COLOURING OF TRIVALENT POLYHEDRA

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By an *Euler polyhedron of valence three* or a *trivalent convex polyhedron* in Euclidean 3-space **(4)** we mean in the present paper an Euler polyhedron in the sense of Steinitz **(8**, p. 113**)**, such that each vertex is incident with exactly three edges.

In the present paper we establish a theorem concerning the colouring of trivalent polyhedra. A specialization of this theorem solves the following problem implicit in Eberhard **(1**, p. 84**)**: Does there exist a trivalent Euler polyhedron with an odd number of faces such that the number of edges incident with any face is divisible by three?

I wish to mention that this problem has recently been solved completely by Motzkin in **(6)**. Previously, Grünbaum and Motzkin solved it, by a different method, in **(4)** for the special case where all faces are triangles or hexagons. The present paper solves it by a third, completely different method.

Let $\mathfrak{P}$ be any Euler polyhedron and let $\mathfrak{V}$ [$\mathfrak{H}$, $\mathfrak{S}$] be the set of its vertices [edges, faces]. By a *colouring* of the faces of $\mathfrak{P}$ we mean a mapping ϕ of the set $\mathfrak{S}$ onto the set $\Phi = \{\alpha, \beta, \gamma, \delta\}$ such that $\phi(S_1) \neq \phi(S_2)$ for any two faces $S_1, S_2 \in \mathfrak{S}$ that have an edge in common. The elements of the set Φ will be called the colours of the faces. By a colouring of the edges of the polyhedron we mean a mapping λ of the set $\mathfrak{H}$ onto the set $\Lambda = \{1, 2, 3\}$ such that $\lambda(H_1) \neq \lambda(H_2)$ for any two edges $H_1, H_2 \in \mathfrak{H}$, incident with the same vertex $\in \mathfrak{V}$.

The following lemma is well known **(2**; **3)**.

LEMMA 1. *A colouring of the faces of the polyhedron $\mathfrak{P}$ exists if and only if there exists a colouring of its edges.*

Given a colouring ϕ of the faces of $\mathfrak{P}$, a colouring λ of its edges can easily be found in the following way. Let H be any edge incident with the faces S_1, S_2. Define $F(H) = \{\phi(S_1), \phi(S_2)\}$ and set:

$$(*)\begin{cases}\lambda(H) = 1 & \text{if } F(H) = \{\alpha, \beta\} \text{ or if } F(H) = \{\gamma, \delta\},\\ \lambda(H) = 2 & \text{if } F(H) = \{\alpha, \gamma\} \text{ or if } F(H) = \{\beta, \delta\},\\ \lambda(H) = 3 & \text{if } F(H) = \{\alpha, \delta\} \text{ or if } F(H) = \{\beta, \gamma\}.\end{cases}$$

Obviously the mapping λ defined in this way is a colouring of the edges of $\mathfrak{P}$. Conversely, given a colouring of the edges of $\mathfrak{P}$, we can always find exactly four colourings of the faces of $\mathfrak{P}$ with colours from Φ such that the conditions $(*)$ are fulfilled. It is sufficient to choose for one face $S \in \mathfrak{S}$ the value $\phi(S) \in \Phi$;

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the colours of the other faces in $\mathfrak{S}$ are then uniquely determined by the conditions $(*)$.

Remark 1. The problem of colouring maps, which has resisted solution for a century, known as "the four-colour problem," can be reduced to the question of whether there exists a colouring of edges for every trivalent Euler polyhedron.

For a given colouring λ of the edges of $\mathfrak{P}$, we define a mapping λ^* of $\mathfrak{B}$ into the set $\{-1, 1\}$ by letting $\lambda^*(V) = 1$ if the edges VV_i such that $\lambda(VV_i) = i$ $(i = 1, 2, 3)$ follow each other in the positive sense of rotation; otherwise put $\lambda^*(V) = -1$.

The mapping λ^* will be said to be the *valuation* of vertices with respect to the colouring λ, and the number $\lambda^*(V)$ will be said to be the *value* of the vertex V for this colouring.

We know (**7**, p. 18, Theorem 5) the following lemma.

LEMMA 2. *Let $\mathfrak{P}$ be any Euler polyhedron of valence three. A colouring λ of the edges of this polyhedron exists if and only if there exists a mapping κ of the set $\mathfrak{B}$ into the set $\{-1, 1\}$ such that the sum of the values $\kappa(V)$ for all the vertices incident with any given face is divisible by three. If κ is such a mapping, then there exists a colouring λ such that $\lambda^* = \kappa$.*

The following lemma is a direct consequence of Lemma 2.

LEMMA 3. *Let $\mathfrak{P}$ be any Euler polyhedron of valence three, such that the number of edges incident with any face is divisible by three. Then there exists a colouring λ of the edges of the polyhedron such that $\lambda^*(V) = 1$ for each vertex $V \in \mathfrak{B}$.*

Let ϕ be a colouring of the faces of an Euler polyhedron $\mathfrak{P}$ of valence three and let λ be a colouring of its edges such that λ and ϕ fulfil the conditions $(*)$. Let λ^* be the corresponding valuation of the vertices in $\mathfrak{B}$. Let $\xi \neq \eta$ be any two colours in Φ. Denote by $\mathfrak{H}(\xi, \eta)$ the set of all those edges in $\mathfrak{H}$ which are incident with faces, one of which has the colour ξ, the other the colour η. Further, denote by the symbol $\mathfrak{B}(\xi, \eta)$ the set of all the vertices in $\mathfrak{B}$ that are incident with edges in $\mathfrak{H}(\xi, \eta)$ and put

$$\Lambda^*(\xi, \eta) = \sum_{V \in \mathfrak{B}(\xi, \eta)} \lambda^*(V).$$

We evidently have

$$\mathfrak{B} = \mathfrak{B}(\alpha, \beta) \cup \mathfrak{B}(\gamma, \delta) = \mathfrak{B}(\alpha, \gamma) \cup \mathfrak{B}(\beta, \delta) = \mathfrak{B}(\alpha, \delta) \cup \mathfrak{B}(\beta, \gamma),$$
$$\emptyset = \mathfrak{B}(\alpha, \beta) \cap \mathfrak{B}(\gamma, \delta) = \mathfrak{B}(\alpha, \gamma) \cap \mathfrak{B}(\beta, \delta) = \mathfrak{B}(\alpha, \delta) \cap \mathfrak{B}(\beta, \gamma).$$

This yields:

LEMMA 4. *A colouring ϕ of the faces of the polyhedron $\mathfrak{P}$ and the corresponding valuation of the vertices λ^* always satisfy*

$$\Lambda^*(\alpha, \beta) = \Lambda^*(\gamma, \delta), \qquad \Lambda^*(\alpha, \gamma) = \Lambda^*(\beta, \delta), \qquad \Lambda^*(\alpha, \delta) = \Lambda^*(\beta, \gamma).$$

Proof. The set $\mathfrak{H}(\alpha, \beta) \cup \mathfrak{H}(\gamma, \delta) = \mathfrak{H}(1)$ is evidently the set of all the edges of $\mathfrak{P}$ which for the colouring λ have the colour 1. Similarly

$$\mathfrak{H}(\alpha, \gamma) \cup \mathfrak{H}(\beta, \delta) = \mathfrak{H}(2) \quad [\mathfrak{H}(\alpha, \delta) \cup \mathfrak{H}(\beta, \gamma) = \mathfrak{H}(3)]$$

is the set of all the edges in $\mathfrak{H}$ with the colour 2 [3]. For each $i \in \{1, 2, 3\}$ we have: $\mathfrak{H}(i)$ is the set of edges of a linear factor $L(i)$ of the graph G of $\mathfrak{P}$ (we obtain the graph of $\mathfrak{P}$ from $\mathfrak{P}$ by deleting all the faces while preserving the vertices and edges as well as their incidence); from $i \neq j$ it follows that $\mathfrak{H}(i) \cap \mathfrak{H}(j) = \emptyset$. Then $\mathfrak{H}(2) \cup \mathfrak{H}(3)$ is the set of the edges of a quadratic factor $Q_{2,3}$ of G; hence each component of $Q_{2,3}$ is, in the sense of the theory of finite graphs, a circuit (**5**, p. 155) with an even number of vertices, where the edges of colour 2 alternate with edges of colour 3. Let S_1 be any faces in $\mathfrak{S}$. To every circuit K of $Q_{2,3}$ we can assign a partition of the set $\mathfrak{S}$ into two classes

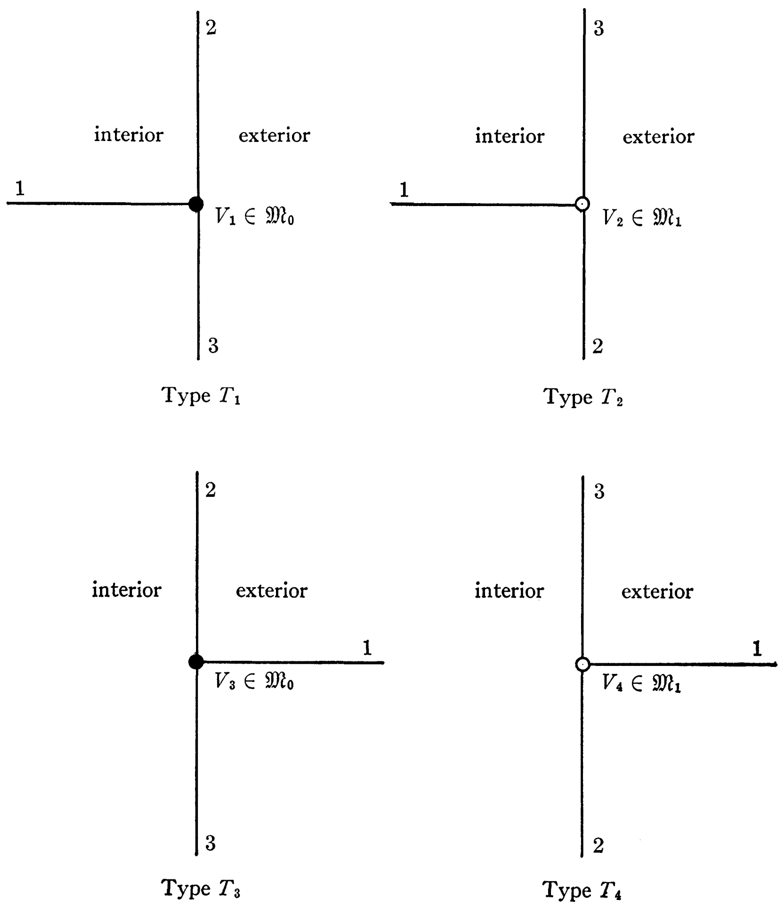

Figure 1

$\mathfrak{S}_0(K)$ and $\mathfrak{S}_1(K)$ in the following way: the face $S \in \mathfrak{S}$ belongs to $\mathfrak{S}_1(K)$ if and only if we can move from S on the surface of $\mathfrak{P}$ to S_1 without crossing the circuit K; $\mathfrak{S}_0(K) = \mathfrak{S} - \mathfrak{S}_1(K)$. The set $\mathfrak{S}_0(K)$ will be the *exterior* and the set $\mathfrak{S}_1(K)$ the *interior* of K. Any edge in $\mathfrak{H}(1)$ incident with a vertex of K (hence not belonging to K) is incident either with two faces of the interior or with two faces of the exterior. In addition, the following holds: either all the edges in $\mathfrak{H}(1)$, incident with a vertex from K and with a face in $\mathfrak{S}_0(K)$, belong to $\mathfrak{H}(\alpha, \beta)$ (first case) or all of them belong to $\mathfrak{H}(\gamma, \delta)$ (second case). In the first case all the edges in $\mathfrak{H}(1)$ incident with a vertex from K and with a face in $\mathfrak{S}_1(K)$ belong to $\mathfrak{H}(\gamma, \delta)$; in the second case they all belong to $\mathfrak{H}(\alpha, \beta)$.

Let us partition the set of vertices of the circuit K alternating into classes $\mathfrak{W}_0$, $\mathfrak{W}_1$ so that an observer moving along K on the surface of $\mathfrak{P}$ in such a way that the interior of K stays on his left must, after passing a vertex in $\mathfrak{W}_0$ [a vertex in $\mathfrak{W}_1$], move through an edge of colour 2 [an edge of colour 3]. Evidently only four types of vertices can exist on K, viz. the types T_1, T_2, T_3, T_4 illustrated in Fig. 1 (the vertices in $\mathfrak{W}_0$ are indicated by disks; those in $\mathfrak{W}_1$ by circles; the number of an edge denotes its colour). Let us denote by $\mathfrak{A}$ [by $\mathfrak{B}$] the set of all the vertices in K incident with an edge in $\mathfrak{H}(1)$ which is incident with a face in the interior [the exterior] of K. Evidently

$$\lambda^*(V_1) = -1, \qquad \lambda^*(V_2) = 1, \qquad \lambda^*(V_3) = 1, \qquad \lambda^*(V_4) = -1;$$

cf. Fig. 1.

Denote by τ_i the number of those vertices of K that belong to the type T_i $(i = 1, 2, 3, 4)$. Since the number of vertices belonging to $\mathfrak{W}_0$ must be equal to the number of vertices belonging to $\mathfrak{W}_1$, we have

$$\tau_1 + \tau_3 = \tau_2 + \tau_4,$$

and further

$$\sum_{V \in \mathfrak{A}} \lambda^*(V) = -\tau_1 + \tau_2, \qquad \sum_{V \in \mathfrak{B}} \lambda^*(V) = \tau_3 - \tau_4;$$

hence

$$\sum_{V \in \mathfrak{A}} \lambda^*(V) = \sum_{V \in \mathfrak{B}} \lambda^*(V).$$

Thus the sum of the values of the vertices in $\mathfrak{B}(\alpha, \beta)$ belonging to K is equal to the sum of the vertices in $\mathfrak{B}(\gamma, \delta)$ belonging to K. Since K was any circuit of $Q_{2,3}$, this holds for each component of $Q_{2,3}$ and so for $Q_{2,3}$ itself. Since each vertex in $\mathfrak{B}$ belongs to $Q_{2,3}$, it follows that

$$\Lambda^*(\alpha, \beta) = \Lambda^*(\gamma, \delta).$$

Symmetrically, we obtain in

$$\Lambda^*(\alpha, \gamma) = \Lambda^*(\beta, \delta) \quad \text{and} \quad \Lambda^*(\alpha, \delta) = \Lambda^*(\beta, \gamma).$$

THEOREM. *Let $\mathfrak{P}$ be any Euler polyhedron of valence three and let ϕ be a colouring of its faces. Let λ be the colouring of its edges defined by the conditions $(*)$ and let λ^* be the corresponding valuation of the vertices of $\mathfrak{P}$. Let $\mathfrak{V}(\xi)$ (where ξ is any colour in Φ) be the set of all the vertices in $\mathfrak{V}$ incident with a face of colour ξ and let*

$$M(\xi) = \sum_{V \in \mathfrak{V}(\xi)} \lambda^*(V).$$

Then $M(\xi) = M$ for all $\xi \in \Phi$, where M is an integer divisible by three.

Proof. Put $\Omega = \sum_{V \in \mathfrak{V}} \lambda^*(V)$. Then

$$M(\alpha) + M(\beta) = \Omega + \Lambda^*(\alpha, \beta).$$

This assertion follows at once from the fact that in the sum $M(\alpha) + M(\beta)$ the value of a vertex V not incident with an edge in $\mathfrak{H}(\alpha, \beta)$ is counted exactly once and the value of a vertex V incident with such an edge twice (because V is incident both with a face of colour α and with a face of colour β). Similarly

$$M(\xi) + M(\eta) = \Omega + \Lambda^*(\xi, \eta)$$

for any two colours ξ and η. From Lemma 4 it follows, therefore, that:

$$\begin{aligned} M(\alpha) + M(\beta) &= M(\gamma) + M(\delta), \\ M(\alpha) + M(\gamma) &= M(\beta) + M(\delta), \\ M(\alpha) + M(\delta) &= M(\beta) + M(\gamma). \end{aligned}$$

Hence

$$M(\alpha) = M(\beta) = M(\gamma) = M(\delta) = M.$$

It follows from the definition of the sum $M(\alpha)$ and from Lemma 2 that the number M is divisible by three. This proves the theorem.

We deduce some corollaries.

COROLLARY 1. *The sum $\Omega = \sum_{V \in \mathfrak{V}} \lambda^*(V)$ is divisible by four.*

Proof. In the sum $M(\alpha) + M(\beta) + M(\gamma) + M(\delta) = 4M$ the value of each vertex in $\mathfrak{V}$ is counted exactly three times. Thus $4M = 3\Omega$, which proves our assertion.

COROLLARY 2. *If the number of edges incident with any face of an Euler polyhedron $\mathfrak{P}$ of valence three is divisible by three, then the number of vertices is divisible by four and the number of faces is even.*

Proof. Let π_0 denote the number of vertices, π_1 that of the edges, and π_2 that of the faces of $\mathfrak{P}$. By Lemma 3, there exists a colouring λ of the edges of $\mathfrak{P}$ such that $\lambda^*(V) = 1$ for each vertex V. Hence $\Omega = \pi_0$ and $\pi_0 = 4N$, where N is a positive integer. Since $\mathfrak{P}$ has valence three, we have $2\pi_1 = 3\pi_0$; hence $\pi_1 = 6N$. By Euler's equation (valid for all Euler polyhedra)

$$\pi_0 - \pi_1 + \pi_2 = 2.$$

This yields $\pi_2 = 2N + 2$; thus the number of faces of $\mathfrak{P}$ is even. This proves both assertions.

Remark 2. Corollary 2 contains a new solution of the problem of Eberhard **(1)**, quoted in **(4)**; Motzkin gives a complete solution in **(6)**.

Remark 3. We can readily show that a similar theorem holds for all trivalent planar regular graphs which are decomposable into three linear factors. It is therefore not necessary to restrict the above considerations to graphs of (trivalent) Euler polyhedra.

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