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Martingale central limit theorems without uniform asymptotic negligibility: Corrigendum

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The following is a correct proof of the main theorem of [1]. It should be substituted for the published Section 3, which, as pointed out by Professor B.L.S. Prakasa Rao, contains an error in the equations following (15) on page 49.

3. Proof of the theorem

We shall use the notation $\sum'_{j\le k}$ for the sum over all $j \le k$, $j \in U_n$, and $\sum''_{j\le k}$ for the sum over all $j \le k$, $j \in \overline{U}_n$. Our first step is to reduce the problem without loss of generality. Note first that we need only show that for any subsequence $\{n'\}$ there exists a further subsequence $\{n''\}$ along which the convergence to normality holds. We may thus assume that

$$\sum_k{}'' \sigma_k^2(n) \to L \quad \text{as} \quad n \to \infty , \tag{13}$$

for some $0 \le L \le 1$. Then we observe that $\sum'_{j\le k} X_j(n)$ is a martingale difference array satisfying the conditions of McLeish's Theorem 2.3, [5], with $\sum'_j X_j^2(n) \xrightarrow{p} 1 - L$ instead of 1 . We may assume also, by replacing $X_k(n)$ for $k \in U_n$ by $X_k(n) I\left\{\sum_{j=1}^{k-1}{}' X_j^2(n) \le 2\right\}$, that when $\sum_{j=1}^{k}{}' X_j^2(n) > 2$,

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all subsequent $X_j(n)$ terms for $j \in U_n$ are zero. The argument for this is exactly the same as that of McLeish [5, p. 622]. To reduce the problem further we use Theorem 4.2 of Billingsley [2]. Set

(14) $X_j^*(n) = X_j^*(n, M) =$

$$= X_j(n)I(|X_j(n)| < M) - E\{X_j(n)I(|X_j(n)| < M) \mid F_{j-1}(n)\}$$

for $j \in U_n$, while $X_j^*(n) = X_j(n)$ for $j \in \overline{U}_n$. Then if $S_k^*(n) = \sum_{j=0}^{k} X_j^*(n)$, $\{S_k^*(n), F_k(n)\}$ is a martingale array and condition (3) of the theorem is still satisfied. We show (6) is satisfied also, but in addition, the condition

$$\max_{j \in U_n} \cdot|X_j^*(n)| \xrightarrow{L_2} 0 , \tag{15}$$

which implies (4) and (5) is also satisfied, for this new array. Clearly $\max_{j \in U_n} |X_j(n)|I(|X_j(n)| < M) \xrightarrow{p} 0$ as $n \to \infty$, and in addition, by boundedness, the convergence is in L_2 also. Now (McLeish [5], p. 621)

$$\sum{}' X_j^2(n)I(\varepsilon < |X_j(n)| < M) \xrightarrow{p} 0 \text{ for each } \varepsilon > 0 , \tag{16}$$

and this being bounded by $2 + M^2$, the convergence is in L_2 again, implying

$$\sum_j{}' E\left\{X_j^2(n)I(\varepsilon < |X_j(n)| < M) \mid F_{j-1}(n)\right\} \xrightarrow{p} 0 .$$

Then, since $\varepsilon > 0$ is arbitrary,

$$\left(\max_{j \in U_n} E\{X_j(n)I(|X_j(n)| < M) \mid F_{j-1}(n)\}\right)^2$$
$$\le \sum_j{}' E\left\{X_j^2(n)I(\varepsilon < |X_j(n)| < M) \mid F_{j-1}(n)\right\} + \varepsilon^2$$

implies

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(17) $$\max_{j \in U_n} E\{X_j(n)I(|X_j(n)| < M) \mid F_{j-1}(n)\} \xrightarrow{L_2} 0 .$$

Clearly (15) is satisfied. To show that (6) is satisfied we observe first that

$$P\Big(\sum_j{}' X_j^2(n) \neq \sum_j{}' X_j^2(n)I(|X_j(n)| < M)\Big) \leq P\big(\max_{j \in U_n} |X_j(n)| > M\big) ,$$

so that $\sum_j{}' X_j^2(n)I(|X_j(n)| < M) \xrightarrow{p} 1 - L$, and we must show only

(18) $$\sum_j{}' X_j(n)I(|X_j(n)| < M)E\{X_j(n)I(|X_j(n)| < M) \mid F_{j-1}(n)\} \xrightarrow{p} 0$$

and

(19) $$\sum_j E\{X_j(n)I(|X_j(n)| < M) \mid F_{j-1}(n)\}^2 \xrightarrow{p} 0 .$$

Now

(20) $$\begin{aligned} &\Big|\sum_j{}' X_j(n)I(|X_j(n)| < M)E\{X_j(n)I(|X_j(n)| < M) \mid F_{j-1}(n)\}\Big| \\ &= \Big|\sum_j{}' X_j(n)I(|X_j(n)| < M)E\{X_j(n)I(|X_j(n)| \geq M) \mid F_{j-1}(n)\}\Big| \\ &\leq \frac{1}{M}\sum_j{}' |X_j(n)|I(|X_j(n)| < M)E\Big\{X^2(n)I(|X_j(n)| \geq M) \mid F_{j-1}(n)\Big\} \\ &\leq \frac{1}{M} \max_{j \in U_n} |X_j(n)| \sum_j{}' E\Big\{X_j^2(n)I(|X_j(n)| \geq M) \mid F_{j-1}(n)\Big\} . \end{aligned}$$

But

$$\begin{aligned} E \sum_j{}' E\Big\{X_j^2(n)I(|X_j(n)| \geq M) \mid F_{j-1}(n)\Big\} &= E\Big(\sum{}' X_j^2(n)I(|X_j(n)| \geq M)\Big) \\ &\leq E\Big(2 + \max_{j \in U_n} X_j^2(n)\Big) \\ &\leq 2 + K_1 , \end{aligned}$$

where K_1 is bound in (5). Hence by Markov's inequality, $\sum_j{}' E\Big\{X_j^2(n)I(|X_j(n)| \geq M) \mid F_{j-1}(n)\Big\}$ is bounded in probability and (18)

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then follows from (20) and (4). Similar reasoning involving (17), rather than (4), gives (19), and so $\{S_k^*(n)\}$ satisfies (3), (6), and (15).

We must now show that the conditions of Theorem 4.2 of [2] are satisfied; that is to say we want, for $\varepsilon > 0$,

$$\limsup_{M\to\infty} \lim_{n\to\infty} P\left(|S^*_{k_n}(n) - S_{k_n}(n)| < \varepsilon\right) = 0 \ .$$

Now

$$|S^*_{k_n}(n) - S_{k_n}(n)| \le \left|\sum_j{}' X_j(n) I\left(|X_j(n)| \ge M\right)\right| + \left|\sum_j{}' E\{X_j(n) I\left(|X_j(n)| \quad M\right) \mid F_{j-1}(n)\}\right| ,$$

and

$$P\left\{\left|\sum_j{}' X_j(n) I\left(|X_j(n)| \ge M\right)\right| \ne 0\right\} \le P\left(\max_{j\in U_n} |X_j(n)| \ge M\right) ,$$

which converges to zero for each fixed M as $n \to \infty$. Also

$$\begin{aligned} E\left|\sum_j{}' E\{X_j(n) I\left(|X_j(n)| < M\right) \mid F_{j-1}(n)\}\right| &\le E \sum_j{}' E\{|X_j(n)| I\left(|X_j(n)| \ge M\right) \mid F_{j-1}(n)\} \\ &\le \frac{1}{M} E\left(\sum_j{}' X_j(n) I\left(|X_j(n)| \ge M\right)\right) \\ &\le \frac{1}{M}\left(2+K_1\right) , \end{aligned}$$

and using Markov's inequality

$$\limsup_{M\to\infty} \lim_{n\to\infty} P\left\{\left|\sum_j{}' E\{X_j(n) I\left(|X_j(n)| < M\right) \mid F_{j-1}(n)\}\right| > \varepsilon\right\} \le \limsup_{M\to\infty} \frac{1}{M\varepsilon}\left(2+K_1\right) = 0 \ .$$

We have thus shown that to prove the theorem we may assume $\left\{S_k^{(n)}, F_k(n)\right\}$ is a martingale triangular array satisfying

$$\text{(21)} \qquad \sum_j{}'' \sigma_j^2(n) \to L , \quad 0 \le L \le 1 ,$$

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(22) $$\max_{j \in U_n} |X_j(n)| \xrightarrow{L_2} 0 ,$$

(23) $$\sum_j{}' X_j^2(n) \to 1 - L ,$$

and for some $M > 0$,

$$\sum_j{}' X_j^2(n) \le 2 + 2M^2 ,$$

so that

(24) $$\sum_j{}' X_j^2(n) \xrightarrow{L_1} 1 - L .$$

Using the techniques of either McLeish ([5], pp. 621-622) or Scott ([6], §3),

(26) $$E\left\{\sum_j{}' X_j^2(n) I\left(|X_j(n)| > \varepsilon\right)\right\} \to 0 \text{ for each } \varepsilon > 0 ,$$

and

(27) $$\sum_j{}' E\left\{X_j^2(n) \mid F_{j-1}(n)\right\} \xrightarrow{p} 1 - L ,$$

so that we may assume

$$\sum_j{}' E\left\{X_j^2(n) \mid F_{j-1}(n)\right\} < C < \infty$$

for all n . (Otherwise replace $X_k(n)$ by

$$X_k(n) I\left\{\sum_{j \le k}{}' E\left\{X_j^2(n) \mid F_{j-1}(n)\right\} < C\right\} .)$$

We wish to show then, for each real t ,

(28) $$\lim_{n\to\infty} E e^{itS_{k_n}(n)} = e^{-t^2/2} .$$

We put for $n \ge 1$, $j = 1, 2, \ldots, k_n$,

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$$\tau_j^2 = \begin{cases} \sigma_j^2(n) , & j \in \bar{U}_n , \\ \tilde{\sigma}_j^2(n) , & j \in U_n , \end{cases}$$

and for $k = 1, 2, \ldots, k_n$,

(29) $$U_k^2(n) = \sum_{j=1}^{k} \tau_j^2(n) .$$

We will show

(30) $$\lim_{n\to\infty} E\left|\exp\left\{itS_{k_n}(n)+\tfrac{1}{2}t^2U_{k_n}^2(n)\right\}-1\right| = 0$$

and

(31) $$\lim_{n\to\infty} E\left|\exp\left\{\tfrac{1}{2}t^2U_{k_n}^2(n)\right\}-\exp\left(-t^2/2\right)\right| = 0 ,$$

from which (28) follows without difficulty.

Set

$$Z_j(n) = \left\{\exp itS_{j-1}(n)+\tfrac{1}{2}t^2U_j^2(n)\right\}\left(e^{itX_j(n)}-e^{-\frac{1}{2}t^2\tau_j^2(n)}\right) ,$$

so that

$$\left|E\left\{\exp\left\{itS_{k_n}(n)+t^2U_{k_n}^2(n)\right\}-1\right| = \left|E\sum_{j=1}^{k_n} Z_j(n)\right|$$
$$\le E\left|\sum_{j=1}^{k_n} E\{Z_j(n) \mid F_{j-1}(n)\}\right| .$$

If $j \in U_n$, then

$$|E\{Z_j(n) \mid F_{j-1}(n)\}| \le \tfrac{1}{2}t^2e^{\frac{1}{2}t^2C}\left[E\left\{X_j^2(n)M(|tX_j(n)|) \mid F_{j-1}(n)\right\} + \tfrac{1}{4}t^2\sigma_j^2(n)\left\{\max_{j\in U_n} \sigma_j^2(n)\right\}\right]$$

for $M(\cdot)$ defined by $M(x) = \min\left\{\frac{1}{3}x, 2\right\}$, as in Brown [3], p. 64. If

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$j \in \bar{U}_n$ then let $Y_j(n)$ be $N\left(0, \sigma_j^2(n)\right)$, independent of each other and the σ-field generated by $\bigcup_{j=1}^{k_n} F_j(n)$. Then

$$|E\{Z_j(n) \mid F_{j-1}(n)\}| \le e^{\frac{1}{2}t^2C}\left|E\left\{e^{itX_j(n)} - e^{-\frac{1}{2}t^2\tau_j^2(n)} \mid F_{j-1}(n)\right\}\right|$$
$$\le e^{\frac{1}{2}t^2C}\left|E\left\{e^{itX_j(n)} - e^{itY_j(n)} \mid F_{j-1}(n)\right\}\right| .$$

Combining these last three inequalities,

$$(32) \quad \left|E \exp\left(itS_{k_n}(n)+t^2U_{k_n}^2(n)\right)-1\right|$$
$$\le E \sum_j{}' \left[\tfrac{1}{2}t^2e^{\frac{1}{2}t^2C}E\left\{X_j^2(n)M(|tX_j(n)|) \mid F_{j-1}(n)\right\}+\tfrac{1}{2}t^2\sigma_j^2(n) \max_{j\in U_n} \sigma_j^2(n)\right]$$
$$+ E \sum_j{}'' e^{\frac{1}{2}t^2C}\left|E\left\{e^{itX_j(n)} - e^{itY_j(n)} \mid F_{j-1}(n)\right\}\right| .$$

The first sum goes to zero with n using (2.4), (26), and (22) as in Brown [3], p. 64. For the second term we may use the argument on pp. 50-51 of [1], which for convenience is repeated here. Define a sequence of numbers A_n by $A_n = \sqrt{2\log\alpha_n^{-1}}$. By Feller [4] (page 175) we have

$$\Phi\left(-A_n\right) = 1 - \Phi\left(A_n\right) < A_n^{-1}e^{-A_n^2/2} = \alpha_n\left(2 \log \alpha_n^{-1}\right)^{-\frac{1}{2}} .$$

Since for $j \in \bar{U}_n$, $\sigma_j(n) < 1$, it follows that

$$(33) \qquad \Phi\left(-A_n/\sigma_j(n)\right) = 1 - \Phi\left(A_n/\sigma_j(n)\right) < \alpha_n\left(2 \log \alpha_n^{-1}\right)^{-\frac{1}{2}} .$$

We have thus

$$E\left|E\left\{e^{itX_j(n)}-e^{itY_j(n)} \mid F_{j-1}(n)\right\}\right|$$
$$= E\left|\int e^{itx}d\Delta_j^{(n)}(x)\right|$$
$$\leq E\left|\int_{-}^{-A_n} e^{itx}d\Delta_j^{(n)}(x)\right| + E\left|\int_{-A_n}^{A_n} e^{itx}d\Delta_j^{(n)}(x)\right| + E\left|\int_{A_n}^{\infty} e^{itx}d\Delta_j^{(n)}(x)\right|$$
$$= I_1 + I_2 + I_3 \ .$$

Treating these terms separately,

$$(34) \quad I_1 \leq E\int_{-A}^{-A_n}\left|d\Delta_j^{(n)}(x)\right| \leq E\left(P\{X_j(n) \leq -A_n \mid F_{j-1}\} + \Phi(-A_n/\sigma_j(n))\right)$$
$$\leq 2\Phi(-A_n/\sigma_j(n)) + \alpha_n$$
$$\leq 2\alpha_n\left(2 \log \alpha_n^{-1}\right)^{-\frac{1}{2}} + \alpha_n \ ,$$

using (2) and (33). Furthermore

$$(33) \qquad I_2 \leq E\left\{\left|\int_{-A_n}^{A_n} ite^{itx}\Delta_j^{(n)}(x)dx\right| + \left|\left[e^{itx}\Delta_j^{(n)}(x)\right]_{-A_n}^{A_n}\right|\right\}$$
$$\leq tE\int_{-A_n}^{A_n}\left|\Delta_j^{(n)}(x)\right|dx + 2\alpha_n$$
$$\leq 2tA_n\alpha_n + 2\alpha_n$$
$$= 2\alpha_n\left\{1 + t\sqrt{2\log\alpha_n^{-1}}\right\} \ .$$

But $j \in \overline{U}_n$ entails $\sigma_j^2(n) \leq \gamma_n$, and since $\sum_j'' \sigma_j^2(n) \leq 1$, there are at most γ_n^{-1} indices in $\overline{U}_n$. Combining this with (34), (35), and a similar bound for I_3, we obtain

$$\sum_j'' E\left|E\left\{e^{itX_j(n)}-e^{itY_j(n)} \mid F_{j-1}(n)\right\}\right|$$
$$\leq \gamma_n^{-1}\left\{4\alpha_n + 2t\alpha_n\sqrt{2\log\alpha_n^{-1}} + 4\alpha_n\left(2 \log \alpha_n^{-1}\right)^{-\frac{1}{2}}\right\}$$
$$\to 0 \text{ as } n \to \infty \ ,$$

using (1); so we have completed the proof of (30).

The proof of (31) is relatively simple:

$$\exp\left\{\tfrac{1}{2}t^2U^2_{k_n}(n)\right\} \xrightarrow{p} \exp\left(\tfrac{1}{2}t^2\right)$$

from (21) and (27), and

$$\exp\left\{\tfrac{1}{2}t^2U^2_{k_n}(n)\right\} \le \exp\left(\tfrac{1}{2}t^2[C+1]\right) ,$$

so the convergence is in L_1 also, which is just (31).

References

[1] R.J. Adler and D.J. Scott, "Martingale central limit theorems without uniform asymptotic negligibility", *Bull. Austral. Math. Soc.* 13 (1975), 45-55.

[2] Patrick Billingsley, *Convergence of probability measures* (John Wiley & Sons, New York, London, Sydney, Toronto, 1968).

[3] B.M. Brown, "Martingale central limit theorems", *Ann. Math. Statist.* 42 (1971), 59-66.

[4] William Feller, *An introduction to probability theory and its applications*, Volume I, 3rd ed. (John Wiley & Sons, New York, London, Sydney, 1968).

[5] D.L. McLeish, "Dependent central limit theorems and invariance principles", *Ann. Probability* 2 (1974), 620-628.

[6] D.J. Scott, "Central limit theorems for martingales and for processes with stationary increments using a Skorokhod representation approach", *Adv. in Appl. Probability* 5 (1973), 119-137.

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