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Properties of the solution set of a generalized differential equation

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We prove that the solution set of a generalized differential equation is connected and that points on the boundary of the solution funnel are peripherally attainable. This is done without the additional assumption of continuity in the state variable required in previous results. The result on upper semicontinuity of the solution set with respect to initial conditions is extended to include variations of initial time.

1. Introduction

In this paper we study the generalized differential equation

$$x'(t) \in F\big(t, x(t)\big) \text{ almost everywhere } t \in I ,$$

$$x(t_0) = x_0 .$$

$F(t, x)$ is a convex compact set valued function which is upper semicontinuous in x and bounded by an integrable function on the compact interval I. We assume that for a given x there exists a measurable selector $f_x(t)$ contained in $F(t, x)$. The solution set of the equation, denoted by $H\big(t_0, x_0\big)$, is a nonempty compact subset of $C(I)$, ([*11*], [*10*], [3], [7]).

$H\big(t_0, x_0\big)$ is upper semicontinuous in x_0, ([3], [*8*]). We prove that $H\big(t_0, x_0\big)$ is upper semicontinuous in $\big(t_0, x_0\big)$. If F is also

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continuous in x_0 , $H(t_0, x_0)$ is a connected subset of $C(I)$, every point on the boundary of the solution funnel is peripherally attainable and the generalized differential equation has the bang-bang property ([8], [9], [12]). We show that connectedness and peripheral attainability follow without the additional assumption of continuity. We give an example to show that the bang-bang property does not hold without continuity.

Filippov's paper [6] contains results similar to ours, but assumes the existence of a measurable selector $f(t, x)$ contained in $F(t, x)$ and does not contain proofs.

To make the paper self-contained we include results on upper semicontinuity due to Berge [2] and a proof of the existence of solutions.

2. Upper semicontinuity

Upper semicontinuity of compact set valued functions is a generalization of the concept of continuity of point valued functions. A point valued function is upper semicontinuous (as a set valued function) if and only if it is continuous. This section is based on Berge [2].

X and Y are topological spaces and $\Omega(Y)$ is the set of nonempty compact subsets of Y . F maps from X to $\Omega(Y)$. Let A be a subset of X . Then

$$F(A) = \bigcup_{x \in A} F(x) .$$

F is upper semicontinuous at x_0 if for all open sets G containing $F(x_0)$, there exists a neighbourhood $U(x_0)$ of x_0 such that $F(U(x_0)) \subseteq G$. F is upper semicontinuous if F is upper semicontinuous at every point of X .

THEOREM 2.1. *F is upper semicontinuous if and only if $F^+(G) = \{x \mid F(x) \subseteq G\}$ is an open set for all open sets G in Y .*

Proof. (a) Assume F is upper semicontinuous. Let $x_0 \in F^+(G)$. There exists a neighbourhood $U(x_0)$ of x_0 such that $F(U(x_0)) \subseteq G$. Thus $U(x_0) \subseteq F^+(G)$ and $F^+(G)$ is an open set.

(b) Assume G is open implies $F^+(G)$ is open. Let $x_0 \in X$ and let

G be an open set containing $F(x_0)$. $F^+(G)$ is a neighbourhood of x_0 and $F\left(F^+(G)\right) \subseteq G$. //

THEOREM 2.2. *If F is upper semicontinuous the image $F(K)$ of a compact set K is compact.*

Proof. Let $\{G_i \mid i \in I\}$ be an open covering of $F(K)$. If $x \in K$, the set $F(x)$, which is compact, is covered by a finite number of G_i . Let their union be denoted by G_x . $\left\{F^+\left(G_x\right) \mid x \in K\right\}$ is an open covering of K . Thus, there exists a finite subcovering $F^+\left(G_{x_1}\right), \ldots, F^+\left(G_{x_n}\right)$. $G_{x_1}, \ldots, G_{x_n}$ cover $F(K)$ and each G_{x_j} is the union of a finite number of G_i . Therefore $F(K)$ is covered by a finite number of G_i and thus is compact. //

THEOREM 2.3. *If F is upper semicontinuous, K is a connected subset of X and $F(x)$ is a connected subset of Y for each $x \in K$ then $F(K)$ is a connected subset of Y .*

Proof. Suppose $F(K)$ is not connected. That is, there exist two open disjoint sets A_1, A_2 of Y such that $F(K) \subseteq A_1 \cup A_2$ and $F(K) \cap A_1 \neq \emptyset$ and $F(K) \cap A_2 \neq \emptyset$. Then $F^+(A_1)$ and $F^+(A_2)$ are open sets of X . Let $x \in K$; then $F(x) \subseteq A_1 \cup A_2$, and since $F(x)$ is connected, it is contained in A_1 or in A_2 . Thus $K \subseteq F^+(A_1) \cup F^+(A_2)$. It is obvious that $F^+(A_1) \cap F^+(A_2) = \emptyset$, and $K \cap F^+(A_1) \neq \emptyset$ and $K \cap F^+(A_2) \neq \emptyset$. Thus K is not connected, which is a contradiction. Therefore $F(K)$ is connected. //

THEOREM 2.4. *Let $F_1 : X \to \Omega(Y)$ and $F_2 : Y \to \Omega(Z)$ be upper semicontinuous. Define $F_2 \circ F_1(x) = F_2\left(F_1(x)\right)$. Then $F_2 \circ F_1$ maps X to $\Omega(Z)$ and is upper semicontinuous.*

Proof. By Theorem 2.2, $F_2\left(F_1(x)\right) \in \Omega(Z)$. Let G be an open set of Z .

$$\begin{aligned}(F_2 \circ F_1)^+(G) &= \{x \mid F_2 \circ F_1(x) \subseteq G\} \\ &= \{x \mid F_1(x) \subseteq F_2^+(G)\} \\ &= F_1^+\left[F_2^+(G)\right] ,\end{aligned}$$

which is open in X by Theorem 2.1. Thus $F_2 \circ F_1$ is upper semicontinuous by Theorem 2.1. //

Let X be a metric space, ε a positive real number and $x \in X$. Define $B_\varepsilon(x) = \{y \mid d(x, y) < \varepsilon\}$. Let A be a subset of X. Define $A^\varepsilon = \bigcup_{x \in A} B_\varepsilon(x)$.

THEOREM 2.5. *Let X and Y be metric spaces and $F : X \to \Omega(Y)$. F is upper semicontinuous at x_0 if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that* $F\left(B_\delta(x_0)\right) \subseteq F^\varepsilon(x_0)$.

Proof. (a) Suppose F is upper semicontinuous at x_0. Choose a neighbourhood $U(x_0)$ of x_0 such that $F\left(U(x_0)\right) \subseteq F^\varepsilon(x_0)$. Choose $\delta > 0$ such that $B_\delta(x_0) \subseteq U(x_0)$. Therefore $F\left(B_\delta(x_0)\right) \subseteq F^\varepsilon(x_0)$.

(b) Let G be an open set of Y containing $F(x_0)$. Suppose there does not exist $\varepsilon > 0$ such that $F^\varepsilon(x_0) \subseteq G$. Then there exists $x_k \in F^{1/k}(x_0)$ such that $x_k \notin G$. Now $x_k \in \overline{F^1(x_0)}$ which is a compact set. Therefore there exists a subsequence, also denoted by $\{x_k\}$, converging to a point $x_\infty \in \overline{F^1(x_0)}$. But $x_k \in F^{1/p}(x_0)$ for all $k \geq p$. Thus $x_\infty \in \overline{F^{1/p}(x_0)}$ for all p. Therefore x_∞ is contained in $F(x_0)$. But $x_k \in G^c$, which is closed. Hence $x_\infty \in G^c$. This contradiction implies that there exists $\varepsilon > 0$ such that $F^\varepsilon(x_0) \subseteq G$. Choose $\delta > 0$ such that $F\left(B_\delta(x_0)\right) \subseteq F^\varepsilon(x_0) \subseteq G$. Put $U(x_0) = B_\delta(x_0)$. //

Let X be a normed linear space. A subset K of X is said to be

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convex if, given x_1 and $x_2 \in K$, all points of the form $\alpha x_1 + (1-\alpha)x_2$ with $0 \le \alpha \le 1$ are in K. If A is a subset of X, the convex hull of A, denoted by $\mathrm{co}A$, is the smallest convex set containing A. Since the intersection of any collection of convex sets is a convex set, $\mathrm{co}A$ is the intersection of all the convex sets containing A. The closed convex hull of A, denoted by $\overline{\mathrm{co}}A$, is defined by $\overline{\mathrm{co}}A = \overline{(\mathrm{co}A)}$.

LEMMA 2.6. $\mathrm{co}\left(A^{\varepsilon}\right) = (\mathrm{co}A)^{\varepsilon}$.

Proof. (a) Let $x \in \mathrm{co}\left(A^{\varepsilon}\right)$. Then $x = \lambda x_1 + (1-\lambda)x_2$ where $x_1, x_2 \in A^{\varepsilon}$ and $0 \le \lambda \le 1$. Thus there exist y_1 and $y_2 \in A$ such that $\|x_1-y_1\| < \varepsilon$ and $\|x_2-y_2\| < \varepsilon$. Put $y = \lambda y_1 + (1-\lambda)y_2$. Then $y \in \mathrm{co}A$ and

$$\|x-y\| = \|\lambda\left(x_1-y_1\right)+(1-\lambda)\left(x_2-y_2\right)\| < \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon .$$

Therefore $x \in (\mathrm{co}A)^{\varepsilon}$.

(b) Let $x \in (\mathrm{co}A)^{\varepsilon}$. Then there exists $y \in \mathrm{co}A$ such that $\|x-y\| < \varepsilon$. Therefore $y = \lambda y_1 + (1-\lambda)y_2$ where y_1 and $y_2 \in A$ and $0 \le \lambda \le 1$. Put $z = x - y$, $x_1 = y_1 + z$ and $x_2 = y_2 + z$. Then x_1 and $x_2 \in A^{\varepsilon}$ and $\lambda x_1 + (1-\lambda)x_2 = x$. Thus $x \in \mathrm{co}\left(A^{\varepsilon}\right)$. //

COROLLARY 2.7. $\overline{\mathrm{co}}A = \mathrm{co}(\overline{A})$.

THEOREM 2.8. *Let X be a metric space and Y a normed linear space. Suppose $F : X \to \Omega(Y)$ is upper semicontinuous at x_0. If x_k tends to x_0 then*

$$\bigcap_{i=1}^{\infty} \overline{\mathrm{co}} \bigcup_{k=i}^{\infty} F\left(x_k\right) \subseteq \mathrm{co}F\left(x_0\right) .$$

Proof. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $F\left[B_{\delta}\left(x_0\right)\right] \subseteq F^{\varepsilon}\left(x_0\right)$. Choose k_0 such that $k \ge k_0$ implies $x_k \in B_{\delta}\left(x_0\right)$. Therefore $k \ge k_0$

implies $F(x_k) \subseteq F^{\varepsilon}(x_0)$ and $\bigcup_{k=k_0}^{\infty} F(x_k) \subseteq F^{\varepsilon}(x_0)$. Thus

$$\operatorname{co} \bigcup_{k=k_0}^{\infty} F(x_k) \subseteq \operatorname{co}F^{\varepsilon}(x_0) = \left(\operatorname{co}F(x_0)\right)^{\varepsilon} .$$

Therefore

$$\overline{\operatorname{co}} \bigcup_{k=k_0}^{\infty} F(x_k) \subseteq \overline{\left(\operatorname{co}F(x_0)\right)^{\varepsilon}} .$$

Hence

$$\bigcap_{i=1}^{\infty} \overline{\operatorname{co}} \bigcup_{k=i}^{\infty} F(x_k) \subseteq \overline{\left(\operatorname{co}F(x_0)\right)^{\varepsilon}} .$$

Thus

$$\bigcap_{i=1}^{\infty} \overline{\operatorname{co}} \bigcup_{k=i}^{\infty} F(x_k) \subseteq \operatorname{co}F(x_0)$$

since $\operatorname{co}F(x_0)$ is closed by Corollary 2.7. //

3. Generalized differential equations

We will consider the generalized differential equation

$$x'(t) \in F(t, x(t)) \text{ almost everywhere } t \in I ,$$

$$x(t_0) = x_0 ,$$

where F satisfies the following conditions:

1. F maps from $I \times R^n$ into $\Omega(R^n)$ where I is the compact interval $[a, b]$;
2. $F(t, x)$ is convex;
3. for all $t \in I$, $x \mapsto F(t, x)$ is upper semicontinuous on R^n ;
4. for all $x \in R^n$ there exists $f_x : I \to R^n$ such that f_x is measurable and $f_x(t) \in F(t, x)$;

5. there exists $g \in L^1(I)$ such that $y \in F(t, x)$ implies $|y| \le g(t)$.

Let $t_0 \in I$ and $x_0 \in R^n$. The function x is a solution of the generalized differential equation if and only if

(i) $x : I \to R^n$ is absolutely continuous,

(ii) $x'(t) \in F(t, x(t))$ almost everywhere $t \in I$,

(iii) $x(t_0) = x_0$.

The set of all solutions is called the solution set and is denoted by $H(t_0, x_0)$. In this paper we will study the properties of the solution set.

4. Existence

In this section we give Aumann's proof [1] of a result on the convergence of absolutely continuous functions and use this result in Kikuchi's proof [7] of the existence of a solution.

THEOREM 4.1. *Let $\{x_k\}$ be a sequence of absolutely continuous functions $x_k : I \to R^n$. We suppose that*

(i) $x_k(t) \to x(t)$ as $k \to \infty$ for all $t \in I$ where $x : I \to R^n$,

(ii) $|x_k'(t)| \le g(t)$ almost everywhere $t \in I$ where $g : I \to R$ is an integrable function.

Then x is an absolutely continuous function such that

$$x'(t) \in \bigcap_{i=1}^{\infty} \overline{\text{co}} \bigcup_{k=i}^{\infty} x_k'(t) \text{ almost everywhere } t \in I .$$

Proof. Since x_k is absolutely continuous, $x_k' \in L^1(I)$. Also $|x_k'(t)| \le g(t)$ almost everywhere $t \in I$. Therefore by Theorem IV.8.9 on page 292 of Dunford and Schwartz [5] there exists a subsequence $\{x_{k(j)}'\}$

converging weakly to $f \in L^1(I)$. Thus

$$\begin{aligned} x(t) &= \lim_{j\to\infty} x_{k(j)}(t) \\ &= \lim_{j\to\infty} \left[x_{k(j)}(a) + \int_a^t x'_{k(j)}\right] \\ &= x(a) + \int_a^t f . \end{aligned}$$

Therefore x is absolutely continuous and $x'(t) = f(t)$ almost everywhere $t \in I$.

By Corollary V.3.14 on page 422 of Dunford and Schwartz [5], there exists a sequence $\{g_m\}$ of convex combinations of $\{x'_{k(1)}, x'_{k(2)}, \ldots\}$ converging strongly to f. There exists a subsequence, also denoted by $\{g_m\}$, such that $g_m(t) \to f(t)$ almost everywhere $t \in I$. But

$$\begin{aligned} g_m(t) &\in \operatorname{co} \bigcup_{j=1}^{\infty} x'_{k(j)}(t) \\ &\subseteq \operatorname{co} \bigcup_{k=1}^{\infty} x'_k(t) . \end{aligned}$$

Hence

$$f(t) \in \overline{\operatorname{co}} \bigcup_{k=1}^{\infty} x'_k(t) \quad \text{almost everywhere} \quad t \in I .$$

But $\{x_i, x_{i+1}, \ldots\}$ also tends to x for all positive integers i. Thus

$$f(t) \in \overline{\operatorname{co}} \bigcup_{k=i}^{\infty} x'_k(t) \quad \text{almost everywhere} \quad t \in I .$$

Therefore

$$f(t) \in \bigcap_{i=1}^{\infty} \overline{\operatorname{co}} \bigcup_{k=i}^{\infty} x'_k(t) \quad \text{almost everywhere} \quad t \in I . \qquad //$$

THEOREM 4.2. *$H(t_0, x_0)$ is nonempty.*

Proof. Subdivide $[t_0, b]$ into k equal parts by $t_i = t_0 + i\,\frac{b-t_0}{k}$. We define $x_k : [t_0, b] \to R^n$ inductively. First $x_k(t_0) = x_0$. Suppose x_k is defined up to t_i where $0 \le i < k$. Select a measurable function $f_i : [t_i, t_{i+1}] \to R^n$ such that $f_i(t) \in F\big(t, x_k(t_i)\big)$ for all $t \in [t_i, t_{i+1}]$. Define

$$x_k(t) = x_k(t_i) + \int_{t_i}^{t} f_i \quad \text{for all} \quad t \in [t_i, t_{i+1}] .$$

Define $f : [t_0, b] \to R^n$ by $f(t) = f_i(t)$ for all $t \in [t_i, t_{i+1}]$. Hence

$$x_k(t) = x_0 + \int_{t_0}^{t} f ,$$

$$|x_k(t)-x_0| \le \int_{t_0}^{t} g ,$$

$$|x_k(t)| \le |x_0| + \int_{t_0}^{b} g .$$

Therefore x_k is well defined.

Now

$$x_k'(t) = f(t) \quad \text{almost everywhere} \quad t \in [t_0, b] .$$

Thus

$$|x_k'(t)| \le g(t) \quad \text{almost everywhere} \quad t \in [t_0, b] .$$

Therefore $\{x_k\}$ is equicontinuous. Define $y_k : [t_0, b] \to R^n$ by $y_k(t) = x_k(t_i)$ if $t \in [t_i, t_{i+1}]$. Then $x_k'(t) \in F\big(t, y_k(t)\big)$ almost everywhere $t \in [t_0, b]$.

Consider the sequence $\{x_k\}$. It is bounded and equicontinuous. Thus

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by Ascoli's Theorem it has a convergent subsequence, also denoted by $\{x_k\}$, converging to $x \in C(I)$. By Theorem 4.1, x is absolutely continuous and

$$\begin{aligned} x'(t) &\in \bigcap_{i=1}^{\infty} \overline{\mathrm{co}} \bigcup_{k=i}^{\infty} x_k'(t) \text{ almost everywhere } t \in [t_0, b] \\ &\subseteq \bigcap_{i=1}^{\infty} \overline{\mathrm{co}} \bigcup_{k=i}^{\infty} F(t, y_k(t)) \text{ almost everywhere } t \in [t_0, b] \\ &\subseteq F(t, x(t)) \text{ almost everywhere } t \in [t_0, b] \end{aligned}$$

by Theorem 2.8 since $y_k(t)$ tends to $x(t)$. Also

$$x(t_0) = \lim_{k\to\infty} x_k(t_0) = x_0 .$$

Similarly we can find $x : [a, t_0] \to R^n$ such that $x(t_0) = x_0$ and $x'(t) \in F(t, x(t))$ almost everywhere $t \in [a, t_0]$. Putting these two functions together we have $x : [a, b] \to R^n$ and $x \in H(t_0, x_0)$. Thus $H(t_0, x_0)$ is nonempty. //

5. Compactness and upper semicontinuity

In this section we prove that the solution set is compact and upper semicontinuous.

THEOREM 5.1. *If* $M \in \Omega(I \times R^n)$ *, then* $H(M)$ *is a compact subset of* $C(I)$.

Proof. (a) Let $x \in H(M)$. There exists $t_0 \in I$ such that $(t_0, x(t_0)) \in M$. Let M' be the projection of M into R^n. M' is compact and $x(t_0) \in M'$. Thus $|x(t_0)| < d$, where d is a bound for M'. Now $x(t) = x(t_0) + \int_{t_0}^{t} x'$. Therefore $|x(t)-x(t_0)| \le \int_I g$. Hence $\|x\| \le d + \int_I g$. Thus $H(M)$ is bounded.

(b) If $x \in H(M)$ we have $|x'(t)| \le g(t)$ almost everywhere $t \in I$.

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Thus $|x(t)-x(t_0)| \le \int_{t_0}^{t} g$. Therefore $H(M)$ is equicontinuous.

(c) Let $x_k \to x$ where $x_k \in H(M)$. By Theorem 4.1, x is absolutely continuous and

$$\begin{aligned} x'(t) &\in \bigcap_{i=1}^{\infty} \overline{\text{co}} \bigcup_{k=i}^{\infty} x_k'(t) \text{ almost everywhere } t \in I \\ &\subseteq \bigcap_{i=1}^{\infty} \overline{\text{co}} \bigcup_{k=i}^{\infty} F(t, x_k(t)) \text{ almost everywhere } t \in I \\ &\subseteq F(t, x(t)) \text{ almost everywhere } t \in I \end{aligned}$$

by Theorem 2.8.

For each x_k there exists $t_k \in I$ such that $\left(t_k, x_k(t_k)\right) \in M$. Since M is compact there exists a subsequence, also denoted by $\left\{\left(t_k, x_k(t_k)\right)\right\}$, such that $\left(t_k, x_k(t_k)\right)$ tends to $(t_0, x_0) \in M$. Let $\varepsilon > 0$. Choose k_0 such that $k \ge k_0$ implies $\|x-x_k\| < \frac{\varepsilon}{3}$, and $\delta > 0$ such that $|t-t_0| < \delta$ implies $|x(t)-x(t_0)| < \frac{\varepsilon}{3}$. Choose $k_1 \ge k_0$ such that $k \ge k_1$ implies $|t_k-t_0| < \delta$ and $|x_k(t_k)-x_0| < \frac{\varepsilon}{3}$. Therefore

$$\begin{aligned} |x(t_0)-x_0| &\le |x(t_0)-x(t_k)| + |x(t_k)-x_k(t_k)| + |x_k(t_k)-x_0| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon . \end{aligned}$$

Since ε is any positive number we have $x(t_0) = x_0$. Thus $x \in H(M)$ and $H(M)$ is closed in $C(I)$.

(d) Thus by Ascoli's Theorem, $H(M)$ is compact in $C(I)$. //

COROLLARY 5.2. *H maps from $I \times R^n$ into $\Omega(C(I))$.*

Proof. Theorems 4.2 and 5.1 give the result. //

We now use Theorem 5.1 to prove that H is upper semicontinuous.

THEOREM 5.3. *$H : I \times R^n \to \Omega(C(I))$ is an upper semicontinuous map.*

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Proof. Let $(t_0, x_0) \in I \times R^n$. Assume that H is not upper semicontinuous at (t_0, x_0); that is, there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, $H\left(B_\delta(t_0, x_0)\right) \nsubseteq H^{\varepsilon_0}(t_0, x_0)$. Choose x_k such that $x_k \in H\left(B_{1/k}(t_0, x_0)\right)$ and $x_k \notin H^{\varepsilon_0}(t_0, x_0)$. Now $x_k \in \overline{H\left(B_1(t_0, x_0)\right)}$, which is compact by Theorem 5.1. There exists a subsequence, also denoted by $\{x_k\}$, such that x_k converges to $x \in \overline{H\left(B_1(t_0, x_0)\right)}$. But there exists $t_k \in I$ such that $\left(t_k, x_k(t_k)\right) \in B_{1/k}(t_0, x_0)$. Let $\varepsilon > 0$. Choose k_0 such that $k \geq k_0$ implies $\|x-x_k\| < \frac{\varepsilon}{3}$ and $\delta > 0$ such that $|t-t_0| < \delta$ implies $|x(t)-x(t_0)| < \frac{\varepsilon}{3}$. Choose $k_1 \geq k_0$ such that $k \geq k_1$ implies $|t_k-t_0| < \delta$ and $|x_k(t_k)-x_0| < \frac{\varepsilon}{3}$. Therefore

$$|x(t_0)-x_0| \leq |x(t_0)-x(t_k)| + |x(t_k)-x_k(t_k)| + |x_k(t_k)-x_0| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Since ε is any positive number we have $x(t_0) = x_0$. Thus $x \in H(t_0, x_0)$.

But $x_k \notin H^{\varepsilon_0}(t_0, x_0)$. Therefore $x \notin H(t_0, x_0)$. From this contradiction we conclude that H is upper semicontinuous. //

6. Connectedness

In this section we prove that the solution set is connected.

LEMMA 6.1. *Let $(t_0, x_0) \in I \times R^n$. Suppose F does not depend on x. Then $H(t_0, x_0)$ is convex.*

Proof. Let x_1 and $x_2 \in H(t_0, x_0)$; that is $x_i(t_0) = x_0$ and

$$x_i'(t) \in F(t) \text{ almost everywhere } t \in I \text{ for } i = 1, 2 .$$

Consider $x = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$. We have

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$$x(t_0) = \lambda x_0 + (1-\lambda)x_0 = x_0$$

and

$$x'(t) = \lambda x_1'(t) + (1-\lambda)x_2'(t) ,$$
$$\in F(t) \text{ almost everywhere } t \in I ,$$

since F is convex. Thus $H(t_0, x_0)$ is convex. //

LEMMA 6.2. *Suppose F depends on a parameter $u \in R^m$ and that F is upper semicontinuous in (x, u). Let $H(t_0, x_0, u_0)$ denote the solution set passing through (t_0, x_0) when $u = u_0$. Then $H : I \times R^{n+m} \to C(I)$ is upper semicontinuous.*

Proof. We consider the system

$$x' \in F(t, x)$$
$$u' = 0 ;$$

that is, $\tilde{x}' \in \tilde{F}(t, x)$ where $\tilde{x} = \begin{pmatrix} x \\ u \end{pmatrix}$ and $\tilde{F} = \begin{pmatrix} F \\ 0 \end{pmatrix}$. The solution set of this system $\tilde{H}(t_0, x_0, u_0)$ is upper semicontinuous by Theorem 5.3. Define $p : C_{R^{n+m}}(I) \to C_{R^n}(I)$ by $p(\tilde{x})(t) = q \circ \tilde{x}(t)$ where q is the projection operator $q : R^{n+m} \to R^n$. Then p is continuous, and $H(t_0, x_0, u_0) = p \circ \tilde{H}(t_0, x_0, u_0)$. Therefore H is upper semicontinuous by Theorem 2.4. //

Subdivide $I = [a, b]$ into k equal parts by $t_i = a + i\frac{b-a}{k}$. We define a function $A_i : C[t_0, t_i] \to \Omega(C[t_0, t_{i+1}])$, $i = 0, \ldots, k-1$. Let $x \in C[t_0, t_i]$. Then $y \in A_i(x)$ if and only if

(i) $y : [t_0, t_{i+1}] \to R^n$;

(ii) $y(t) = x(t)$ for all $t \in [t_0, t_i]$;

(iii) y is absolutely continuous on $[t_i, t_{i+1}]$;

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(iv) $y'(t) \in F\left(t, x\left(t_i\right)\right)$ almost everywhere $t \in \left[t_i, t_{i+1}\right]$.

LEMMA 6.3. *$A_i(x)$ is a compact convex nonempty set.* *$A_i : C\left[t_0, t_i\right] \to \Omega\left(C\left[t_0, t_{i+1}\right]\right)$ is upper semicontinuous.*

Proof. Consider the equation

$$y'(t) \in F\left(t, x\left(t_i\right)\right) \text{ almost everywhere } t \in \left[t_i, t_{i+1}\right] ,$$

$$y\left(t_i\right) = x\left(t_i\right) .$$

The right-hand side does not depend on y . Thus by Lemma 6.1 and Theorems 4.2 and 5.1, $A_i(x)$ is a compact convex nonempty set.

By Lemma 6.2, $H\left(t_i, x\left(t_i\right), x\left(t_i\right)\right)$ is upper semicontinuous. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $\left|(t, u, v) - \left(t_i, x_0\left(t_i\right), x_0\left(t_i\right)\right)\right| < 2\delta$ implies $H(t, u, v) \subseteq H^{\varepsilon}\left(t_i, x_0\left(t_i\right), x_0\left(t_i\right)\right)$ and such that $\delta < \varepsilon$. Therefore $\|x-x_0\| < \delta$ implies $\left|x(t)-x_0(t)\right| < \varepsilon$ for all $t \in \left[t_0, t_i\right]$ and $H\left(t_i, x\left(t_i\right), x\left(t_i\right)\right) \subseteq H^{\varepsilon}\left(t_i, x_0\left(t_i\right), x_0\left(t_i\right)\right)$; that is, $\|x-x_0\| < \delta$ implies $A_i(x) \subseteq A_i^{\varepsilon}\left(x_0\right)$. //

Let $y \in C(I)$. Define a function

$$b_{yi} : \left[t_i, t_{i+1}\right] \to \Omega\left(C\left[t_0, t_{i+1}\right]\right) , \quad i = 0, \ldots, k-1 .$$

Let $r \in \left[t_i, t_{i+1}\right]$. Then $z \in b_{yi}(r)$ if and only if

(i) $z : \left[t_0, t_{i+1}\right] \to R^n$;

(ii) $z(t) = y(t)$ if $t \in \left[t_0, r\right]$;

(iii) z is absolutely continuous on $\left[r, t_{i+1}\right]$;

(iv) $z'(t) \in F\left(t, y\left(t_i\right)\right)$ almost everywhere $t \in \left[r, t_{i+1}\right]$.

LEMMA 6.4. *$b_{yi}(r)$ is a compact convex nonempty set.*

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$b_{yi} : [t_i, t_{i+1}] \to \Omega(C[t_0, t_{i+1}])$ *is upper semicontinuous.*

Proof. Consider the equation

$$z'(t) \in F\left(t, y(t_i)\right) \text{ almost everywhere } t \in [t_i, t_{i+1}] ,$$

$$z(r) = y(r) .$$

The right hand side does not depend on z. Thus by Lemma 6.1 and Theorems 4.2 and 5.1, $b_{yi}(r)$ is a compact convex nonempty set. By Theorem 5.3, $H(r, a)$ is upper semicontinuous. Let $\varepsilon > 0$. Choose $\gamma > 0$ such that $|(r, a)-(r_0, a_0)| < 2\gamma$ implies $H(r, a) \subseteq H^\varepsilon(r_0, a_0)$. Choose $\delta_1 > 0$ such that $|r-r_0| < \delta_1$ implies $|y(r)-y(r_0)| < \gamma$ and such that $\delta_1 < \gamma$. Thus $|r-r_0| < \delta_1$ implies $H(r, y(r)) \subseteq H^\varepsilon\left(r_0, y(r_0)\right)$. The set of all solutions is an equicontinuous family. Choose $\delta_2 > 0$ such that $|r-r_0| < \delta_2$ implies $|z(r)-z(r_0)| < \frac{\varepsilon}{2}$ for all solutions z. Choose $\delta_3 > 0$ such that $|r-r_0| < \delta_3$ implies $|y(r)-y(r_0)| < \frac{\varepsilon}{2}$. Put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $|r-r_0| < \delta$ implies $b_{yi}(r) \subseteq b^\varepsilon_{yi}(r)$. //

Define a function $B_{yi} : [t_i, t_{i+1}] \to \Omega(C(I))$, $i = 0, \ldots, k-1$, by $B_{yi}(t) = A_{k-1} \circ \ldots \circ A_{i+1} \circ b_{yi}(t)$. Define a function $B_y : I \to \Omega(C(I))$ by $B_y(t) = B_{yi}(t)$ for all $t \in [t_i, t_{i+1}]$.

LEMMA 6.5. $B_y(t)$ *is a compact connected nonempty set.* $B_y : I \to \Omega(C(I))$ *is upper semicontinuous.*

Proof. First we note that a convex set is connected. By Lemmas 6.3 and 6.4 and Theorems 2.2, 2.3 and 2.4 we have that $B_{yi}(t)$ is a compact connected nonempty set and that B_{yi} is upper semicontinuous. Now $B_{yi}(t_{i+1}) = B_{y(i+1)}(t_{i+1})$, $i = 0, \ldots, k-2$. Hence B_y is well defined and upper semicontinuous. //

Having defined the operator B_y we can now mimic the standard proof of Kneser's Theorem, (see Coppel [4]).

THEOREM 6.6. *Let* $(t_0, x_0) \in I \times R^n$. *Then* $H(t_0, x_0)$ *is a connected subset of* $C(I)$.

Proof. If $H(t_0, x_0)\big|_{[t_0,b]}$ and $H(t_0, x_0)\big|_{[a,t_0]}$ are connected then $H(r_0, x_0)$ is connected. So without loss of generality we can assume that $t_0 = a$.

Suppose that $H(a, x_0)$ is disconnected; that is, $H(a, x_0) = H_1 \cup H_2$ where H_1 and H_2 are disjoint nonempty closed sets. Let $x_1 \in H_1$ and $x_2 \in H_2$. Subdivide I into k equal parts. By Theorems 2.2 and 2.3 and Lemma 6.5, $B_{x_1}(I)$ and $B_{x_2}(I)$ are compact connected nonempty sets. Further $B_{x_1}(a) = B_{x_2}(a)$. Thus $B_{x_1}(I) \cap B_{x_2}(I) \neq \emptyset$ and $X_k = B_{x_1}(I) \cup B_{x_2}(I)$ is connected. Now $B_{x_1}(b) = \{x_1\}$ and $B_{x_2}(b) = \{x_2\}$. Thus x_1 and $x_2 \in X_k$. Let G_1 and G_2 be disjoint open sets in $C(I)$ such that $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$. Then there exists a function $y_k \in X_k$ which is not contained in either G_1 or G_2 .

Now y_k is of the form

(i) $y_k(t) = x_i(t)$ for all $t \in [a, r_k]$,

(ii) $y_k'(t) \in F\left[t, y_k(t_i)\right]$ almost everywhere $t \in [r_k, b]$ where $t \in [t_i, t_{i+1})$.

Thus $y_k(a) = x_0$ and $|y_k'(t)| \le g(t)$. We now let k (the number of subdivisions of I) vary to obtain a sequence $\{y_k\}$. These y_k are bounded and equicontinuous. By Ascoli's Theorem there is a convergent subsequence, also denoted by $\{y_k\}$, such that y_k tends to y . Therefore y is not contained in $H(a, x_0)$.

But $y(a) = \lim_{k\to\infty} y_k(a) = x_0$. Define $z_k : I \to R^n$ by

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(i) $z_k(t) = y_k(t)$ if $t \in [a, r_k]$,

(ii) $z_k(t) = y_k(t_i)$ if $t \in [r_k, b]$ and $t \in [t_i, t_{i+1})$.

Thus $y_k'(\ ) \in F(t, z_k(t))$ almost everywhere $t \in I$ and z_k tends to y . Therefore

$$\begin{aligned} y'(t) &\in \bigcap_{i=1}^{\infty} \overline{\text{co}} \bigcup_{k=i}^{\infty} y_k'(t) \text{ almost everywhere } t \in I \\ &\subseteq \bigcap_{i=1}^{\infty} \overline{\text{co}} \bigcup_{k=i}^{\infty} F(t, z_k(t)) \text{ almost everywhere } t \in I \\ &\subseteq F(t, y(t)) \text{ almost everywhere } t \in I , \end{aligned}$$

by Theorems 4.1 and 2.8. Thus $y \in H(a, x_0)$ which gives us a contradiction. Therefore $H(a, x_0)$ is connected. //

COROLLARY 6.7. *Let* M *be a connected subset of* $I \times R^n$. *Then* $H(M)$ *is a connected subset of* $C(I)$.

Proof. Theorems 6.6, 5.3 and 2.3 give the required result. //

7. Peripheral attainability and the bang-bang property

If a point on the boundary of the funnel can be reached by a solution lying on the boundary of the funnel, the point is said to be peripherally attainable. In this section we prove that every point on the boundary of the funnel is peripherally attainable.

Let $t_0 \in I$ and $x_0 \in R^n$. Define

$$Z(t_0, x_0) = \{(t, x(t)) : t \in I, x \in H(t_0, x_0)\} .$$

Let $t \in I$. Define

$$A(t, t_0, x_0) = \{x(t) : x \in H(t_0, x_0)\} .$$

$Z(t_0, x_0)$ is called the solution funnel and $A(t, t_0, x_0)$ is called the solution cross-section.

The next two theorems follow easily from these definitions and the preceding theorems.

THEOREM 7.1. *Z maps from $I \times R^n$ to $\Omega(I \times R^n)$ and is upper semicontinuous. If M is a connected subset of $I \times R^n$ then $Z(M)$ is connected.*

THEOREM 7.2. *A maps from $I \times I \times R^n$ to $\Omega(R^n)$. For all $t \in I$, the map $(t_0, x_0) \mapsto A(t, t_0, x_0)$ is upper semicontinuous. For all $(t_0, x_0) \in I \times R^n$ the map $t \mapsto A(t, t_0, x_0)$ is continuous. If M is a connected subset of $I \times R^n$, then $A(t, M)$ is connected.*

We now use Corollary 6.7 to show that every point on the boundary of the funnel is peripherally attainable.

THEOREM 7.3. *Let $K \in \Omega(R^n)$. If $q \in \partial A(b, a, K)$, then there exists $x \in H(a, K)$ such that $x(b) = q$ and $x(t) \in \partial A(t, a, K)$ for all $t \in I$.*

Proof. (a) We first prove that there exists $x \in H(a, K)$ such that $x(b) = q$ and $x(a) \in \partial K$. Suppose that $\operatorname{int} K$ is nonempty (otherwise there is nothing to prove). Let y_k be an exterior point of $A(b, a, K)$ such that $|y_k - q| < 1/k$. Let $\overline{y_k q}$ denote the closed line segment joining y_k and q. $\overline{y_k q}$ is connected. Therefore $K_1 = A(a, b, \overline{y_k q})$ is connected. Now $K_1 \cap K^c \neq \emptyset$ since $y_k \notin A(b, a, K)$. Also $K_1 \cap K \neq \emptyset$ since $q \in A(b, a, K)$. If $K_1 \cap \partial K = \emptyset$, then $K_1 \cap \operatorname{int} K \neq \emptyset$ and $K_1 \subseteq K^c \cap \operatorname{int} K$. But K^c and $\operatorname{int} K$ are disjoint nonempty open sets. This is a contradiction since K_1 is connected. Therefore $K_1 \cap \partial K \neq \emptyset$. Thus there exists $x_k \in H(a, K)$ such that $x_k(a) \in \partial K$ and $x_k(b) \in \overline{y_k q}$. Hence $|x_k(b) - q| < 1/k$. By Theorem 5.1 there exists a convergent subsequence, also denoted by $\{x_k\}$, converging to $x \in H(a, K)$. Therefore $x(b) = q$. Also $x(a) \in \partial K$ since ∂K is closed.

(b) Subdivide $I = [a, b]$ into k equal parts by $t_i = a + i\,\frac{b-a}{k}$.

Using part (a) we can find $x_k \in H(a, k)\big|_{[t_{k-1},b]}$ such that $x_k(b) = q$, $x_k(t_{k-1}) \in \partial A(t_{k-1}, a, K)$. Again using (a) we can extend to $x_k \in H(a, K)\big|_{[t_{k-2},b]}$ such that $x_k(t_{k-2}) \in \partial A(t_{k-2}, a, K)$. Continuing in this manner we obtain $x_k \in H(a, K)$ such that $x_k(t_i) \in \partial A(t_i, a, K)$, $i = 0, \ldots, k-1$ and $x_k(b) = q$. By Theorem 5.1 there exists a subsequence, also denoted by $\{x_k\}$, converging to $x \in H(a, K)$. Therefore $x(t) \in \partial A(t, a, K)$ for all $t \in I$ and $x(b) = q$. //

If $x(t) \in \partial A(t, M)$ for all $t \in I$ implies that $x'(t) \in \partial F(t, x(t))$ almost everywhere $t \in I$, the generalized differential equation $x'(t) \in F(t, x(t))$ is said to have the bang-bang property. In [9] Kikuchi proves that if F is continuous in x then F has the bang-bang property.

The assumption of continuity is needed, as is shown by the following example. Let $I = [0, 1]$. Define

$$F(t, x) = \begin{cases} [0, 2] & \text{if } x = t , \\ \{0\} & \text{otherwise.} \end{cases}$$

Consider

$$x'(t) \in F(t, x(t)) \quad \text{almost everywhere} \quad t \in [0, 1] .$$

$$x(0) = 0 .$$

The peripheral solutions are $x_1(t) = t$ and $x_2(t) = 0$. Thus $x_1'(t) = 1$ is not on the boundary of $[0, 2]$.

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