h(z) = h \otimes \mathrm{cth}^n \pi\eta(z - c), \quad n \geqslant 0, \quad c \in \mathbb{C} \tag{2.3}$$

satisfy the conditions $(\mathfrak{i})$ and $(\mathfrak{ii})$ and their finite linear combinations form a Lie algebra denoted by $\tilde{\mathfrak{a}}_\eta^0(\widehat{\mathfrak{sl}}_2)$.

The algebra $\tilde{\mathfrak{a}}_\eta^0(\widehat{\mathfrak{sl}}_2)$ can be described by means of the generating functions depending on the generating parameter u:

$$\begin{aligned} \tilde{e}_\eta(u) &= e \otimes \frac{i\pi\eta}{\mathrm{sh}\,\pi\eta(z - u)}, \qquad \tilde{h}_\eta(u) = h \otimes i\pi\eta\,\mathrm{cth}\,\pi\eta(z - u), \\ \tilde{f}_\eta(u) &= f \otimes \frac{i\pi\eta}{\mathrm{sh}\,\pi\eta(z - u)}, \end{aligned} \tag{2.4}$$

which satisfy the commutation relations

$$[\tilde{h}_\eta(u_1), \tilde{e}_\eta(u_2)] = 2i\pi\eta\,\mathrm{cth}\,\pi\eta(u_1 - u_2)\tilde{e}_\eta(u_2) - \frac{2i\pi\eta}{\mathrm{sh}\,\pi\eta(u_1 - u_2)}\tilde{e}_\eta(u_1), \tag{2.5}$$

$$[\tilde{h}_\eta(u_1), \tilde{f}_\eta(u_2)]$$
$$= -2i\pi\eta \operatorname{cth} \pi\eta(u_1 - u_2)\tilde{f}_\eta(u_2) + \frac{2i\pi\eta}{\operatorname{sh} \pi\eta(u_1 - u_2)}\tilde{f}_\eta(u_1), \tag{2.6}$$

$$[\tilde{e}_\eta(u_1), \tilde{f}_\eta(u_2)] = \frac{i\pi\eta}{\operatorname{sh} \pi\eta(u_1 - u_2)}\left(\tilde{h}_\eta(u_1) - \tilde{h}_\eta(u_2)\right). \tag{2.7}$$

The Lie algebras $\tilde{\mathfrak{a}}^0_\eta(\widehat{\mathfrak{sl}}_2)$ are isomorphic for different (finite) values of parameter η. An isomorphism

$$T_{\eta,\eta'}\colon \tilde{\mathfrak{a}}^0_\eta(\widehat{\mathfrak{sl}}_2) \to \tilde{\mathfrak{a}}^0_{\eta'}(\widehat{\mathfrak{sl}}_2) \tag{2.8}$$

comes from the gauge transformation

$$a(u) \to \lambda a(\lambda u). \tag{2.9}$$

In terms of generating functions (2.4) the isomorphism (2.8) looks as follows:

$$\begin{aligned} &\tilde{e}_{\eta'}(u) = \frac{\eta'}{\eta}\tilde{e}_\eta\left(\frac{\eta'}{\eta}u\right), \qquad \tilde{h}_{\eta'}(u) = \frac{\eta'}{\eta}\tilde{h}_\eta\left(\frac{\eta'}{\eta}u\right), \\ &\tilde{f}_{\eta'}(u) = \frac{\eta'}{\eta}\tilde{f}_\eta\left(\frac{\eta'}{\eta}u\right). \end{aligned} \tag{2.10}$$

Let $\Pi^+ \subset \mathbb{C}$ be a strip $-1/\eta < \operatorname{Im} z < 0$. In the sequel we need an interpretation of the elements of the algebra $\tilde{\mathfrak{a}}^0_\eta(\widehat{\mathfrak{sl}}_2)$ as of the generalized $\mathfrak{sl}_2$-valued function on the strip Π^+. This description reads as follows. Let K be the space of basic functions $s(z)$ analytical in a strip Π^+, continuous in the closure $\overline{\Pi}^+$ of Π^+ and decreasing in the closed strip $\overline{\Pi}^+$ faster than some exponential function:

$$|s(x+iy)| \underset{x\to\pm\infty}{<} C\,\mathrm{e}^{-\alpha|x|}, \quad \alpha > 0.$$

We treat sl_2-valued functions (2.1)–(2.3) as sl_2-valued functionals on the space K and denote them by $e_+(z), h_+(z), f_+(z)$. A pairing $(a_+, s) \in \mathfrak{sl}_2$, $a \in \tilde{\mathfrak{a}}^0_\eta(\widehat{\mathfrak{sl}}_2)$, $s \in K$ is defined as

$$\int_{-\infty}^{\infty} \mathrm{d}x\, a(x)s(x). \tag{2.11}$$

Let us denote the described space of generalized functions as $\mathfrak{a}^0_\eta(\widehat{\mathfrak{sl}}_2)$. It inherits the structure of Lie algebra $\tilde{\mathfrak{a}}^0_\eta(\widehat{\mathfrak{sl}}_2)$. As before, we compose the distributions into generating functions which we now denote by $e_+(u), h_+(u)$, and $f_+(u)$:

$$\begin{aligned} &e_+(u) = e \otimes \frac{i\pi\eta}{\operatorname{sh} \pi\eta(z-u)}, \qquad h_+(u) = h \otimes i\pi\eta \operatorname{cth} \pi\eta(z-u), \\ &f_+(u) = f \otimes \frac{i\pi\eta}{\operatorname{sh} \pi\eta(z-u)}. \end{aligned} \tag{2.12}$$

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An index $+$ now reminds us that the contour of integration in the pairing (2.11) lies above the pole u. For a fixed $u \in \Pi^+$ these generating functions are distributions from the space $\mathfrak{a}_\eta^0(\widehat{\mathfrak{sl}}_2)$.

The algebra $\mathfrak{a}_\eta^0(\widehat{\mathfrak{sl}}_2)$ is a Lie bialgebra. The cobracket δ is given by the relations

$$\begin{aligned} \delta(e_+(u)) &= h_+(u) \wedge e_+(u), \\ \delta(f_+(u)) &= f_+(u) \wedge h_+(u), \\ \delta(h_+(u)) &= 2e_+(u) \wedge f_+(u). \end{aligned} \tag{2.13}$$

The algebra $\mathfrak{a}_\eta^0(\widehat{\mathfrak{sl}}_2)$ possesses an invariant scalar product $\langle , \rangle$:

$$\begin{aligned} \langle e_+(u_1), f_+(u_2)\rangle &= \frac{\pi\eta^2(u_1-u_2)}{\operatorname{sh}\pi\eta(u_1-u_2)}, \langle h_+(u_1), h_+(u_2)\rangle \\ &= 2\pi\eta^2(u_1-u_2)\operatorname{cth}\pi\eta(u_1-u_2). \end{aligned} \tag{2.14}$$

Note that this scalar product differs from the one used in [RS]. We will specify below the subalgebras of $\mathfrak{a}_\eta^0(\widehat{\mathfrak{sl}}_2)$ which become isotropic in the rational limit $\eta \to 0$.

The Lie algebra $\mathfrak{a}_\eta^0(\widehat{\mathfrak{sl}}_2)$ of generalized automorphic $\mathfrak{sl}_2$-valued functions on the strip admits a central extension. It can be defined by a ‘strange’ cocycle

$$\begin{aligned} &B(x \otimes \varphi(z), y \otimes \psi(z)) \\ &\quad = \frac{\eta^2}{4\pi}\int_{\partial\Pi} \mathrm{d}z \left(\frac{\mathrm{d}\psi(z)}{\mathrm{d}\eta}\varphi(z) - \psi(z)\frac{\mathrm{d}\varphi(z)}{\mathrm{d}\eta}\right)\langle x, y\rangle, \end{aligned} \tag{2.15}$$

where $\langle\, , \rangle$ is the Killing form, $x, y \in \mathfrak{sl}_2$. The integration in (2.15) goes anticlockwise along the boundary $\partial\Pi$ of the strip Π which consists of the real axis and of the line $\operatorname{Im} z = -1/\eta$. The commutation relations of the extended algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ are ($u = u_1 - u_2$)

$$\begin{aligned} [h_+(u_1), e_+(u_2)] &= 2i\pi\eta\operatorname{cth}(\pi\eta u)e_+(u_2) - \frac{2i\pi\eta}{\operatorname{sh}\pi\eta u}e_+(u_1), \\ [h_+(u_1), f_+(u_2)] &= -2i\pi\eta\operatorname{cth}(\pi\eta u)f_+(u_2) + \frac{2i\pi\eta}{\operatorname{sh}\pi\eta u}f_+(u_1), \\ [e_+(u_1), f_+(u_2)] &= \frac{i\pi\eta}{\operatorname{sh}\pi\eta u}(h_+(u_1) - h_+(u_2)) \\ &\quad + ic\pi\eta^2\left(\frac{\pi\eta u\operatorname{ch}\pi\eta u}{\operatorname{sh}^2\pi\eta u} - \frac{1}{\operatorname{sh}\pi\eta u}\right), \\ [h_+(u_1), h_+(u_2)] &= 2ic\pi\eta^2\left(\frac{\pi\eta u}{\operatorname{sh}^2\pi\eta u} - \operatorname{cth}\pi\eta u\right). \end{aligned} \tag{2.16}$$

Unfortunately, we do not know how to extend the Lie coalgebra structure (2.13) from the algebra $\mathfrak{a}^0_\eta(\widehat{\mathfrak{sl}}_2)$ to its central extension $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. We return to this point later in Section 4.

The isomorphism (2.8) induced by the gauge transformation (2.9) has a natural prolongation to the central extended algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. It preserves the pairing (2.14) and multiply the cobracket $\delta\colon \mathfrak{a}^0_\eta(\widehat{\mathfrak{sl}}_2) \to \mathfrak{a}^0_\eta(\widehat{\mathfrak{sl}}_2) \to \mathfrak{a}^0_\eta(\widehat{\mathfrak{sl}}_2)$ by a factor η/η'. So we have actually the unique algebraic structure realized in different spaces of distributions.

2.2. Analytical continuation and Sokhotsky's formulas

For a periodic function (over variable z) of the type

$$a(u) = e \otimes \frac{i\pi\eta}{\operatorname{sh}\pi\eta(z-u)} \quad \text{or} \quad a(u) = h \otimes i\pi\eta \operatorname{cth}\pi\eta(z-u) \quad \text{or}$$
$$a(u) = f \otimes \frac{i\pi\eta}{\operatorname{sh}\pi\eta(z-u)},$$

let us denote by $a_-(u)$ the distribution whose value on a basic function $s(z) \in K$ is given by the integral

$$\int_\Gamma \mathrm{d}z a(z) s(z),$$

where the contour Γ is a line parallel to the real line such that the pole u is the first pole of $a(z)$ located above the contour Γ. Then for $u \in \Pi^- = \{0 < \operatorname{Im} z < 1/\eta\}$ we have by definition

$$\begin{aligned} &e_-(u) = -e_+(u - i/\eta), \quad h_-(u) = h_+(u - i/\eta), \\ &f_-(u) = -f_+(u - i/\eta). \end{aligned} \tag{2.17}$$

The distributions (over the variable z) $e_\pm(u)$, $f_\pm(u)$ and $h_\pm(u)$ admit analytical continuations over the parameter u. For instance, the analytical continuation of $e_-(u)$, $f_-(u)$ and of $h_-(u)$ to the region of parameter $u \in \Pi^+$ are the distributions of the type $a_-(z)$ defined as

$$(a_-, s) = \int_{-i/\eta-\infty}^{-i/\eta+\infty} \mathrm{d}z a(z) s(z).$$

Due to the relations (2.17) this analytical continuation preserves the structure of Lie algebra. The analytical continuation of the commutation relations (2.16) yields:

$$\begin{aligned} &[e_\pm(u_1), f_\pm(u_2)] \\ &\quad = \frac{i\pi\eta}{\operatorname{sh}\pi\eta(u_1-u_2)} \left(h_\pm(u_1) - h_\pm(u_2)\right) + B(e_\pm(u_1), f_\pm(u_2)) \cdot c, \end{aligned}$$

$$[h_\pm(u_1), h_\pm(u_2)] \;=\; B(h_\pm(u_1), h_\pm(u_2)) \cdot c,$$

$$[h_\pm(u_1), e_\pm(u_2)] \;=\; 2i\pi\eta \,\mathrm{cth}\, \pi\eta(u_1 - u_2) e_\pm(u_2) - \frac{2i\pi\eta}{\mathrm{sh}\, \pi\eta(u_1 - u_2)} e_\pm(u_1),$$

$$[h_\pm(u_1), f_\pm(u_2)] \;=\; -2i\pi\eta \,\mathrm{cth}\, \pi\eta(u_1 - u_2) f_\pm(u_2) + \frac{2i\pi\eta}{\mathrm{sh}\, \pi\eta(u_1 - u_2)} f_\pm(u_1),$$

$$[e_\pm(u_1), f_\mp(u_2)] = \frac{i\pi\eta}{\mathrm{sh}\, \pi\eta(u_1 - u_2)} (h_\mp(u_1) - h_\pm(u_2)) + B(e_\pm(u_1), f_\mp(u_2)) \cdot c,$$

$$[h_\pm(u_1), h_\mp(u_2)] \;=\; B(h_\pm(u_1), h_\pm(u_2)) \cdot c,$$

$$[h_\pm(u_1), e_\mp(u_2)] \;=\; 2i\pi\eta \,\mathrm{cth}\, \pi\eta(u_1 - u_2) e_\mp(u_2) - \frac{2i\pi\eta}{\mathrm{sh}\, \pi\eta(u_1 - u_2)} e_\pm(u_1),$$

$$[h_\pm(u_1), f_\mp(u_2)] \;=\; -2i\pi\eta \,\mathrm{cth}\, \pi\eta(u_1 - u_2) f_\mp(u_2) + \frac{2i\pi\eta}{\mathrm{sh}\, \pi\eta(u_1 - u_2)} f_\pm(u_1), \tag{2.18}$$

where $(u = u_1 - u_2)$

$$B(e_\pm(u_1), f_\pm(u_2)) = i\pi\eta^2 \left(\frac{\pi\eta u \,\mathrm{ch}\, \pi\eta u}{\mathrm{sh}^2\, \pi\eta u} - \frac{1}{\mathrm{sh}\, \pi\eta u} \right),$$

$$B(e_\pm(u_1), f_\mp(u_2)) = i\pi\eta^2 \left(\frac{\pi\eta(u \pm i/\eta) \,\mathrm{ch}\, \pi\eta u}{\mathrm{sh}^2\, \pi\eta u} - \frac{1}{\mathrm{sh}\, \pi\eta u} \right),$$

$$B(h_\pm(u_1), h_\pm(u_2)) = 2i\pi\eta^2 \left(\frac{\pi\eta u}{\mathrm{sh}^2\, \pi\eta u} - \mathrm{cth}\, \pi\eta u \right),$$

$$B(h_\pm(u_1), h_\mp(u_2)) = 2i\pi\eta^2 \left(\frac{\pi\eta(u \pm i/\eta)}{\mathrm{sh}^2\, \pi\eta u} - \mathrm{cth}\, \pi\eta u \right).$$

The pairing (2.14) also admits an analytical continuation:

$$\langle e_\pm(u_1), f_\mp(u_2) \rangle = \frac{\pi\eta^2(u_1 - u_2 \pm i/\eta)}{\mathrm{sh}\, \pi\eta(u_1 - u_2)},$$

$$\langle h_\pm(u_1), h_\mp(u_2) \rangle = 2\pi\eta^2(u_1 - u_2 \pm i/\eta) \,\mathrm{cth}\, \pi\eta(u_1 - u_2). \tag{2.19}$$

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The definition of the distributions $e_\pm(u)$, $f_\pm(u)$, $h_\pm(u)$ gives possibility to apply Sokhotsky's formulas. They can be written as follows:

$$\begin{aligned}&e(u)=e_+(u)-e_-(u),\quad f(u)=f_+(u)-f_-(u),\\&h(u)=h_+(u)-h_-(u),\end{aligned}\tag{2.20}$$

where

$$\begin{aligned}&e(u)=2\pi e\otimes\delta(u),\quad f(u)=2\pi f\otimes\delta(u),\\&h(u)=2\pi h\otimes\delta(u).\end{aligned}\tag{2.21}$$

On the other hand, we also have the relations

$$\begin{aligned}&e_\pm(u)=\frac{i\eta}{2}\int_{\Gamma_\pm}\mathrm{d}z\frac{e(z)}{\operatorname{sh}\pi\eta(z-u)},\\&f_\pm(u)=\frac{i\eta}{2}\int_{\Gamma_\pm}\mathrm{d}z\frac{f(z)}{\operatorname{sh}\pi\eta(z-u)},\\&h_\pm(u)=\frac{i\eta}{2}\int_{\Gamma_\pm}\mathrm{d}z h(z)\operatorname{cth}\pi\eta(z-u),\end{aligned}\tag{2.22}$$

where the contour Γ_+ is a line parallel to the real axis and lying above the point u and Γ_- is also a line parallel to the real axis but the point u is above it.

The relations (2.20)–(2.22) show that the algebra of formal sl_2-valued currents on the line is embedded into the analytical continuation of the Lie algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$:

$$\begin{aligned}&[h(u),e(v)]=2\delta(u-v)e(v),\\&[h(u),f(v)]=-2\delta(u-v)f(v),\\&[e(u),f(v)]=\delta(u-v)h(v)+c\cdot\delta'(u-v),\\&[h(u),h(v)]=2c\cdot\delta'(u-v).\end{aligned}\tag{2.23}$$

3. The Lie algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ in terms of Fourier harmonics

Let $\hat e_\lambda$, $\hat f_\lambda$ and $\hat f_\lambda$, $\lambda\in\mathbb{R}$ be the symbols satisfying the following relations

$$\begin{aligned}&[\hat h_\lambda,\hat e_\mu]=2\hat e_{\lambda+\mu},\\&[\hat h_\lambda,\hat f_\mu]=-2\hat f_{\lambda+\mu},\\&[\hat e_\lambda,\hat f_\mu]=\hat h_{\lambda+\mu}+c\cdot\lambda\delta(\lambda+\mu),\end{aligned}$$

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$$[\hat{h}_\lambda, \hat{h}_\mu] = 2c \cdot \lambda\delta(\lambda + \mu). \tag{3.1}$$

Let $\bar{\mathfrak{a}}_\eta$ be a vector space which consists of the expressions of the type

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\hat{e}_\lambda g(\lambda)}{1 + \mathrm{e}^{\lambda/\eta}}, \quad \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\hat{f}_\lambda g'(\lambda)}{1 + \mathrm{e}^{\lambda/\eta}}, \quad ⨍_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\hat{h}_\lambda g''(\lambda)}{1 - \mathrm{e}^{\lambda/\eta}}, \tag{3.2}$$

where $g(\lambda)$, $g'(\lambda)$, $g''(\lambda)$ are quasi-polynomials

$$\begin{aligned} &g(\lambda) = \sum_j P_j(\lambda)\mathrm{e}^{i\lambda u_j}, \qquad g'(\lambda) = \sum_j Q_j(\lambda)\mathrm{e}^{i\lambda u_j}, \\ &g''(\lambda) = \sum_j R_j(\lambda)\mathrm{e}^{i\lambda u_j}, \end{aligned} \tag{3.3}$$

$u_j \in \Pi^+$ and $P_j(\lambda)$, $Q_j(\lambda)$, $R_j(\lambda)$ are polynomials.

We state that the brackets (3.1) define a Lie algebra structure on the space $\bar{\mathfrak{a}}_\eta$ which is isomorphic to $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$.

The isomorphism can be established in the following way. We realize the symbols $\hat{e}_\lambda$, $\hat{f}_\lambda$ and $\hat{h}_\lambda$ as the following functionals on the space K of basic functions $s(z)$:

$$(\hat{e}_\lambda, f(z)) = e \otimes \int_{-\infty}^{\infty} \mathrm{d}z s(z)\, \mathrm{e}^{-i\lambda z}, \tag{3.4}$$

$$(\hat{f}_\lambda, f(z)) = f \otimes \int_{-\infty}^{\infty} \mathrm{d}z s(z)\, \mathrm{e}^{-i\lambda z}, \tag{3.5}$$

$$(\hat{h}_\lambda, f(z)) = h \otimes \int_{-\infty}^{\infty} \mathrm{d}z s(z)\, \mathrm{e}^{-i\lambda z}. \tag{3.6}$$

Then the expressions (3.2) are in one-to-one correspondence with distributions from $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ defined by the functions (2.1)–(2.3). Since both brackets (2.5)–(2.7) and (3.1) for $c = 0$ are pointwise brackets of sl_2-valued functions, they coincide for $c = 0$. One can easily check that the cocycles in (2.16) and in (3.1) also coincide.

In the language of Fourier harmonics we can define a natural extension of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. This extension consists of the following expressions

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda \hat{e}_\lambda \tilde{g}(\lambda), \quad \int_{-\infty}^{\infty} \mathrm{d}\lambda \hat{f}_\lambda \tilde{g}'(\lambda), \quad ⨍_{-\infty}^{\infty} \mathrm{d}\lambda \hat{h}_\lambda \tilde{g}''(\lambda), \tag{3.7}$$

where these functions satisfy the following conditions:

$$\tilde{g}(\lambda) \text{ and } \tilde{g}'(\lambda) \text{ are analytical in the strip } -\pi\eta < \mathrm{Im}(\lambda) < \pi\eta,$$
$$\tilde{g}''(\mu) \text{ is analytical in the strip } -2\pi\eta < \mathrm{Im}\,\mu < 2\pi\eta,$$

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except the point $\mu = 0$ where the function $\tilde{g}''(\mu)$ has a simple pole. Besides, the functions $\tilde{g}(\lambda)$, $\tilde{g}'(\lambda)$, $\tilde{g}''(\lambda)$ should decrease faster than some exponent when $\operatorname{Re}\lambda \to \pm\infty$:

$$\tilde{g}(\lambda) < C\mathrm{e}^{-\alpha|\operatorname{Re}\lambda|}, \quad \tilde{g}'(\lambda) < C\mathrm{e}^{-\beta|\operatorname{Re}\lambda|}, \quad \tilde{g}''(\lambda) < C\mathrm{e}^{-\gamma|\operatorname{Re}\lambda|}$$

for some $\alpha, \beta, \gamma > 0$.

This extended algebra is well-defined and the arguments are the same as in [KLP]. We do not reserve any special notation for the extended algebra. It plays the same role as $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$.

All the structures of the Lie algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ can be reformulated in the language of Fourier harmonics. The invariant scalar product (2.14) now looks standard:

$$\langle \hat{e}_\lambda, \hat{f}_\mu \rangle = \delta(\lambda + \mu), \quad \langle \hat{h}_\lambda, \hat{h}_\mu \rangle = 2\delta(\lambda + \mu) \tag{3.8}$$

and the cobracket for level 0 algebra $\mathfrak{a}^0_\eta(\widehat{\mathfrak{sl}}_2)$ is:

$$\begin{aligned}
\delta \hat{e}_\lambda &= -\frac{1}{2} \fint_{-\infty}^{\infty} \mathrm{d}\tau \hat{h}_\tau \wedge \hat{e}_{\lambda-\tau} \left(\operatorname{cth} \tau/2\eta + \operatorname{th}(\lambda-\tau)/2\eta\right), \\
\delta \hat{f}_\lambda &= -\frac{1}{2} \fint_{-\infty}^{\infty} \mathrm{d}\tau \hat{f}_{\lambda-\tau} \wedge \hat{h}_\tau \left(\operatorname{cth} \tau/2\eta + \operatorname{th}(\lambda-\tau)/2\eta\right), \\
\delta \hat{h}_\lambda &= - \fint_{-\infty}^{\infty} \mathrm{d}\tau \hat{e}_\tau \wedge \hat{f}_{\lambda-\tau} \left(\operatorname{th} \tau/2\eta + \operatorname{th}(\lambda-\tau)/2\eta\right).
\end{aligned} \tag{3.9}$$

The generating functions $e_\pm(u)$, $f_\pm(u)$ and $h_\pm(u)$ are treated now as generating integrals for $\hat{e}_\lambda, \hat{f}_\lambda, \hat{h}_\lambda$. As follows from (3.4)–(3.6) and from (2.12),

$$\begin{aligned}
e_\pm(u) &= \pm \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda u} \frac{\hat{e}_\lambda}{1+\mathrm{e}^{\pm\lambda/\eta}}, \\
f_\pm(u) &= \pm \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda u} \frac{\hat{f}_\lambda}{1+\mathrm{e}^{\pm\lambda/\eta}}, \\
h_\pm(u) &= \pm \fint_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda u} \frac{\hat{h}_\lambda}{1-\mathrm{e}^{\pm\lambda/\eta}}.
\end{aligned} \tag{3.10}$$

and the currents $e(u)$, $f(u)$ and $h(u)$ are total Fourier images of $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$:

$$\begin{aligned}
e(u) &= \int_{-\infty}^{\infty} \mathrm{d}\lambda \hat{e}_\lambda\, \mathrm{e}^{i\lambda u}, \quad f(u) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \hat{f}_\lambda\, \mathrm{e}^{i\lambda u}, \\
h(u) &= \int_{-\infty}^{\infty} \mathrm{d}\lambda \hat{h}_\lambda\, \mathrm{e}^{i\lambda u}.
\end{aligned} \tag{3.11}$$

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4. The representation theory

As we already mentioned in the Introduction, the matrix elements of the generating functions $h_+(u)$, $e_+(u)$, $f_+(u)$ become the meromorphic functions in the representations of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. Let us define representations of this algebra using the description given by (3.7). Assign to each element defined by (3.7), for example, $\int_{-\infty}^{\infty} \mathrm{d}\lambda g(\lambda)\hat{e}_\lambda$, the operator-valued function:

$$\tilde{e}(u) = \int_{-\infty}^{\infty} \mathrm{d}\lambda g(\lambda)\hat{e}_\lambda \, \mathrm{e}^{i\lambda u}.$$

We will say that a representation of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ is well-defined if the operator-valued function $\tilde{e}(u)$ becomes a meromorphic function in the variable $u \in \mathbb{C}$ in some neighbourhood of zero.

4.1. Representation of the algebra $\mathfrak{a}_{\eta,c}(\widehat{\mathfrak{sl}}_2)$ at the level 1

The goal of this subsection is to construct an infinite-dimensional representation of the algebra $\mathfrak{a}_{\eta,c}(\widehat{\mathfrak{sl}}_2)$ at level $c = 1$. For a description of this representation we need a definition of the Fock space generated by a continuous family of free bosons. We borrow this definition from [KLP].

Let a_λ, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ be free bosons which satisfy the commutation relations

$$[a_\lambda, a_\mu] = a(\lambda)\, \delta(+\mu), \quad a(\lambda) = \frac{\lambda}{2}.$$

We define a (right) Fock space $\mathcal{H}_{a(\lambda)}$ as follows. $\mathcal{H}_{a(\lambda)}$ is generated as a vector space by the expressions

$$\int_{-\infty}^{0} f_1(\lambda_1) a_{\lambda_1} \,\mathrm{d}\lambda_1 \ldots \int_{-\infty}^{0} f_n(\lambda_n) a_{\lambda_n} \,\mathrm{d}\lambda_n |\mathrm{vac}\rangle,$$

where the functions $f_i(\lambda)$ satisfy the condition

$$f_i(\lambda) < C\mathrm{e}^{\epsilon\lambda}, \quad \lambda \to -\infty,$$

for $\epsilon > 0$ and $f_i(\lambda)$ are analytical functions in a neighbourhood of $\mathbb{R}_+$ except $\lambda = 0$, where they have a simple pole.

The left Fock space $\mathcal{H}^*_{a(\lambda)}$ is generated by the expressions

$$\langle\mathrm{vac}| \int_{0}^{+\infty} g_1(\lambda_1) a_{\lambda_1} \,\mathrm{d}\lambda_1 \ldots \int_{0}^{+\infty} g_n(\lambda_n) a_{\lambda_n} \,\mathrm{d}\lambda_n,$$

where the functions $g_i(\lambda)$ satisfy the conditions

$$g_i(\lambda) < C\, \mathrm{e}^{-\epsilon\lambda}, \quad \lambda \to +\infty,$$

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and $g_i(t)$ are analytical functions in a neighbourhood of $\mathbb{R}_-$ except $\lambda = 0$, where they also have a simple pole.

The pairing $(,) : \mathcal{H}^*_{a(\lambda)} \otimes \mathcal{H}_{a(\lambda)} \to \mathbb{C}$ is uniquely defined by the following prescriptions:

(i) $(\langle\text{vac}|, |\text{vac}\rangle = 1$,
(ii) $(\langle\text{vac}| \int_0^{+\infty} \mathrm{d}\lambda g(\lambda) a_\lambda, \quad \int_{-\infty}^0 \mathrm{d}\mu f(\mu) a_\mu |\text{vac}\rangle) = \int_{\tilde{C}} \frac{\mathrm{d}\lambda \ln(-\lambda)}{2\pi i} g(\lambda) f(-\lambda) a(\lambda)$,
(iii) the Wick theorem

and contour $\tilde{C}$ shown in Figure 1.

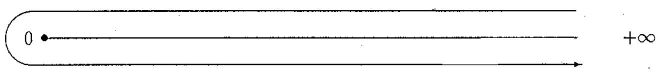

Figure 1.

Let the vacuums $\langle\text{vac}|$ and $|\text{vac}\rangle$ satisfy the conditions

$$a_\lambda|\text{vac} = 0, \quad \lambda > 0, \quad \langle\text{vac}|a_\lambda = 0, \quad \lambda < 0,$$

and $f(\lambda)$ be a function analytical in some neighbourhood of the real line with possible simple pole at $\lambda = 0$ and which has the following asymptotical behaviour:

$$f(\lambda) < \mathrm{e}^{-\epsilon|t|}, \quad \lambda \to \pm\infty$$

for some $\epsilon > 0$. Then, by definition, an operator

$$F = \ :\!\exp\left(\int_{-\infty}^{+\infty} \mathrm{d}\lambda f(\lambda) a_\lambda\right)\!:$$

acts on the right Fock space $\mathcal{H}_{a(\lambda)}$ as follows. $F = F_- F_+$, where

$$F_- = \exp\left(\int_{-\infty}^{0} \mathrm{d}\lambda f(\lambda) a_\lambda\right) \quad \text{and}$$

$$F_+ = \lim_{\epsilon\to 0} \epsilon^{\epsilon f(\epsilon) a_\epsilon} \exp\left(\int_{\epsilon}^{\infty} \mathrm{d}\lambda f(\lambda) a_\lambda\right).$$

An action of operator F on the left Fock space $\mathcal{H}^*_{a(\lambda)}$ is defined via another decomposition $F = \tilde{F}_- \tilde{F}_+$, where

$$\tilde{F}_+ = \exp\left(\int_{0}^{+\infty} \mathrm{d}\lambda f(\lambda) a_\lambda\right) \quad \text{and}$$

$$\tilde{F}_- = \lim_{\epsilon\to 0} \epsilon^{\epsilon f(-\epsilon) a_{-\epsilon}} \exp\left(\int_{-\infty}^{-\epsilon} \mathrm{d}\lambda f(\lambda) a_\lambda\right).$$

These definitions imply that the above-defined-actions of the operator

$$F = :\exp\left(\int_{-\infty}^{+\infty} \mathrm{d}\lambda f(\lambda) a_\lambda\right):$$

on the Fock spaces $\mathcal{H}_{a(\lambda)}$ and $\mathcal{H}^*_{a(\lambda)}$ are adjoint and the product of normally ordered operators satisfy the property [JKM]

$$\begin{aligned}
&:\exp\left(\int_{-\infty}^{\infty} \mathrm{d}\lambda g_1(\lambda)\, a_\lambda\right): \cdot :\exp\left(\int_{-\infty}^{\infty} \mathrm{d}\mu g_2(\mu)\, a_\mu\right): \\
&\quad = \exp\left(\int_{\tilde{C}} \frac{\mathrm{d}\lambda \ln(-\lambda)}{2\pi i} a(\lambda) g_1(\lambda) g_2(-\lambda)\right) \\
&\qquad \times :\exp\left(\int_{-\infty}^{\infty} \mathrm{d}\lambda (g_1(\lambda) + g_2(\lambda)) a_\lambda\right):. \qquad (4.1)
\end{aligned}$$

PROPOSITION. *The generating functions* (4.2) *satisfy the commutation relations* (2.18) *and define a highest weight, level* 1 *representation of the algebra* $\mathfrak{a}_{\eta,c}(\widehat{\mathfrak{sl}}_2)$.

$$\begin{aligned}
e_\pm(u) &= \frac{i\eta\, \mathrm{e}^\gamma}{2} \int_{\Gamma_\pm} \frac{\mathrm{d}z}{\mathrm{sh}\, \pi\eta(z-u)} :\exp\left(\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda z} \frac{2a_\lambda}{\lambda}\right):, \\
f_\pm(u) &= \frac{i\eta\, \mathrm{e}^\gamma}{2} \int_{\Gamma_\pm} \frac{\mathrm{d}z}{\mathrm{sh}\, \pi\eta(z-u)} :\exp\left(-\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda z} \frac{2a_\lambda}{\lambda}\right):, \qquad (4.2) \\
h_\pm(u) &= \pm \fint_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda u} \frac{2a_\lambda}{1 - \mathrm{e}^{\pm\lambda/\eta}}.
\end{aligned}$$

The contours $\Gamma_\pm$ in (4.2) are the same as in (2.22) and γ is Euler constant. The action of the total currents $e(u)$, $f(u)$ and $h(u)$ has a simple form

$$\begin{aligned}
e(u) &= \mathrm{e}^\gamma :\exp\left(\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda u} \frac{2a_\lambda}{\lambda}\right):, \\
f(u) &= \mathrm{e}^\gamma :\exp\left(-\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{e}^{i\lambda u} \frac{2a_\lambda}{\lambda}\right):, \\
h(u) &= 2\int_{-\infty}^{\infty} \mathrm{d}\lambda a_\lambda\, \mathrm{e}^{i\lambda u}.
\end{aligned}$$

5. The algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$ and its classical limit

5.1. The definition of $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$

The algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$ is a scaling limit of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ [FIJKMY] at $p, q \to 1$. This algebra has been investigated in [KLP]. $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$ is generated by

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the central element c and by certain integrals over Fourier harmonics $\hat{e}_\lambda, \hat{f}_\lambda, \hat{h}_\lambda$ for $c = 0$ and $\hat{e}_\lambda, \hat{f}_\lambda, \hat{t}_\lambda$ for $c \neq 0$ (see [KLP]) of the matrix elements of the L-operator $L^+(u,\eta)$ or its analytical continuation $L^-(u,\eta)$:

$$L^-(u,\eta) = \sigma_z L^+(u - i(1/\eta + \hbar c/2), \eta)\sigma_z. \tag{5.1}$$

Operator $L^+(u,\eta)$ satisfies the relations

$$\begin{aligned} &R^+(u_1 - u_2, \eta')L_1^+(u_1,\eta)L_2^+(u_2,\eta) \\ &\quad = L_2^+(u_2,\eta)L_1^+(u_1,\eta)R^+(u_1 - u_2,\eta), \qquad (5.2) \\ &q\text{-det}\, L(u,\eta) = 1, \end{aligned}$$

where

$$\eta' = \frac{\eta}{1 + \eta c\hbar}, \quad c\hbar > 0.$$

The last inequality means that in the representations the central element c is equal to some number c such that $\hbar c > 0$. The R-matrix in (5.2) reads as

$$R^+(u,\eta) = \tau^+(u)R(u,\eta), \quad R(u,\eta) = \varrho(u,\eta)\overline{R}(u,\eta),$$

$$\overline{R}(u,\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u,\eta) & c(u,\eta) & 0 \\ 0 & c(u,\eta) & b(u,\eta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\varrho(u,\eta) = \frac{\Gamma(\hbar\eta)\Gamma(1 + i\eta u)}{\Gamma(\hbar\eta + i\eta u)} \prod_{p=1}^{\infty} \frac{R_p(u,\eta)R_p(i\hbar - u,\eta)}{R_p(0,\eta)R_p(i\hbar,\eta)},$$

$$R_p(u,\eta) = \frac{\Gamma(2p\hbar\eta + i\eta u)\Gamma(1 + 2p\hbar\eta + i\eta u)}{\Gamma((2p+1)\hbar\eta + i\eta u)\Gamma(1 + (2p-1)\hbar\eta + i\eta u)},$$

$$b(u,\eta) = \frac{\text{sh}\, \pi\eta u}{\text{sh}\, \pi\eta(u - i\hbar)},$$

$$c(u,\eta) = \frac{-\text{sh}\, i\pi\eta\hbar}{\text{sh}\, \pi\eta(u - i\hbar)}, \quad \tau^+(u) = \text{cth}\left(\frac{\pi u}{2\hbar}\right) \tag{5.3}$$

and the quantum determinant is

$$q\text{-det}\, L(z,\eta) = L_{11}(z - i\hbar,\eta)L_{22}(z,\eta) - L_{12}(z - i\hbar,\eta)L_{21}(z,\eta). \tag{5.4}$$

Relations between $L^+(u)$ and $L^-(u)$ can be obtained by means of the analytical continuations of the relations (5.2).

In terms of the coordinates of the Gauss decomposition

$$L^+(u) = \begin{pmatrix} 1 & f_+(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (k_+(u+i\hbar))^{-1} & 0 \\ 0 & k_+(u) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e_+(u) & 1 \end{pmatrix}, \tag{5.5}$$

the relations (5.2) read as follows ($u = u_1 - u_2$):

$$e_+(u_1)f_+(u_2) - f_+(u_2)e_+(u_1) = \frac{\text{sh}\, i\pi\eta'\hbar}{\text{sh}\,\pi\eta' u} h_+(u_1) - \frac{\text{sh}\, i\pi\eta\hbar}{\text{sh}\,\pi\eta u}\tilde{h}_+(u_2),$$

$$\begin{aligned}&\text{sh}\,\pi\eta(u+i\hbar)h_+(u_1)e_+(u_2) - \text{sh}\,\pi\eta(u-i\hbar)e_+(u_2)h_+(u_1)\\ &\quad = \text{sh}(i\pi\eta\hbar)\{h_+(u_1), e_+(u_1)\},\end{aligned}$$

$$\begin{aligned}&\text{sh}\,\pi\eta'(u-i\hbar)h_+(u_1)f_+(u_2) - \text{sh}\,\pi\eta'(u+i\hbar)f_+(u_2)h_+(u_1)\\ &\quad = -\text{sh}(i\pi\eta'\hbar)\{h_+(u_1), f_+(u_1)\},\end{aligned}$$

$$\begin{aligned}&\text{sh}\,\pi\eta(u+i\hbar)e_+(u_1)e_+(u_2) - \text{sh}\,\pi\eta(u-i\hbar)e_+(u_2)e_+(u_1)\\ &\quad = \text{sh}(i\pi\eta\hbar)(e_+(u_1)^2 + e_+(u_2)^2),\end{aligned}$$

$$\begin{aligned}&\text{sh}\,\pi\eta'(u-i\hbar)f_+(u_1)f_+(u_2) - \text{sh}\,\pi\eta'(u+i\hbar)f_+(u_2)f_+(u_1)\\ &\quad = -\text{sh}(i\pi\eta'\hbar)(f_+(u_1)^2 + f_+(u_2)^2),\end{aligned}$$

$$\begin{aligned}&\text{sh}\,\pi\eta(u+i\hbar)\text{sh}\,\pi\eta'(u-i\hbar)h_+(u_1)h_+(u_2)\\ &\quad = h_+(u_2)h_+(u_1)\text{sh}\,\pi\eta'(u+i\hbar)\text{sh}\,\pi\eta(u-i\hbar),\end{aligned} \tag{5.6}$$

where

$$h_+(u) = k_+(u)^{-1}k_+(u+i\hbar)^{-1},$$

$$\tilde{h}_+(u) = k_+(u+i\hbar)^{-1}k_+(u)^{-1} = \frac{\eta}{\eta'}\frac{\sin\,\pi\eta'\hbar}{\sin\,\pi\eta\hbar}h_+(u).$$

The commutation relations (5.6) should be treated as the relations for the generating integrals $e_+(u)$, $f_+(u)$, $h_+(u)$ of elements $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$.

The commutation relations for the operator $L^-(u)$ entries can be obtained from (5.6) by analytical continuation as was done in case of commutation relations (2.18).

The Hopf structure on $\mathcal{A}_{\hbar,1/\xi}(\widehat{\mathfrak{sl}}_2)$ is defined in the following sense. Consider the family of the algebras $\mathcal{A}_{\hbar,1/\xi}(\widehat{\mathfrak{sl}}_2)$ with fixed $\hbar 0$ and variable $\xi = 1/\eta$. Then the operations

$$\Delta c = c_1 + c_2 = c \otimes 1 + 1 \otimes c,$$

$$\Delta L^+_{ij}(u,\xi) = \sum_{k=1}^{2} L^+_{kj}(u + ic_2\hbar/2, \xi) \otimes L^+_{ik}(u - ic_1\hbar/2, \xi + \hbar c_1) \tag{5.7}$$

define a map

$$\mathcal{A}_{\hbar,1/\xi}(\widehat{\mathfrak{sl}}_2) \to \mathcal{A}_{\hbar,1/\xi}(\widehat{\mathfrak{sl}}_2) \otimes \mathcal{A}_{\hbar,1/(\xi+\hbar c_1)(\widehat{\mathfrak{sl}}_2)}, \tag{5.8}$$

which is coassociative and is compatible with the commutation relations (5.2). The coproduct in terms of the generating functions $h_+(u)$, $e_+(u)$, $f_+(u)$ is given in [KLP].

5.2. Classical Limit ($\hbar \to 0$) of the Algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$

Let $L^\pm(u,\eta) = 1 + \hbar\mathcal{L}^\pm(u,\eta) + o(\hbar^2)$. It is easy to calculate that

$$\overline{R}(u,\eta) = 1 + \hbar r_0(u,\eta) + \hbar(i\pi\eta \operatorname{cth} \pi\eta u)\mathrm{id} \otimes \mathrm{id} + \hbar^2 r_1(u,\eta) + o(\hbar^2),$$

$$\overline{R}(u,\eta') = 1 + \hbar r_0(u,\eta) + \hbar(i\pi\eta \operatorname{cth} \pi\eta u)\mathrm{id} \otimes \mathrm{id} + \hbar^2 r'_1(u,\eta) + o(\hbar^2),$$

$$\frac{\varrho(u,\eta')}{\varrho(u,\eta)} = 1 + \hbar^2 \varrho_0(u,\eta) + o(\hbar^2),$$

where

$$r_0(u,\eta) = -i\pi\eta \begin{pmatrix} \operatorname{cth}\pi\eta u & 0 & 0 & \\ 0 & 0 & (\operatorname{sh}\pi\eta u)^{-1} & 0 \\ 0 & (\operatorname{sh}\pi\eta u)^{-1} & 0 & 0 \\ 0 & 0 & 0 & \operatorname{cth}\pi\eta u \end{pmatrix} \tag{5.9}$$

is a trigonometric solution to the classical Yang–Baxter equations,

$$r_1(u,\eta) - r'_1(u,\eta)$$
$$= i\pi\eta^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \operatorname{cth}\pi\eta u - \dfrac{\pi\eta u}{\operatorname{sh}^2\pi\eta u} & \dfrac{\pi\eta u \operatorname{ch}\pi\eta u}{\operatorname{sh}^2\pi\eta u} - \dfrac{1}{\operatorname{sh}\pi\eta u} & 0 \\ 0 & \dfrac{\pi\eta u \operatorname{ch}\pi\eta u}{\operatorname{sh}^2\pi\eta u} - \dfrac{1}{\operatorname{sh}\pi\eta u} & \operatorname{cth}\pi\eta u - \dfrac{\pi\eta u}{\operatorname{sh}^2\pi\eta u} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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and

$$\varrho_0(u,\eta) = \frac{i\pi\eta^2}{2}\left(\operatorname{cth}\pi\eta u - \frac{\pi\eta u}{\operatorname{sh}^2\pi\eta u}\right).$$

This expression can be found from the integral representation of the factor $\varrho(u,\eta)$.

We now obtain from (5.2) the relation

$$\begin{aligned}[\mathcal{L}_1^+(u_1), \mathcal{L}_2^+(u_2)] &= [\mathcal{L}_1^+(u_1) + \mathcal{L}_2^+(u_2), r_0(u_1-u_2,\eta)] \\ &\quad + (r_1(u_1-u_2,\eta) - r_1'(u_1-u_2,\eta))c \\ &\quad - \varrho_0(u_1-u_2,\eta)c\cdot \mathrm{id}\otimes\mathrm{id}. \end{aligned} \tag{5.10}$$

These commutation relations can be found without calculation of the expansion of the ratio of scalar factors $\varrho(u,\eta')/\varrho(u,\eta)$. The role of the factor $\varrho_0(u,\eta)$ is to transform the matrix in front of the central element to a traceless matrix. This condition fixes the factor $\varrho_0(u,\eta)$ uniquely. Using the freedom

$$r_0(u,\eta) \to \tilde{r}_0(u,\eta) = r_0(u,\eta) + \kappa(u)\cdot \mathrm{id}\otimes\mathrm{id},$$

let us make the new r-matrix $\tilde{r}_0(u,\eta)$ traceless. Then the commutation relations (5.10) can be written in the form

$$[\mathcal{L}_1^+(u_1), \mathcal{L}_2^+(u_2)] = [\mathcal{L}_1^+(u_1) + \mathcal{L}_2^+(u_2), \tilde{r}_0(u_1-u_2,\eta)] + \eta^2\frac{d\tilde{r}_0(u_1-u_2,\eta)}{d\eta}.$$

We will use this observation in the last section calculating the classical limit of the quantum elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Let $e_\pm(u)$, $f_\pm(u)$ and $h_\pm(u)$ be formal integrals of the symbols $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$ given by the formulas (3.10) with the spectral parameter being a complex number $u\in\mathbb{C}$. By direct verification, we can check that if the complex number u is inside the strip Π^+, then elements $e_+(u)$, $f_+(u)$, $h_+(u)$ belong to $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. If the complex number u is inside the strip Π^-, then the elements $e_-(u)$, $f_-(u)$, $h_-(u)$ also belong to $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. Thus we can treat the integrals $e_\pm(u)$, $f_\pm(u)$, $h_\pm(u)$ as generating functions of the elements of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$, analytical in the strips $\Pi^\pm$. We can state the following:

PROPOSITION. *The commutation relations* (5.10) *for the Gauss coordinates of the L-operator* $\mathcal{L}^+(u,\eta)$

$$\mathcal{L}^+(u,\xi) = \begin{pmatrix} h_+(u)/2 & f_+(u) \\ e_+(u) & -h_+(u)/2 \end{pmatrix}$$

are isomorphic to the commutation relations (2.16) *and the generating functions* $e_+(u)$, $f_+(u)$, $h_+(u)$ *satisfy these commutation relations if* $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$ *satisfy the relations* (3.1).

In order to prove this proposition we should use the Fourier transform calculations and fix in (5.10) either $\mathrm{Im}\, u_1 \langle \mathrm{Im}\, u_2$ or $\mathrm{Im}\, u_1 \rangle \mathrm{Im}\, u_2$.

We conclude that the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ is a classical limit of the algebra $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$. Simple calculations show that for $c = 0$ the cobracket on $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ defined by the usual prescription

$$\delta(x) = \lim_{\hbar\to 0} \frac{\Delta(x) - \Delta'(x)}{\hbar} \tag{5.11}$$

coincides with cobracket (2.13).

For $c \neq 0$ there is no reason to define an object like (5.11) since we have no ways of identifying the tensor components in the image of Δ. Nevertheless, if we follow the standard prescription (5.11) we get the following map:

$$\begin{aligned}
\delta(e_+(u)) &= h_+(u) \wedge e_+(u) + c \wedge \left(\frac{1}{2i}\frac{\mathrm{d}e_+(u)}{\mathrm{d}u} - \eta^2 \frac{\mathrm{d}e_+(u)}{\mathrm{d}\eta}\right),\\
\delta(f_+(u)) &= f_+(u) \wedge h_+(u) + c \wedge \left(\frac{1}{2i}\frac{\mathrm{d}f_+(u)}{\mathrm{d}u} - \eta^2 \frac{\mathrm{d}f_+(u)}{\mathrm{d}\eta}\right),\\
\delta(h_+(u)) &= 2e_+(u) \wedge f_+(u) + c \wedge \left(\frac{1}{2i}\frac{\mathrm{d}h_+(u)}{\mathrm{d}u} - \eta^2 \frac{\mathrm{d}h_+(u)}{\mathrm{d}\eta}\right)
\end{aligned} \tag{5.12}$$

which can be also rewritten in Fourier harmonics as follows:

$$\begin{aligned}
\delta\hat{e}_\lambda &= -\frac{1}{2} \fint_{-\infty}^{\infty} \mathrm{d}\tau\, \hat{h}_\tau \wedge \hat{e}_{\lambda-\tau} (\mathrm{cth}\, \tau/2\eta + \mathrm{th}\,(\lambda-\tau)/2\eta)\\
&\quad + \frac{\lambda}{2}\mathrm{th}\left(\frac{\lambda}{2\eta}\right) \hat{e}_\lambda \wedge c,\\
\delta\hat{f}_\lambda &= -\frac{1}{2} \fint_{-\infty}^{\infty} \mathrm{d}\tau\, \hat{f}_{\lambda-\tau} \wedge \hat{h}_\tau (\mathrm{cth}\, \tau/2\eta + \mathrm{th}\,(\lambda-\tau)/2\eta)\\
&\quad + \frac{\lambda}{2}\mathrm{th}\left(\frac{\lambda}{2\eta}\right) \hat{f}_\lambda \wedge c,\\
\delta\hat{h}_\lambda &= - \fint_{-\infty}^{\infty} \mathrm{d}\tau\, \hat{e}_\tau \wedge \hat{f}_{\lambda-\tau} (\mathrm{th}\, \tau/2\eta + \mathrm{th}\,(\lambda-\tau)/2\eta)\\
&\quad + \frac{\lambda}{2}\mathrm{cth}\left(\frac{\lambda}{2\eta}\right) \hat{h}_\lambda \wedge c.
\end{aligned} \tag{5.13}$$

6. The rational degeneration

The affine algebras with a bialgebra structure usually appear as classical doubles and thus are factorized into a sum of isotropic subalgebras. It is not true for the algebra $\mathfrak{a}^0_\eta(\widehat{\mathfrak{sl}}_2)$. Nevertheless, as follows from the definition of the elements of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ (3.2) and the generating functions (3.10), each substrip of the strips

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$\Pi^{\pm}$ defines a subalgebra of the algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$. It is clear from (3.2) that in terms of Fourier components, these subalgebras are distinguished by different asymptotics of the functions $g(\lambda)$, $g'(\lambda)$, $g''(\lambda)$ at $\lambda \to \pm\infty$. Let us consider the substrips $\tilde{\Pi}^{\pm} \subset \Pi^{\pm}$

$$\tilde{\Pi}^{+} = \left\{-\frac{1}{2\eta} < \operatorname{Im} u < 0\right\}, \quad \tilde{\Pi}^{-} = \left\{0 < \operatorname{Im} u < \frac{1}{2\eta}\right\} \tag{6.1}$$

and restrict the generating functions $e^{+}(u)$, $f^{+}(u)$, $h^{+}(u)$ onto $\tilde{\Pi}^{+}$ and $e^{-}(u)$, $f^{-}(u)$, $h^{-}(u)$ onto $\tilde{\Pi}^{-}$ respectively. Let us denote corresponding subalgebras of $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ as $\mathfrak{a}^{+}_\eta(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{a}^{-}_\eta(\widehat{\mathfrak{sl}}_2)$.

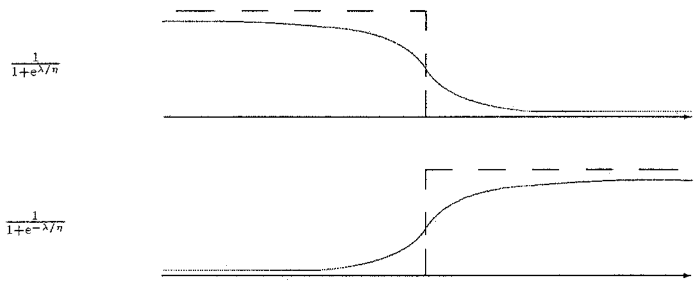

Figure 2.

Then in the limit $\eta \to +0$ these subalgebras become isotropic subalgebras of the loop algebra $\widehat{\mathfrak{sl}}_2$, the generating functions of these subalgebras will be defined in lower and upper half-planes and their expressions turns into the Laplace transform via formal generators (see Figure 2).

In this limit the family $\mathcal{A}_{\hbar,\eta}(\widehat{\mathfrak{sl}}_2)$ turns into the central extended Yangian double $\widehat{DY(\mathfrak{sl}_2)}$ [K, IK] defined in contrast to [K, IK] by means of L-operators $L^{\pm}(u)$ which are analytical in lower and upper half-planes respectively and generated by the continuous family of formal generators $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$ which are formal Fourier harmonics of the elements of the L-operators $L^{\pm}(u)$ [KLP1]. The Lie algebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$ turns at this limit into the central extension of sl_2-valued rational functions vanishing at ∞; subalgebras $\mathfrak{a}^{+}_\eta(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{a}^{-}_\eta(\widehat{\mathfrak{sl}}_2)$ turn into subalgebras of rational functions analytical in lower and upper half-planes respectively.

The elements of the algebra $\mathfrak{a}_0(\widehat{\mathfrak{sl}}_2)$ can be identified with $\mathfrak{sl}_2$-valued distributions

$$e \otimes \frac{P(z)}{Q(z)}, \quad f \otimes \frac{P'(z)}{Q'(z)}, \quad h \otimes \frac{P''(z)}{Q''(z)}\,, \tag{6.2}$$

where $P(z)$, etc., are polynomials such that $\deg P(z) < \deg Q(z)$, $\deg P'(z) < \deg Q'(z)$ and $\deg P''(z) < \deg Q''(z)$. Subalgebras $\mathfrak{a}^{+}_0(\widehat{\mathfrak{sl}}_2)$ and $\mathfrak{a}^{-}_0(\widehat{\mathfrak{sl}}_2)$ are distinguished by the contours in pairing the distributions (6.2) and the elements of the space of the basic functions on the complex plane analytical in the lower and the upper half-planes respectively and vanishing at infinity. For the elements of

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the subalgebra $\mathfrak{a}_0^+(\widehat{\mathfrak{sl}}_2)$, this contour is parallel to the real axis and lies above all zeros of the polynomials $Q(z)$, $Q'(z)$ and $Q''(z)$ and vice versa for the subalgebra $\mathfrak{a}_0^-(\widehat{\mathfrak{sl}}_2)$.

The central extention of the algebra $\mathfrak{a}_0^+(\widehat{\mathfrak{sl}}_2)$ is defined by the standard cocycle and the commutation relations between the generating functions of the elements of the algebra $\mathfrak{a}_{0,c}^+(\widehat{\mathfrak{sl}}_2)$ and can be formally obtained from the commutation relations (2.18) at the limit $\eta \to +0$:

$$
\begin{aligned}
&[e_\pm(u_1), f_\pm(u_2)] = \frac{i}{u_1-u_2}(h_\pm(u_1) - h_\pm(u_2)),\\
&[h_\pm(u_1), h_\pm(u_2)] = 0,\\
&[h_\pm(u_1), e_\pm(u_2)] = \frac{2i}{u_1-u_2}(e_\pm(u_2) - e_\pm(u_1)),\\
&[h_\pm(u_1), f_\pm(u_2)] = \frac{-2i}{u_1-u_2}(f_\pm(u_2) - f_\pm(u_1)),\\
&[e_\pm(u_1), f_\mp(u_2)] = \frac{i}{u_1-u_2}(h_\mp(u_1) - h_\pm(u_2)) \mp \frac{1}{(u_1-u_2)^2}\cdot c,\\
&[h_\pm(u_1), h_\mp(u_2)] = \mp\frac{2}{(u_1-u_2)^2}\cdot c,\\
&[h_\pm(u_1), e_\mp(u_2)] = \frac{2i}{u_1-u_2}(e_\mp(u_2) - e_\pm(u_1)),\\
&[h_\pm(u_1), f_\mp(u_2)] = \frac{-2i}{u_1-u_2}(f_\mp(u_2) - f_\pm(u_1)).
\end{aligned}
$$

Subalgebras $\mathfrak{a}_0^\pm(\widehat{\mathfrak{sl}}_2)$ become isotropic and there is a nontrivial pairing between these subalgebras:

$$
\langle e_+(u_1), f_-(u_2)\rangle = \frac{i}{u_1-u_2}, \quad \langle h_+(u_1), h_-(u_2)\rangle = \frac{2i}{u_1-u_2}.
$$

The algebra $\mathfrak{a}_0(\widehat{\mathfrak{sl}}_2)$ is a bialgebra with the cobracket:

$$
\begin{aligned}
&\delta(e_\pm(u)) = \pm h_\pm(u)\wedge e_\pm(u) \pm i\frac{de_\pm(u)}{du}\wedge c,\\
&\delta(f_\pm(u)) = \pm f_\pm(u)\wedge h_\pm(u) \pm i\frac{df_\pm(u)}{du}\wedge c,\\
&\delta(h_\pm(u)) = \pm 2e_\pm(u)\wedge f_\pm(u) \pm i\frac{dh_\pm(u)}{du}\wedge c,
\end{aligned}
$$

The representation (3.10) in the form of Fourier integrals becomes the Laplace transform (this can be visualized in Figure 1)

$$e_\pm(u) = \pm \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{\mp i\lambda u} \hat{e}_{\mp\lambda}\theta(\lambda), \qquad f_\pm(u) = \pm \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{\mp i\lambda u} \hat{f}_{\mp\lambda}\theta(\lambda),$$

$$h_\pm(u) = \pm \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{\mp i\lambda u} \hat{h}_{\mp\lambda}\theta(\lambda),$$

so the formal generators $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$ at $\lambda \leqslant 0$ form the subalgebra $\mathfrak{a}_0^+(\widehat{\mathfrak{sl}}_2)$ and $\hat{e}_\lambda$, $\hat{f}_\lambda$, $\hat{h}_\lambda$ at $\lambda \geqslant 0$ the subalgebra $\mathfrak{a}_0^-(\widehat{\mathfrak{sl}}_2)$. The map (5.13) now becomes a cobracket

$$\delta\hat{e}_\lambda = -\int_0^\lambda \mathrm{d}\tau[\theta(\tau) - \theta(\tau-\lambda)]\hat{h}_\tau \wedge \hat{e}_{\lambda-\tau} + \frac{\lambda}{2}\,\mathrm{sgn}(\lambda)\hat{e}_\lambda \wedge c,$$

$$\delta\hat{f}_\lambda = -\int_0^\lambda \mathrm{d}\tau[\theta(\tau) - \theta(\tau-\lambda)]\hat{f}_{\lambda-\tau} \wedge \hat{h}_\tau + \frac{\lambda}{2}\,\mathrm{sgn}(\lambda)\hat{f}_\lambda \wedge c,$$

$$\delta\hat{h}_\lambda = -2\int_0^\lambda \mathrm{d}\tau[\theta(\tau) - \theta(\tau-\lambda)]\hat{e}_\tau \wedge \hat{f}_{\lambda-\tau} + \frac{\lambda}{2}\,\mathrm{sgn}(\lambda)\hat{h}_\lambda \wedge c,$$

where $\theta(\lambda)$ is the step function

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0 \\ \frac{1}{2}, & \lambda = 0 \\ 0, & \lambda < 0 \end{cases}$$

and defines a Lie bialgebra structure on $\mathfrak{a}_0(\widehat{\mathfrak{sl}}_2)$ at arbitrary central element c. This bialgebra is a classical double of one of subalgebras $\mathfrak{a}_0^\pm(\widehat{\mathfrak{sl}}_2)$.

7. The Lie algebra $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$

The structure of the Lie bialgebra $\mathfrak{a}_\eta(\widehat{\mathfrak{sl}}_2)$(at level 0) from Sections 1 and 2 can be automatically generalized to the elliptic case. Using the Lie algebra of double periodic automorphic functions which take values in $\mathfrak{sl}_2$ [RS], one can construct on the half-parallelogram of the periods $\Pi\ (0, 2K, iK', 2K + iK')$ the Lie algebra of $\mathfrak{sl}_2$-valued generalized functions $\mathfrak{a}_p^0(p = \mathrm{e}^{\pi i\tau} = \mathrm{e}^{-\pi K'/K})$. Let us introduce the generating functions $\sigma_a^\pm(u)$ of the elements of the algebra $\mathfrak{a}_p^0(\widehat{\mathfrak{sl}}_2)$:

$$\sigma_a^+(u) = \sigma_a \otimes \omega_a(z-u), \quad \sigma_a^-(u) = \sigma_a^+(u - K'), \quad a = 1, 2, 3, \tag{7.1}$$

where σ_a are the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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and functions $\omega_a(u)$ are ratios of elliptic Jacobi functions of modulus k [BE]:

$$\omega_1(u) = \frac{1}{\mathrm{sn}\,(u,k)}, \quad \omega_2(u) = \frac{\mathrm{dn}\,(u,k)}{\mathrm{sn}\,(u,k)}, \quad \omega_3(u) = \frac{\mathrm{cn}\,(u,k)}{\mathrm{sn}\,(u,k)}.$$

Because of the addition theorem for elliptic functions

$$\omega_a(u-v)\omega_c(v) - \omega_b(u-v)\omega_c(u) = \omega_a(u)\omega_b(v) \tag{7.2}$$

the generating functions $\sigma_a^{\pm}(u)$ satisfy the commutation relations:

$$\begin{aligned}[][\sigma_a^{\pm}(u_1), \sigma_b^{\pm}(u_2)] &= 2\left[i\omega_a(u_1-u_2)\sigma_c^{\pm}(u_2) - i\omega_b(u_1-u_2)\sigma_c^{\pm}(u_1)\right], \\ [\sigma_a^{\pm}(u_1), \sigma_b^{\mp}(u_2)] &= 2\left[i\omega_a(u_1-u_2)\sigma_c^{\mp}(u_2) - i\omega_b(u_1-u_2)\sigma_c^{\pm}(u_1)\right], \end{aligned} \tag{7.3}$$

where a, b, c are cyclic permutations of 1, 2, 3. The cobracket

$$\delta\sigma_a(u) = \sigma_b(u) \wedge \sigma_c(u) \tag{7.4}$$

defines on $\mathfrak{a}_p^0(\widehat{\mathfrak{sl}}_2)$ a Lie bialgebra structure.

In analogy with the trigonometric case the algebra $\mathfrak{a}_p^0(\widehat{\mathfrak{sl}}_2)$ admits a central extension given by the two-cocycle:

$$B(x\otimes\varphi(z), y\otimes\psi(z)) = \frac{1}{2K}\int_{\partial\Pi}\frac{\mathrm{d}z}{2\pi i}\left(\frac{\mathrm{d}\psi(z)}{\mathrm{d}\tau}\varphi(z) - \psi(z)\frac{\mathrm{d}\varphi(z)}{\mathrm{d}\tau}\right)\langle x, y\rangle,$$

where $\partial\Pi$ is the boundary of the period's half-parallelogram $(0, 2K, iK', 2K + iK')$. The cocycle property is a consequence of the addition theorem (7.2). The values of the cocycle on the generating functions are

$$B(\sigma_a^{\pm}(u_1), \sigma_b^{\pm}(u_2)) = \frac{\delta_{a,b}}{K}\frac{d\omega_a(u_1-u_2)}{d\tau}, \tag{7.5}$$

the values of the cocycle for the other combinations of generating functions one can find by applying the automorphic conditions (7.1) to (7.5). The central extended Lie algebra $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$ is defined in terms of generating functions subjected to the commutation relations (7.3) together with the relations (7.6) instead of trivial ones.

$$\begin{aligned}[][\sigma_a^{\pm}(u_1), \sigma_a^{\pm}(u_2)] &= c\cdot B(\sigma_a^{\pm}(u_1), \sigma_a^{\pm}(u_2)), \\ [\sigma_a^{\pm}(u_1), \sigma_a^{\mp}(u_2)] &= c\cdot B(\sigma_a^{\pm}(u_1), \sigma_a^{\mp}(u_2)). \end{aligned} \tag{7.6}$$

It should be mentioned that in terms of Fourier harmonics

$$\sigma_1^{+}(u) = \frac{i\pi}{K}\sum_{\ell\ \text{odd}} \sigma_1^{\ell}\frac{p^{\ell}}{p^{\ell}-1}\mathrm{e}^{2\pi i\ell u},$$

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$$\sigma_2^+(u) = \frac{i\pi}{K}\sum_{\ell \text{ odd}} \sigma_2^\ell \frac{p^\ell}{p^\ell+1}\mathrm{e}^{2\pi i\ell u},$$

$$\sigma_3^+(u) = \frac{i\pi}{K}\sum_{\ell \text{ even}} \sigma_3^\ell \frac{p^\ell}{p^\ell+1}\mathrm{e}^{2\pi i\ell u},$$

the commutation relations (7.3), (7.6) become the relations for the generators of central extended loop algebra $\widehat{\mathrm{sl}}_2$:

$$[\sigma_a^k, \sigma_b^\ell] = 2i\varepsilon_{abc}\sigma_c^{k+\ell} + c\cdot B(\sigma_a^k, \sigma_b^\ell) \tag{7.7}$$

with the standard cocycle

$$B(\sigma_a^k, \sigma_b^\ell) = k\delta_{a,b}\delta_{k,-\ell}.$$

But, as well as in the trigonometric case, it is not sufficient to write down the relations (7.7) for the description of the Lie algebra $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$. One should specify the elements of $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$ to be a series of the formal generators σ_a^k with coefficients of the type (3.2) and (3.3), which appear after Fourier decomposition of the generalized functions from the space $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$. In terms of Fourier harmonics, the cobracket (7.4) is given by the relations

$$\delta(\sigma_1^k) = \sum_{\substack{i,j\\ i+j=k}} \frac{p^k-1}{(p^i+1)(p^j+1)}\sigma_2^i\wedge\sigma_3^j,$$

$$\delta(\sigma_2^k) = \sum_{\substack{i,j\\ i+j=k}} \frac{p^k+1}{(p^i+1)(p^j-1)}\sigma_3^i\wedge\sigma_1^j,$$

$$\delta(\sigma_3^k) = \sum_{\substack{i,j\\ i+j=k}} \frac{p^k+1}{(p^i-1)(p^j+1)}\sigma_1^i\wedge\sigma_2^j.$$

The central extended algebra $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$ can be written in r-matrix formalism. Let

$$r(u) = \sum_{a=1}^{3}\omega_a(u)\sigma_a\otimes\sigma_a \tag{7.8}$$

be an elliptic solution of the classical Yang–Baxter equation and

$$\mathcal{L}^+(u) = \begin{pmatrix} \sigma_3^+(u) & \sigma_1^+(u) - i\sigma_2^+(u) \\ \sigma_1^+(u) + i\sigma_2^+(u) & -\sigma_3^+(u)\end{pmatrix},$$

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where $\sigma_a^+(u)$ satisfy the commutation relations (7.3) and (7.6). The r-matrix (7.8) is traceless and the commutation relations (7.8) and (7.6) can be written in the following form:

$$[\mathcal{L}_1^+(u_1), \mathcal{L}_2^+(u_2)] = [\mathcal{L}_1^+(u_1) + \mathcal{L}_2^+(u_2), r(u_1 - u_2)] + \frac{1}{K}\frac{dr(u_1-u_2)}{d\tau} \cdot c. \quad (7.9)$$

This representation allows us to demonstrate that the Lie algebra $\mathfrak{a}_p(\widehat{\mathfrak{sl}}_2)$ is a classical limit of the quantum elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ [FIJKMY]. Indeed, in the 'RLL' formalism, the algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$ is defined by the relations

$$R(v; \hbar, \tilde{\tau}) L_1^+(u_1) L_2^+(u_2) = L_2^+(u_2) L_1^+(u_1) R(v; \hbar, \tilde{\tau}^*), \quad (7.10)$$

where $\hbar$ is a deformation parameter and $R(v; \hbar, \tilde{\tau})$ is the Baxter elliptic R-matrix [B]

$$\rho(v) \begin{pmatrix} a(v) & 0 & 0 & d(v) \\ 0 & b(v) & c(v) & 0 \\ 0 & c(v) & b(v) & 0 \\ d(v) & 0 & 0 & a(v) \end{pmatrix}.$$

The normalization factor $\rho(v)$ provides the crossing symmetry and unitarity conditions and

$$a(v; \hbar, \tilde{\tau}) = \mathrm{sn}(\hbar + iv), \quad b(v; \hbar, \tilde{\tau}) = \mathrm{sn}(iv),$$
$$c(v; \hbar, \tilde{\tau}) = \mathrm{sn}(\hbar), \quad d(v; \hbar, \tilde{\tau}) = \tilde{k}\, \mathrm{sn}(\hbar)\mathrm{sn}(iv)\mathrm{sn}(\hbar + iv). \quad (7.11)$$

In (7.11) $\mathrm{sn}(x) = \mathrm{sn}(x, \tilde{k})$ is Jacobi's elliptic function of modulus $\tilde{k}$. The modular parameter $\tilde{\tau} = i\tilde{K}'/\tilde{K}$ is defined by the half-periods $\tilde{K}$ and $\tilde{K}'$ and related to $\tilde{k}$ in the standard way. In (7.10) $\tilde{\tau}^*$ is the following expression:

$$\tilde{\tau}^* = \tilde{\tau} + \alpha\hbar c, \quad (7.12)$$

where c is the central element and $\mathbb{C}$-number α depends on the half-period $\tilde{K}$ and modulus $\tilde{k}$.

To obtain the commutation relations (7.9) from (7.10) first we have to use the Landen transform

$$k' = \frac{1-\tilde{k}}{1+\tilde{k}}, \quad \zeta = \frac{\hbar(1+\tilde{k})}{2}, \quad u = i(1+\tilde{k})v$$

applied to the matrix elements of the R-matrix and then the imaginary Jacobi transform which connect the elliptic functions of argument iu and supplementary modulus k' with those of argument u and modulus k (see details in [B]). The R-matrix in this new parametrization reads:

$$R(u;\zeta,\tau) = \tilde{\rho}(u)\left[1 + \sum_{a=1}^{3} W_a(u)\sigma_a \otimes \sigma_a\right],$$

where

$$W_1 = \frac{\mathrm{sn}(\zeta,k)}{\mathrm{sn}(u+\zeta,k)}, \qquad W_2 = \frac{\mathrm{sn}(\zeta,k)\,\mathrm{dn}(u+\zeta,k)}{\mathrm{sn}(u+\zeta,k)\,\mathrm{dn}(\zeta,k)},$$

$$W_3 = \frac{\mathrm{sn}(\zeta,k)\,\mathrm{cn}(u+\zeta,k)}{\mathrm{sn}(u+\zeta,k)\,\mathrm{cn}(\zeta,k)}$$

and had been used by E. Sklyanin in [Sk]. Since Landen transform do not affect much the modular parameter τ the prescription of the central extension (7.12) will be the same: $\tau^* = \tau + \zeta c/K$.

After all the transformations, the classical limit of the quantum relations (7.10) means that $\zeta \to 0$. In this limit $L^+(u) = 1 + \zeta\,\mathcal{L}^+(u) + o(\zeta^2)$, $R(u;\zeta,\tau) = [1 + \zeta r(u,\tau) + \zeta^2 r_1(u,\tau) + o(\zeta^3)]$, $R(u;\zeta,\tau^*) = [1 + \zeta r(u,\tau) + \zeta^2\tilde{r}_1(u,\tau) + o(\zeta^3)]$ and

$$\tilde{r}_1(u,\tau) - r_1(u,\tau) = \frac{c}{K}\frac{\mathrm{d}r(u,\tau)}{\mathrm{d}\tau}$$

so (7.9) appears as the first nontrivial coefficient of the Taylor expansion of (7.10) with respect to ζ. Since the classical r-matrix is traceless, the scalar factor $\tilde{\rho}(u)$ does not contribute towards the calculation of (7.9) (compare with Subsection 4.2).

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