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ON THE ARF INVARIANT OF AN INVOLUTION

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Let Σ^{4k+1} denote a smooth manifold homeomorphic to the $(4k+1)$-sphere, S^{4k+1}, $k \geqq 1$, and $T: \Sigma^{4k+1} \to \Sigma^{4k+1}$ a differentiable free involution. Our aim in this note is to derive a connection between the differentiable structure on Σ^{4k+1} and the properties of the free involution T.

To be more specific, recall [**5**] that the h-cobordism classes of smooth manifolds homeomorphic (or, what is the same, homotopy equivalent) to S^{4k+1}, $k \geqq 1$, form a finite abelian group θ_{4k+1} with group operation connected sum. The elements are called homotopy spheres. Those homotopy spheres that bound parallelizable manifolds form a subgroup $bP_{4k+2} \subset \theta_{4k+1}$. It is proved in [**5**, Theorem 8.5] that bP_{4k+2} is either zero or cyclic of order 2. In the latter case the two distinct homotopy spheres are distinguished by the Arf invariant of the parallelizable manifolds they bound.

This Arf invariant may be defined as follows. (For details see [**10**, § 5 or **7**, § 4].) Let Σ^{4k+1} bound a parallelizable manifold. Performing interior surgery [**5**, § 5] we obtain a new parallelizable manifold M^{4k+2} bounding Σ^{4k+1} so that M^{4k+2} is $2k$-connected.

The homology group $H_{2k+1}(M; Z)$ is free of even rank and each element $x \in H_{2k+1}(M; Z)$ may be represented by an immersed framed sphere, S_x. In addition to the usual skew-symmetric bilinear (algebraic) intersection pairing denoted by $x \cdot y$ for $x, y \in H_{2k+1}(M; Z)$, we can define a map

$$\mu: H_{2k+1}(M; Z) \to Z_2$$

to count, modulo 2, the self-intersections of S_x. Equivalently, $\mu(x)$ counts, modulo 2, the number of pairs of points in the intersection $S_x \cdot S_x$.

Note that

$$\mu(x + y) = \mu(x) + \mu(y) + x \cdot y \pmod 2.$$

If we choose a symplectic basis for $H_{2k+1}(M; Z)$:

$$e_1, \ldots, e_r, e_1^*, \ldots, e_r^*; \qquad e_i \cdot e_j = 0, \quad e_i^* \cdot e_j^* = 0, \quad e_i \cdot e_j^* = \delta_{ij},$$

then the Arf invariant associated with μ and $\cdot$ is the integer, modulo 2, defined by

$$c(M) = \sum_{i=1}^{r} \mu(e_i)\mu(e_i^*) \pmod 2.$$

It is called the *Arf invariant of M* since it is independent of the choices above. It represents the obstruction to making M a ball by surgery.

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Let $T: \Sigma^{4k+1} \to \Sigma^{4k+1}$ be a differentiable free involution. In [**2**], Browder and Livesay considered the problem of finding an invariant codimension one sphere $S^{4k} \subset \Sigma^{4k+1}$. The approach is first to construct an invariant submanifold $W^{4k} \subset \Sigma^{4k+1}$ so that $\Sigma^{4k+1} = A \cup TA$, A and TA are compact submanifolds of Σ^{4k+1} and $\partial A = \partial TA = A \cap TA = W$. Next perform *equivariant surgery* (see [**2**; **6**]) on W trying to make it a sphere.

There is no obstruction to making W $(2k - 1)$-connected. In the middle dimension, however, another Arf invariant occurs as the obstruction. This is described as follows.

From the Mayer-Vietoris sequence of $(\Sigma; A, TA)$ we have:

$$0 \to H_{2k}(W) \xrightarrow{(i_A, i_{TA})} H_{2k}(A) \oplus H_{2k}(TA) \to 0;$$

hence $H_{2k}(W) = \ker i_A \oplus \ker i_{TA}$ and $T(\ker i_A) = \ker i_{TA}$. The bilinear form

$$B(x, y) = x \cdot Ty$$

is defined for $x, y \in \ker i_A$ and it is skew-symmetric. A cohomology operation $\psi(x)$ is defined to count, modulo 2, the number of pairs (q, Tq) of points in $S^{2k} \cap TS^{2k}$, where S^{2k} represents an immersion of the Poincaré dual of x. It is used to obtain a function $\psi_0: H_{2k}(W; Z_2) \to Z_2$ with the property

$$\psi_0(x + y) = \psi_0(x) + \psi_0(y) + B_2(x, y),$$

where B_2 is the modulo 2 reduction of B.

Choose a symplectic basis for $\ker i_A$ and define the Arf invariant associated with ψ_0 and B, $c(T, \Sigma)$, to be the *Arf invariant of the involution.*

Browder and Livesay proved [**2**, Theorem 3] that an invariant codimension one subsphere $S^{4k} \subset \Sigma^{4k+1}$ exists if and only if $c(T, \Sigma) = 0$.

Now suppose that $\Sigma^{4k+1} \in bP_{4k+2}$. It has been conjectured that Σ bounds a parallelizable manifold M so that $c(M) = c(T, \Sigma)$. We are not able to establish this relation, but we do establish a sufficient condition under which it holds.† We study the structure of the submanifold W and prove that if $c(T, \Sigma) = c$, then it may be reduced to a standard manifold W_c; W_0 is a sphere and W_1 is given in Lemma 2. Moreover, W_c bounds a standard framed manifold V_c; V_0 is a disk, $V_1 = J \times I$, where J is obtained by plumbing two copies of the tangent D^{2k}-bundle of S^{2k}.

With this notation we prove the following result.

THEOREM. *If $T|W_c$ extends to a framed involution of V_c, then Σ bounds a parallelizable manifold M so that $c(M) = c(T, \Sigma)$.*

As corollaries we obtain known results about free involutions on Brieskorn spheres.

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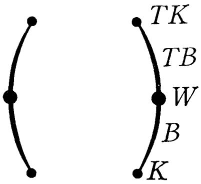
†*Added in proof.* G. Brumfiel gives a counterexample to this conjecture in "Differentiable S^1 actions on homotopy spheres" (preprint). The question arises whether our sufficient condition is also necessary.

The structure of W. Let $T\colon \Sigma^{4k+1} \to \Sigma^{4k+1}$ be a free involution of the homotopy sphere $\Sigma^{4k+1} \in bP_{4k+2}$, $k \geqq 1$.

If $c(T, \Sigma) = 0$, we can perform equivariant surgery on W to obtain an invariant sphere S^{4k}.

Assume now that $c(T, \Sigma) = 1$. In this case we can use equivariant surgery on W to make it $(2k-1)$-connected and $H_{2k}(W) = Z + Z + Z + Z$, but for any symplectic basis e, f, Te, Tf we have

$$\psi_0(e) = \psi_0(f) = \psi_0(Te) = \psi_0(Tf) = 1.$$



We now remove the interior of a disc from A and its image from TA. The resulting manifold B is smooth and since W bounds a parallelizable manifold, B is a framed cobordism between W and the (standard) sphere K^{4k}. We shall first investigate B.

Since $B \cup TB$ is an h-cobordism between K and TK, a Mayer-Vietoris argument shows that B is $(2k-1)$-connected and

$$0 \to H_{2k}(W) \to H_{2k}(B) \oplus H_{2k}(TB) \to 0$$

is an isomorphism. Thus for B, the induced map $i_B\colon W \to B$ is onto in homology and we have the exact sequence of free abelian groups

$$0 \to H_{2k+1}(B, \partial B) \to H_{2k}(W) \xrightarrow{i_{B*}} H_{2k}(B) \to 0.$$

Let $\ker i_{B*} = \ker i_{A*}$ be generated by α and $T\beta$. Then we may assume that $H_{2k}(W)$ is generated by $\{\alpha, \beta, T\alpha, T\beta\}$, a symplectic basis, with $\alpha \cdot \beta = -T\alpha \cdot T\beta = 1$, and all other intersection numbers are zero. Clearly $\psi_0(\alpha) = \psi_0(\beta) = \psi_0(T\alpha) = \psi_0(T\beta) = 1$. That $\alpha \cdot \alpha = T\beta \cdot T\beta = 0$ and $\alpha \cdot T\beta = 0$ is clear since we perform surgery on them. The fact that $\alpha \cdot T\alpha = 0$ follows since α and $T\alpha$ intersect in pairs of points Q, TQ with opposite intersection numbers. Finally, $\alpha \cdot \beta = \pm 1$ since the intersection form is non-singular. We orient β so that $\alpha \cdot \beta = 1$. Since i_{B*} is onto, we can embed the framed surgery on α and $T\beta$ in B and the remaining manifold is an h-cobordism. Let $f_1\colon S^{2k} \times D^{2k} \to W$ represent α and $f_2\colon S^{2k} \times D^{2k} \to W$ represent $T\beta$. Then we have proved the following.

LEMMA 1. $B = W \times I \cup_{f_1} D^{2k+1} \times D^{2k} \cup_{f_2} D^{2k+1} \times D^{2k}$.

Next we give an explicit description of the geometry of W. This will appear in a joint paper with C. P. Rourke (see [**8**]), but since it is crucial for our argument, we sketch it here.

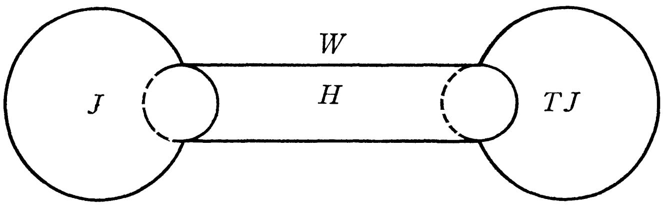
Consider the elements $a = \alpha + \beta - T\beta$, $b = \alpha + \beta - T\alpha$. We can check that

$$a \cdot a = b \cdot b = -Ta \cdot Ta = -Tb \cdot Tb = 2,$$

$$a \cdot b = -Ta \cdot Tb = 1, \quad a \cdot Tb = b \cdot Ta = 0, \quad \text{and} \quad \psi_0(a) = \psi_0(b) = 0.$$

Represent a and b by embedded spheres with the same names such that they meet transversely in one point. Moreover, the above intersection numbers allow us to choose the embedding so that a neighbourhood J of a $\cup b$ in W does not meet TJ. The normal bundles of a and b are equivalent to the tangent bundle of S^{2k} and therefore J may be considered the result of plumbing two copies of the tangent disc bundle of S^{2k}.

Thus we have split W into three parts, J, TJ, and $H = \overline{W - (J \cup TJ)}$.



LEMMA 2 (see also [**8**]). *H is an h-cobordism between ∂J and ∂TJ.*

Proof. H is simply connected, hence it is sufficient to show that $\partial J \to H$ induces isomorphisms in homology. The same will be true of $\partial TJ \to H$. First note that ∂J has reduced homology only in dimension $(2k - 1)$ and $H_{2k-1}(\partial J) = Z_3$. This is so because $H_{2k}(J)$ is free abelian generated by a and b while $H_{2k}(J, \partial J)$ is free abelian generated by u and v. The latter can be thought of as fibres of the normal bundles of a and b, respectively. Looking at intersection numbers we see that $H_{2k}(J) \to H_{2k}(J, \partial J)$ carries a to $2u + v$ and b to $u + 2v$. Hence $H_{2k-1}(\partial J)$ is generated by ∂u and ∂v with $\partial(2u + v) = \partial(u + 2v) = 0$, i.e. $\partial u = \partial v$ has order 3.

Consider the Mayer-Vietoris sequence for $J \cup TJ$ and H. The only non-zero groups are

$$0 \to H_{2k}(H) \oplus \underset{\{\text{free on } a,\, b\}}{\underset{\|}{H_{2k}(J)}} \oplus \underset{\{\text{free on } Ta,\, Tb\}}{\underset{\|}{H_{2k}(TJ)}} \xrightarrow{i}$$

$$\underset{(\text{free on } \alpha,\, \beta,\, T\alpha,\, T\beta\}}{\underset{\|}{H_{2k}(W)}} \xrightarrow{\Delta} \underset{\underset{\{\partial u\}\ \ \{\partial Tu\}}{Z_3 \oplus Z_3}}{\underset{\|}{H_{2k-1}(\partial H)}} \to H_{2k-1}(H) \to 0$$

It follows that $H_{2k}(H) = 0$ and the image of Δ is Z_3 generated by $\operatorname{im} \alpha = \partial u - \partial Tu$, seen by intersection numbers. Thus $H_{2k-1}(H) = Z_3$ and the inclusion of both boundary components give isomorphisms, as required. This completes the proof.

Clearly $\{a, b, Ta, Tb\}$ do not generate $H_{2k}(W)$. In fact, it is easy to see directly what may complete a set of generators. Let u denote the typical fibre of the tangent disc bundle over a as above.

Then $c = u \cup \partial u \times I \cup Tu$ is a $2k$-cycle not homologous to any linear combination of $\{a, b, Ta, Tb\}$. Similarly, if v is a typical fibre over b, then

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$d = v \cup \partial v \times I \cup Tv$ is another such element. Our previous computations show that $\{a, b, Ta, Tb, c, d\}$ generate $H_{2k}(W)$ with the relations

$$a - Ta = 2c + d, \qquad b - Tb = c + 2d.$$

It is a matter of direct substitution of $a = \alpha + \beta - T\beta$, $b = \alpha + \beta - T\alpha$ into these relations to find that $c = \beta - T\beta$, $d = \alpha - T\alpha$.

We can summarize the above as follows. Let T be a free involution on Σ with characteristic submanifold W. Then repeated equivariant surgery yields the submanifold W_c, $c = 0, 1$:

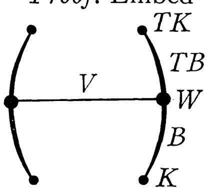
$$\begin{array}{ll} \text{if } c(T, \Sigma) = 0, & W_0 = S^{4k}, \\ \text{if } c(T, \Sigma) = 1, & W_1 \text{ is as in Lemma 2.} \end{array}$$

In the first case, W_0 bounds a disc, V_0, and in the second W_1 bounds $V_1 = J \times I$.

The Arf invariants. Now assume that $\Sigma^{4k+1} \in bP_{4k+2}$ and if Σ bounds a parallelizable manifold M, let $c(M)$ be its Arf invariant.

THEOREM. *Let T be a free involution on $\Sigma^{4k+1} \in bP_{4k+2}$, let $c(T, \Sigma) = c$ and let W_c be a characteristic submanifold as above, bounding V_c. If $T|W_c$ extends to a framed involution on V_c, then Σ bounds a parallelizable manifold M so that $c(M) = c(T, \Sigma)$.*

Proof. Embed V_c in R^N along W_c, otherwise disjoint from B and TB.



We claim that $B \cup_{W_c} V_c$ is a disc. This is obvious if $c = 0$. If $c = 1$ note that the only possible homology is $H_{2k}(B \cup_{W_1} V_1)$. But we showed that α and $T\beta$ are killed in B while c and d are killed in V_1. Together they eliminate the middle homology.

Thus we can put a framed disc on K so that the sphere $D \cup_K B \cup_{W_c} V_c$ bounds a framed ball E. Similarly, $TD \cup_{TK} TB \cup_{W_c} V_c$ bounds a framed ball F. Since the involution on W_c extends to V_c, we can sew E and F together along V_c to yield a framed manifold M with boundary Σ. (Note that by sewing new discs onto K and TK we have changed $[\Sigma] \in \theta_{4k+2}$ by an element of order divisible by 2, and since $[\Sigma] \in bP_{4k+2} \subset Z_2$, $[\Sigma]$ is not changed at all.)

For $c = 0$, V_0 is a disc. Clearly $c(M) = 0$.

For $c = 1$, $V_1 = J \times I$, hence it is $(2k - 1)$-connected with $H_{2k}(V_1) = Z + Z$ generated by a and b above. It is an easy consequence that $c(M) = 1$. In fact, we may choose disjoint embedded framed discs D_α and $D_{T\beta}$ in B with boundary α and $T\beta$; $D_{T\alpha}$ and D_β in TB with boundary $T\alpha$ and β and the necessary number of disjoint D_c and D_d in V with boundary c and d. Then the bounded connected sums $D_a = D_\alpha + D_c$, $D_a' = D_{T\alpha} + D_c + D_d$ represent framed discs bounding a in E and F, respectively. Thus $S_a = D_a \cup (-D_a')$ is a generator of $H_{2k+1}(M)$. Similarly we let $S_b = D_b \cup D_b'$, where

$D_b = D_{T\beta} + D_c + D_d$ and $D_b' = D_\beta + D_d$. Then $S_a \cdot S_b = a \cdot b = 1$ and $S_a \cdot S_a$ and $S_b \cdot S_b$ contain an odd pair of points, completing the proof.

In particular, if $\Sigma^{4k+1}(3)$ is the Kervaire sphere of Brieskorn [**4**; **3**] represented by $\Sigma^{4k+1}(3) = \{z_0^3 + z_1^2 + \ldots + z_{2k+1}^2 = 0\} \cap S^{4k+3}$, then the involution $T: \mathbf{C}^{2k+2} \to \mathbf{C}^{2k+2}$, defined by

$$T(z_0, z_1, \ldots, z_{2k+1}) = (z_0, -z_1, \ldots, -z_{2k+1})$$

is free on Σ^{4k+1} with characteristic submanifold the double of our J above (see [**1**; **3**]) and since it bounds $J \times I$ by construction and T extends to it, we recover the following result of Berstein [**1**].

COROLLARY 1. $(T, \Sigma^{4k+1}(3))$ *as defined above is an example of a smooth involution with* $c(T, \Sigma^{4k+1}) = 1$.

More generally, an extension of the theorem to the case when the characteristic submanifold is the double of the result of a linear plumbing of tangent disc bundles of S^{2k} yields the corresponding theorem for all Brieskorn spheres $\Sigma^{4k+1}(2d + 1)$, due to Giffen [**3**] and Browder (unpublished).

COROLLARY 2. $(T, \Sigma^{4k+1}(2d + 1))$ *is an involution such that*

$$c(T, \Sigma^{4k+1}(2d+1)) = 0 \qquad \text{for } 2d + 1 \equiv \pm 1 \pmod 8$$

and

$$c(T, \Sigma^{4k+1}(2d+1)) = 1 \qquad \text{for } 2d + 1 \equiv \pm 3 \pmod 8.$$

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