Bull. Aust. Math. Soc. **97** (2018), 363–366
doi:10.1017/S0004972718000126

A NOTE ON THE ERDŐS–GRAHAM THEOREM

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(Received 10 December 2017; accepted 11 January 2018; first published online 23 April 2018)

Abstract

Let $\mathcal{A} = \{a_1 < a_2 < \cdots\}$ be a set of nonnegative integers. Put $D(\mathcal{A}) = \gcd\{a_{k+1} - a_k : k = 1, 2, \ldots\}$. The set $\mathcal{A}$ is an asymptotic basis if there exists h such that every sufficiently large integer is a sum of at most h (not necessarily distinct) elements of $\mathcal{A}$. We prove that if the difference of consecutive integers of $\mathcal{A}$ is bounded, then $\mathcal{A}$ is an asymptotic basis if and only if there exists an integer $a \in \mathcal{A}$ such that $(a, D(\mathcal{A})) = 1$.

2010 *Mathematics subject classification*: primary 11B13; secondary 11B50.

Keywords and phrases: asymptotic basis, exact order.

1. Introduction

A set $\mathcal{A}$ of nonnegative integers is said to be an asymptotic basis if there exists h such that every sufficiently large integer is a sum of at most h (not necessarily distinct) elements of $\mathcal{A}$. An asymptotic basis $\mathcal{A}$ is said to have an exact order if there exists h' such that every sufficiently large integer is the sum of exactly h' (not necessarily distinct) elements taken from $\mathcal{A}$. Obviously, when $0 \in \mathcal{A}$, an asymptotic basis $\mathcal{A}$ has an exact order.

For the remainder of the paper, we write $\mathcal{A} = \{a_1 < a_2 < \cdots\}$. For a positive integer h, define $h\mathcal{A}$ to be the h-fold sum set of $\mathcal{A}$, that is

$$h\mathcal{A} = \{n : n = a_{i_1} + \cdots + a_{i_h}, a_{i_1}, \ldots, a_{i_h} \in \mathcal{A}\},$$

and define $D(\mathcal{A}) = \gcd\{a_{k+1} - a_k : k = 1, 2, \ldots\}$.

In 1980, Erdős and Graham [2] provided a necessary and sufficient condition for an asymptotic basis of the nonnegative integers to possess an exact order.

THEOREM 1.1. *An asymptotic basis $\mathcal{A} = \{a_1, a_2, \ldots\}$ has an exact order if and only if $D(\mathcal{A}) = 1$.*

REMARK 1.2. Put $\mathcal{A}(x) = |\{a \in \mathcal{A} : 1 \le a \le x\}|$. The density of $\mathcal{A}$ is defined by $d(\mathcal{A}) = \lim_{x\to+\infty} \mathcal{A}(x)/x$. An asymptotic basis $\mathcal{A} = \{a_1, a_2, \ldots\}$ has an exact order r if and only if the density of integers which can be represented as the sum of exactly r elements

This work was supported by the National Natural Science Foundation of China (Grant No.11471017).

taken from $\mathcal{A}$ (allowing repetitions) is 1. The necessity is obvious. To prove the sufficiency, suppose that the density of integers which can be represented as the sum of exactly r elements taken from $\mathcal{A}$ (allowing repetitions) is 1. If $D(\mathcal{A}) > 1$ and n is the sum of exactly r elements of $\mathcal{A}$, then $n \equiv ra_1 \pmod{D(\mathcal{A})}$. The density of such n is $1/D(\mathcal{A}) < 1$, a contradiction. Thus, we have $D(\mathcal{A}) = 1$. By Theorem 1.1, we know that the asymptotic basis $\mathcal{A}$ has an exact order.

For related problems about exact orders and asymptotic bases, one may refer to [1, 3–6].

It is natural to consider the necessary and sufficient condition for a set of nonnegative integers to be an asymptotic basis. The results in this paper arise from two observations. First, if $\mathcal{A} = \{a_1, a_2, \ldots\}$ is an asymptotic basis, then $(a_k, D(\mathcal{A})) = 1$ for all positive integers k (see [7, Lemma 3]). Second, $\mathcal{A} = \{1\} \cup \{2, 2^2, 2^4, \ldots, 2^{2^n}, \ldots\}$ is not an asymptotic basis.

Theorem 1.3. *Let $\mathcal{A} = \{a_1 < a_2 < \ldots\}$ be a set of nonnegative integers such that the difference of consecutive integers of $\mathcal{A}$ is bounded. Then $\mathcal{A}$ is an asymptotic basis if and only if there exists an integer $a \in \mathcal{A}$ such that* $\gcd(a, D(\mathcal{A})) = 1$.

Corollary 1.4. *Let $\mathcal{A} = \{a_1, a_2, \ldots\}$ be a set of nonnegative integers such that the difference of consecutive integers of $\mathcal{A}$ is bounded. Let $\mathcal{F}$ be a subset of $\mathbb{N}$. If $\mathcal{A} \cup \mathcal{F}$ is an asymptotic basis and $D(\mathcal{A} \cup \mathcal{F}) = D(\mathcal{A})$, then $\mathcal{A}$ is an asymptotic basis.*

Example 1.5. Let $\mathcal{A} = \{1, 3, 5, \ldots\}$. Then every positive integer can be represented as the sum of at most two elements of $\mathcal{A}$, and thus $\mathcal{A}$ is an asymptotic basis. If there exists an integer s such that every sufficiently large integer is the sum of exactly s elements of $\mathcal{A}$, then the parity of every sufficiently large integer is the same as the parity of s, and thus $\mathcal{A}$ does not have an exact order.

Example 1.6. Let $\mathcal{A} = \{2, 4, \ldots, 2n, \ldots\}$ and $\mathcal{F} = \{1\}$, we know that every positive integer can be represented as the sum of at most two elements of $\mathcal{A} \cup \mathcal{F}$, and thus $\mathcal{A} \cup \mathcal{F}$ is an asymptotic basis. But $\mathcal{A}$ is not an asymptotic basis because any sum of elements taken from $\mathcal{A}$ is even.

2. Proof of Theorem 1.3

Proof of necessity. If for all positive integers k we have $\gcd(a_k, D(\mathcal{A})) = d > 1$, then $d \mid a_k$. Therefore, any sum of elements taken from $\mathcal{A}$ is a multiple of d, which contradicts the assumption that $\mathcal{A}$ is an asymptotic basis. □

Proof of sufficiency. We write $g_n = \gcd(a_2 - a_1, a_3 - a_2, \ldots, a_{n+1} - a_n)$. Since $g_1 \geq g_2 \geq \cdots$, it follows that $\lim_{n\to+\infty} g_n = D(\mathcal{A})$. Then there exists a positive integer n_0 such that $|g_n - D(\mathcal{A})| < 1$ for $n \geq n_0$. Since the g_n and $D(\mathcal{A})$ are integers, $g_n = D(\mathcal{A})$ for $n \geq n_0$. This means that $\gcd(a_2 - a_1, a_3 - a_2, \ldots, a_{n_0+1} - a_{n_0}) = D(\mathcal{A})$. Moreover, $\gcd(a_i, D(\mathcal{A})) = 1$ for some $a_i \in \mathcal{A}$.

https://doi.org/10.1017/S0004972718000126 Published online by Cambridge University Press

If $1 \le i \le n_0 + 1$, then $D(\mathcal{A}) \mid a_{i+1} - a_i$. Also, $D(\mathcal{A}) \mid a_{n_0+1} - a_1$. Thus,

$$\begin{aligned}\gcd(a_i, D(\mathcal{A})) &= \gcd(a_i, D(\mathcal{A}), a_{n_0+1} - a_1)\\ &= \gcd(a_i, a_2 - a_1, \dots, a_{i+1} - a_i, \dots, a_{n_0+1} - a_{n_0}, a_{n_0+1} - a_1)\\ &= \gcd(a_i, a_2 - a_1, \dots, a_{i+1}, \dots, a_{n_0+1}, a_{n_0+1} - a_1)\\ &= \gcd(a_1, a_2, \dots, a_{n_0+1}).\end{aligned}$$

If $i > n_0 + 1$, then $D(\mathcal{A}) = \gcd(D(\mathcal{A}), a_{n_0+2} - a_{n_0+1}, \dots, a_i - a_{i-1})$, and thus

$$\begin{aligned}\gcd(a_i, D(\mathcal{A})) &= \gcd(a_i, D(\mathcal{A}), a_{n_0+2} - a_{n_0+1}, \dots, a_i - a_{i-1}, a_i - a_1)\\ &= \gcd(a_i, a_2 - a_1, \dots, a_i - a_{i-1}, a_i - a_1)\\ &= \gcd(a_1, a_2, \dots, a_i).\end{aligned}$$

Hence, in both cases, there exists a positive integer t such that

$$\gcd(a_1, \dots, a_t) = 1$$

and there are integer constants c_i with $\sum_{i=1}^{t} c_i a_i = 1$. Let $A = a_1 \cdots a_t$ and $b_i = c_i + k_i A$, where k_i is the smallest nonnegative integer with $b_i > 0$. Then $\sum_{i=1}^{t} b_i a_i = kA + 1$ for some nonnegative integer k. Let $N = \sum_{i=1}^{t} b_i a_i + kA$, then

$$N + 1 = \sum_{i=1}^{t} b_i a_i + kA + 1 = 2 \sum_{i=1}^{t} b_i a_i.$$

Thus, there exists a positive integer h_1 such that

$$\{N, N+1\} \subset \bigcup_{i=1}^{h_1} i\mathcal{A}.$$

Hence, for every positive integer l,

$$\{lN, lN+1, \dots, lN+l\} \subset \bigcup_{i=1}^{lh_1} i\mathcal{A}.$$

Moreover, when $l \ge N$,

$$\{lN, lN+1, \dots, lN+l\} \cap \{(l+1)N, (l+1)N+1, \dots, (l+1)N+l+1\} \ne \emptyset.$$

Therefore, for every positive integer $s \ge N$,

$$\{N^2, N^2+1, \dots, sN+s\} \subset \bigcup_{i=1}^{sh_1} i\mathcal{A}.$$

Since the difference of consecutive integers of $\mathcal{A}$ is bounded, we may assume that

$$\max\{a_{k+1} - a_k : k = 1, 2, \dots\} \le M.$$

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Then there exists a positive integer $q \geq N$ such that $N^2 + M \leq qN + q$. For every integer $n \geq N^2 + a_1$, there must exist an integer k such that $a_k \leq n - N^2 < a_{k+1}$. Thus,

$$N^2 \leq n - a_k = n - a_{k+1} + a_{k+1} - a_k < N^2 + M.$$

Hence,

$$n - a_k \in \{N^2, N^2 + 1, \ldots, qN + q\} \subset \bigcup_{i=1}^{qh_1} i\mathcal{A},$$

that is, $n = n - a_k + a_k \in \bigcup_{i=1}^{qh_1+1} i\mathcal{A}$. □

This completes the proof of Theorem 1.3.

3. Proof of Corollary 1.4

In 2011, Yang and Chen [7, Lemma 3] showed that if $\mathcal{A} = \{a_1, a_2, \ldots\}$ is an asymptotic basis, then $(a_k, D(\mathcal{A})) = 1$ for all positive integers k. By this result, it follows that $\gcd(a_k, D(\mathcal{A} \cup \mathcal{F})) = 1$ for all positive integers k. Since $D(\mathcal{A} \cup \mathcal{F}) = D(\mathcal{A})$, we see that $\gcd(a_k, D(\mathcal{A})) = 1$ for all positive integers k. Thus, by Theorem 1.3, $\mathcal{A}$ is an asymptotic basis.

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