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REDUCTION OF EXPONENTIAL RANK IN DIRECT LIMITS OF C^*-ALGEBRAS

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ABSTRACT. We prove the following result. Let A be a direct limit of direct sums of C^*-algebras of the form $C(X) \otimes M_n$, with the spaces X being compact metric. Suppose that there is a finite upper bound on the dimensions of the spaces involved, and suppose that A is simple. Then the C^* exponential rank of A is at most $1+\varepsilon$, that is, every element of the identity component of the unitary group of A is a limit of exponentials. This is true regardless of whether the real rank of A is 0 or 1.

Introduction. In [22], we showed that the exponential rank $\mathrm{cer}(A)$ of a C^*-algebra A can be arbitrarily large, that is, that for any n there is a unital C^*-algebra A and a unitary $u \in A$ which is a product of some number of exponentials but not of n or fewer. The algebras in [22], however, are not simple, and it remains unknown whether the exponential rank of a simple C^*-algebra can be large. Indeed, no simple C^*-algebra A is known to satisfy $\mathrm{cer}(A) > 1+\varepsilon$. (The condition $\mathrm{cer}(A) \leq 1+\varepsilon$ means that the exponential unitaries are dense in the identity component of the unitary group. See [20].)

It is known that if B_3 is the closed unit ball in $\mathbb{R}^3$, then $\mathrm{cer}\big((B_3) \otimes M_n\big) \geq 2$ for any n. One might therefore hope to find at least a simple C^*-algebra A with $\mathrm{cer}(A) \geq 2$ among the algebras obtained as direct limits of direct sums of algebras of the form $C(X) \otimes M_n$. However, we show in this paper that if the dimensions of the spaces involved are bounded, then the exponential rank is in fact at most $1+\varepsilon$. Specifically, our main theorem (Theorem 6.1) states that if A is a separable simple unital C^*-algebra, obtained as a direct limit $\varinjlim A_i$, where each A_i has the form $\bigoplus_{t=1}^{s(i)} C(X_{it}) \otimes M_{n(i,t)}$, and if $\sup_{i,t} \dim(X_{it}) < \infty$, then $\mathrm{cer}(A) \leq 1+\varepsilon$. This result parallels results of [8], where it is shown that such algebras have stable rank 1, and of [3] and [4], where it is shown that such algebras often have real rank 0. Our result, however, applies to direct limits of this sort even if they have real rank 1, which is the value that direct arguments (based on [1]) suggest. We thus obtain a large class of simple C^*-algebras A which do not have real rank 0 but nevertheless have exponential rank at most $1+\varepsilon$.

Our proof is based on the methods of Gong and Lin [11], where two results on simple algebras of the form $A = \varinjlim C(X) \otimes M_{n(i)}$ are proved. In the first case, Theorem 1.3 of [11], X is arbitrary, but the maps of the system must have the special form studied in [12], and A must have real rank 0. In the second case, Theorem 3.3 of [11], X must be a finite complex, and again A must have real rank 0. It has since been proved by Lin [16] that

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any C^*-algebra A with real rank 0 satisfies $\mathrm{cer}(A) \leq 1 + \varepsilon$. We generalize Theorem 3.3 of [11] in a different direction. We omit the assumptions that there is only one space at each level, that the space is the same at each level, and, most importantly, that the real rank is 0.

The assumption of real rank 0 was used in three places in [11], all contained in the proof of Theorem 2.6 there: bounding exponential length in terms of exponential rank, splitting off parts of a unitary which are exponentials, and approximating an arbitrary selfadjoint element by a direct summand in one with finite spectrum. We must employ a different device to get along without real rank 0 in each of these three places. Unfortunately, two of these devices must be used together in order to make things work, resulting in the very long and technical proof of Lemma 5.3.

This paper is organized as follows. In Section 1, we give some definitions, establish some notation, and give some results that are proved or almost proved elsewhere. We refer to the algebras we consider as having "no dimension growth", in analogy to the condition of "slow dimension growth" in [4]. In Section 2 we prove several general estimates and other results that will be needed. The third section is devoted to the study of determinants in algebras of the form $\bigoplus_{t=1}^{s} C(X_t) \otimes M_{n(t)}$. In the case of real rank 1, unlike the case of real rank 0, the possibility of nontrivial determinants causes major technical difficulties. In Section 4 we prove some lemmas on projections and unitaries in the algebras we consider. Section 5, much the longest section, is the technical heart of our argument; in it we show how to split off from a unitary a large portion which is an exponential, while keeping the determinant under control. In Section 6 we then put together all the pieces and prove our main theorem. We also state various open problems. Finally, in Section 7 we discuss other related invariants: Banach exponential rank, C^* exponential length, and C^* projective length. We prove that, if A is a simple C^*-algebra of the sort considered in our main theorem, and $K_1(A) \neq 0$, then A has Banach exponential rank 2.

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1. **Preliminaries.**

For ease of terminology, we make the following definition to describe the class of algebras we consider. The term "no dimension growth" is derived from the term "slow dimension growth" used in [4].

DEFINITION 1.1. We say that a unital C^*-algebra A has *no dimension growth* if it is infinite dimensional and can be written as

$$A = \varinjlim A_i \quad \text{with} \quad A_i = \bigoplus_{t=1}^{s(i)} C(X_{it}) \otimes M_{n(i,t)},$$

where the X_{it} are compact metric spaces such that $\sup_{i,t} \dim(X_{it}) < \infty$. Dimension is taken to be any of the three usual dimensions; they are the same on compact metric spaces by Theorem 1.7.7 of [10].

The effect of requiring the X_{it} to be metric is to ensure that A is separable. We include metrizability in our definition in order to avoid some technicalities later. Similarly, we require A to be unital to avoid technicalities. The condition that A be infinite dimensional is included to exclude M_n from the collection of simple C^*-algebras with no dimension growth. We do this to avoid specifically excluding M_n from various lemmas; of course, the main result of this paper is trivially true for M_n.

REMARK 1.2. In Definition 1.1 we may assume without loss of generality that all maps in the direct system are unital injective.

To make them unital, we note that the unit of A must be in some A_i, and we delete the terms before A_i. To make them injective, we replace each $C(X_{it}) \otimes M_{n(i,t)}$ by its image in A. This change either eliminates the algebra, or replaces X_{it} by a closed subspace Y_{it}. In the second case, we have $\dim(Y_{it}) \leq \dim(X_{it})$ by Theorem 1.1.2 of [10].

We now fix some notation for C^*-algebras with no dimension growth, which will be used throughout this paper.

NOTATION 1.3. Let A have no dimension growth. We choose a representation

$$A = \varinjlim A_i \quad \text{with} \quad A_i = \bigoplus_{t=1}^{s(i)} C(X_{it}) \otimes M_{n(i,t)},$$

as in Definition 1.1, where the maps $A_i \longrightarrow A_{i+1}$ are unital and injective as in Remark 1.2. The $s(i)$ and $n(i,t)$ are all positive integers. We let X_i be the primitive ideal space of A_i,

$$X_i = \coprod_{t=1}^{s(i)} X_{it}.$$

If $a \in A_i$, then we also regard a as an element of A_j for $j \geq i$. In particular, expressions such as $a(x)$ for $x \in X_j$, and $a|_{X_{jt}} \in C(X_{jt}) \otimes M_{n(j,t)}$, will be given the obvious meanings. We similarly give the obvious meaning to $\operatorname{rank}\bigl(p(x)\bigr)$, where $p \in A_i$ is a projection and $x \in X_j$ for some $j \geq i$.

We also let

$$d = \sup_i \sup_{1 \leq t \leq s(i)} \dim(X_{it}) < \infty,$$

and we let m be the least positive integer satisfying $m^2 - 1 > d$.

We finish this section with two lemmas on $C(X) \otimes M_n$. The first is a slight reformulation of Proposition 3.2 of [8], which we state here for convenience. We refer to [8] for the proof.

LEMMA 1.4. *Let X be a compact metric space of dimension at most d, let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X, and for each α let $p_\alpha \colon U_\alpha \longrightarrow M_n$ be a continuous projection valued map such that* $\operatorname{rank}\bigl(p_\alpha(x)\bigr) \geq k + (d-1)/2$ *for all $x \in U_\alpha$. Then there exists a continuous projection valued map $p \colon X \longrightarrow M_n$ such that* $\operatorname{rank}\bigl(p(x)\bigr) \geq k$ *and* $p(x)\mathbb{C}^n \subset \operatorname{span}\{p_\alpha(x)\mathbb{C}^n : x \in U_\alpha\}$ *for all $x \in X$.*

LEMMA 1.5. *Let X be a compact metric space of dimension at most d, let $m \in \mathbb{N}$ satisfy $m^2 - 1 > d$, let $p \in C(X) \otimes M_n$ be a projection, and let $u \in p[C(X) \otimes M_n]p$ be* *unitary. Then for every $\varepsilon > 0$ there is a unitary $v \in p[C(X) \otimes M_n]p$ such that $\|u - v\| < \varepsilon$ and, for every $x \in X$, the (nonzero) eigenvalues of $v(x)$ have multiplicity at most $m - 1$.*

In this sort of context, we will usually regard v as an element of $p[C(X) \otimes M_n]p$ rather than $C(X) \otimes M_n$, so that we omit 0 from the eigenvalues of $v(x)$.

PROOF OF LEMMA 1.5. Using Theorem 1.13.5 of [10], write $X = \varprojlim X_i$, where the X_i are compact polyhedrons (finite simplicial complexes) of dimension at most d. Then $C(X) \otimes M_n = \varinjlim C(X_i) \otimes M_n$. Therefore there is i, a projection $p_0 \in C(X_i) \otimes M_n$, and unitaries $z \in C(X) \otimes M_n$ and $u_0 \in p_0[C(X_i) \otimes M_n]p_0$ such that, with $\varphi\colon C(X_i) \otimes M_n \longrightarrow C(X) \otimes M_n$ being the canonical map, we have $z\varphi(p_0)z^* = p$ and $\|z\varphi(u_0)z^* - u\| < \varepsilon/2$. If $v \in p_0[C(X_i) \otimes M_n]p_0$ has no eigenvalues of multiplicity at least m at any point of X_i, then $z\varphi(v)z^*$ has no such eigenvalues at any point of X. Therefore it suffices to approximate v on X_i.

We carry out the approximation first on the 0-skeleton, then on the 1-skeleton, *etc.*, finishing with the d-skeleton, using the method of proof of Lemma 2.5 of [20]. In place of the retraction given there, we use the map $a \longmapsto a(a^*a)^{-1/2}$, which for any closed $Y \subset X_i$ defines a retraction from a neighborhood of the unitary group U of $(p_0|_Y)[C(Y) \otimes M_n](p_0|_Y)$ onto U. The method of proof of Lemma 2.5 of [20] now reduces us to the problem of carrying out the required approximation on a single j-cell B without changing the value on the boundary, given that it has already been done on the boundary.

Since a j-cell is contractible, p_0 is trivial over B. Therefore we can work with $C(B) \otimes M_k$, where $k = \operatorname{rank}(p_0|_B)$, rather than with $(p_0|_B)[C(B) \otimes M_n](p_0|_B)$. Furthermore, there is a continuous function $\zeta\colon B \longrightarrow S^1$, where S^1 is the unit circle, such that $\det\bigl(\zeta(x)u_0(x)\bigr) = 1$ for $x \in B$. It is equivalent to perturb $\zeta u_0|_B$, and so we may assume $\det\bigl(u_0(x)\bigr) = 1$ for all $x \in B$. The proof can now be finished as in Step 3 of the proof of Theorem 3.3 in [22]. ∎

2. **Some general lemmas.** In this section we collect three results on general C^*-algebras and one result on M_n which we use at various places in later sections. Two of these are explicit forms of standard continuity results, and another (Lemma 2.3) is a simpler and more explicit version for M_n of Lemma 2.1 of [11]. Having explicit estimates helps keep down the level of complication of several already messy proofs in Sections 5 and 6. The last result shows that any selfadjoint element can be obtained by cutting down a selfadjoint element with finite spectrum in a matrix algebra in a suitable way. It substitutes for the condition of real rank 0 at one point in the proof of the main result, but it works over an arbitrary C^*-algebra.

LEMMA 2.1. *Let A be a unital C^*-algebra, and let $a, b \in A$ satisfy $\|a\|$, $\|b\| \leq M$. Then $\|\exp(a) - \exp(b)\| \leq e^M \|a - b\|$.*

PROOF. Following the proof of Lemma 2 of [7], we have

$$\|a^n - b^n\| \leq \sum_{k=1}^{n} \|a\|^{k-1} \|a - b\| \, \|b\|^{n-k} \leq nM^{n-1}.$$

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Therefore

$$\|\exp(a) - \exp(b)\| \leq \sum_{n=1}^{\infty} \frac{1}{n!}\|a^n - b^n\| \leq \sum_{n=1}^{\infty} \frac{1}{(n-1)!} M^{n-1} = e^M. \qquad \blacksquare$$

LEMMA 2.2. *Let A be a unital C^*-algebra, let $u \in A$ be unitary, and let $\|u-a\| < 1$. Then $v = a(a^*a)^{-1/2}$ is a unitary satisfying $\|u - v\| \leq 2\|u - a\|$.*

PROOF. See Lemma 3 of [7] and its proof. ■

LEMMA 2.3 (Compare [11], Lemma 2.1). *Let $u \in M_n$ be unitary, let $I_1, \ldots, I_k$ be disjoint arcs in S^1 of length at most $\varepsilon < 1/2$, let $\lambda_j \in I_j$, and let $p_1, \ldots, p_k \in M_n$ be projections such that each $p_j\mathbb{C}^n$ is contained in the linear span of the eigenvectors of u with eigenvalues in I_j. Let $p = 1 - \sum_j p_j$. Then $v = pup(pu^*pup)^{-1/2} + \sum_j \lambda_j p_j$ is a unitary such that $\|u - v\| \leq 4\varepsilon$. (The functional calculus is done in pM_np.)*

PROOF. Let q_j be the projection onto the linear span of the eigenvectors of u with eigenvalues in I_j, and let $q = 1 - \sum_j q_j$. Then each q_j commutes with u and $p_j \leq q_j$. Set $e_j = q_j - p_j$. Then $p = q + \sum_j e_j$. For $i \neq j$ we have

$$(e_iue_j)(e_iue_j)^* = e_iue_ju^*e_i \leq e_iuq_ju^*e_i = 0,$$

whence $e_iue_j = 0$. Also $e_iuq = 0$ for all i. Therefore

$$pup = quq + \sum_j e_jue_j.$$

If we let $a = pup + \sum_j \lambda_j p_j$, it follows that

$$\begin{aligned}
\|u - a\| &= \Big\|\sum_j (q_juq_j - e_jue_j - \lambda_jp_j)\Big\| \\
&= \sup_j \|q_juq_j - e_jue_j - \lambda_jp_j\| \\
&\leq \sup_j \big(\|q_juq_j - \lambda_jp_j\| + \|e_j(q_juq_j - \lambda_jp_j)e_j\|\big) \leq 2\varepsilon.
\end{aligned}$$

Therefore $a(a^*a)^{-1/2}$ is a unitary satisfying $\|a(a^*a)^{-1/2} - u\| \leq 4\varepsilon$ by Lemma 2.2, and it is easily checked that $a(a^*a)^{-1/2} = v$. ■

The following lemma is similar to results independently obtained by others; see for example Lemma 8(i) of [17].

LEMMA 2.4. *Let A be a unital C^*-algebra, let $a \in A$ be selfadjoint with spectrum contained in the interval $[\alpha, \beta]$, and let $n \in \mathbb{N}$. Let $b \in M_{n+1}(A)$ be the diagonal matrix*

$$b = \operatorname{diag}\big(\alpha, \alpha + (\beta - \alpha)/n, \alpha + 2(\beta - \alpha)/n, \ldots, \beta\big),$$

and let

$$a_0 = \operatorname{diag}(a, 0, \ldots, 0).$$

Then there exist a projection p and a unitary u in $M_{n+1}(A)$ such that $upu^* = \operatorname{diag}(1, 0, \ldots, 0)$, $\|pb - bp\| \leq (\beta - \alpha)/(2n)$, *and* $upbpu^* = a_0$.

PROOF. An obvious transformation enables us to assume $\alpha = 0$ and $\beta = 1$. Then

$$b = \operatorname{diag}(0, 1/n, 2/n, \ldots, 1),$$

and we want to have $\|pb - bp\| \leq 1/(2n)$. We further note that it suffices to find p and u in $M_{n+1}\big(C^*(a, 1)\big)$. Thus, we can assume $A = C(T)$, where T is some closed subset of $[0, 1]$, and where $a(t) = t$ for $t \in T$.

For $t \in [(\ell - 1)/n, \ell/n] \cap T$, let $\lambda = n\big(t - (\ell - 1)/n\big)$, which is in $[0, 1]$, and define

$$p(t) = \begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1-\lambda & \sqrt{\lambda(1-\lambda)} & & \\ & & & \sqrt{\lambda(1-\lambda)} & \lambda & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ 0 & & & & & & 0 \end{pmatrix},$$

where the nonzero entries are in the (i, j) positions for $i, j \in \{\ell, \ell+1\}$. It is easily checked that p is a continuous projection-valued function from T to M_{n+1}. Furthermore, computations show that

$$\begin{pmatrix} 1-\lambda & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & \lambda \end{pmatrix} \begin{pmatrix} (\ell-1)/n & 0 \\ 0 & \ell/n \end{pmatrix} \begin{pmatrix} 1-\lambda & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & \lambda \end{pmatrix} = \frac{(\ell - 1 + \lambda)}{n} \begin{pmatrix} 1-\lambda & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & \lambda \end{pmatrix}$$

and

$$\begin{aligned} &\begin{pmatrix} 1-\lambda & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & \lambda \end{pmatrix} \begin{pmatrix} (\ell-1)/n & 0 \\ 0 & \ell/n \end{pmatrix} \\ &\qquad - \begin{pmatrix} (\ell-1)/n & 0 \\ 0 & \ell/n \end{pmatrix} \begin{pmatrix} 1-\lambda & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & \lambda \end{pmatrix} \\ &\qquad = \frac{1}{n} \begin{pmatrix} 0 & \sqrt{\lambda(1-\lambda)} \\ \sqrt{\lambda(1-\lambda)} & 0 \end{pmatrix}. \end{aligned}$$

The first of these shows that $p(t)b(t)p(t) = tp(t)$, and the second shows that

$$\|p(t)b(t) - b(t)p(t)\| \leq \frac{1}{n} \sup_{\lambda \in [0,1]} \sqrt{\lambda(1-\lambda)} = \frac{1}{2n}.$$

Since $p(t)$ is a continuously varying rank 1 projection, it is easy to choose a continuously varying unitary $u(t)$ such that $u(t)p(t)u(t)^* = \operatorname{diag}(1, 0, \ldots, 0)$, and this unitary also satisfies $upbpu^* = a_0$. ∎

3. **Determinants.** For a unital C^*-algebra A, let $U(A)$ be its unitary group and let $U_0(A)$ be the connected component of $U(A)$ containing the identity. The C^* exponential length [28] $\mathrm{cel}(A)$ of A is in effect the rectifiable (path length) diameter of $U_0(A)$, while the C^* exponential rank $\mathrm{cer}(A)$ is (roughly) the smallest n such that every element of $U_0(A)$ is a product of n exponentials. (See [20] for details.) One has $\mathrm{cer}(A) \leq \frac{1}{\pi}\,\mathrm{cel}(A)$ [28]; on the other hand, $\mathrm{cer}\bigl(C([0,1])\bigr) = 1$ while $\mathrm{cel}\bigl(C([0,1])\bigr) = \infty$. The discrepancy is due to presence of unitaries with nontrivial determinant.

If A has real rank 0, then it is easy to see that $\mathrm{cel}(A) \leq \pi\,\mathrm{cer}(A)$, and this fact plays a key role in the proof of Theorem 2.6 of [11]. (It gives a finite bound on $\sup\{\mathrm{cel}(pAp) : p \in A$ is a projection$\}$ for the algebras A considered in [11].) In our case, however, if A has real rank 1, then $\mathrm{cel}(A) = \infty$. (See Theorem 7.5.) In order to control lengths of paths, we must keep track of determinants in the proof of our main lemma. This section develops the machinery needed to do that, and to obtain the benefits of having done so.

DEFINITION 3.1. Let X be a compact space, let $B = C(X) \otimes M_n$, and let $p \in B$ be a projection with constant rank k. For $u \in U(pBp)$ let $\det(u)$ be the function from X to S^1 whose value at x is $\det\bigl(u(x)\bigr)$, evaluated in $p(x)M_n p(x) \cong M_k$. Define

$$D(u) = \inf\bigl\{\,\|a\| : a \in pBp \text{ is selfadjoint and } \det\bigl(u\exp(ia)\bigr) = 1\bigr\}.$$

(Take $D(u) = \infty$ if no such a exists.) We denote these quantities by $\det_B(u)$ and $D_B(u)$ if the algebra B is ambiguous.

If B is an arbitrary finite direct sum of algebras of the form above, and p does not necessarily have constant rank, we write $B = \bigoplus_{t=1}^{s} C(X_t) \otimes M_{n(t)}$ in such a way that $p = (p_1, \ldots, p_s)$ and each p_t has constant rank. (We can always replace a space X by some closed and open subsets whose disjoint union is X and on which p has constant rank.) If $u = (u_1, \ldots, u_s) \in pBp$, we define $\det(u) = \bigl(\det(u_1), \ldots, \det(u_s)\bigr)$ and $D(u) = \sup_t D(u_t)$. Clearly $\det(u)$ and $D(u)$ do not depend on how the spaces are subdivided.

LEMMA 3.2. *Let $\alpha \mapsto u(\alpha)$ be a continuous unitary path in M_n. Then the path lengths $\ell(u)$ and $\ell(\det\circ u)$ satisfy $\ell(\det\circ u) \leq n\ell(u)$.*

PROOF. It suffices to show that the determinant, regarded as a function on the unitary group, has a derivative of norm at most n. We need only compute the derivative at 1. The tangent space at 1 is $i(M_n)_{\mathrm{sa}}$. So let $h \in (M_n)_{\mathrm{sa}}$; choosing an appropriate orthonormal basis, we may assume $h = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Let $a_j \in M_n$ agree with ih in the j-th row and with the identity elsewhere. Regarding det as a multilinear function of its rows, we then get

$$|D\det(1)(ih)| = \Bigl|\sum_{j=1}^{n}\det(a_j)\Bigr| \leq \sum_{j=1}^{n}|\lambda_j| \leq n\|ih\|,$$

that is, $\|D\det(1)\| \leq n$ as desired. ∎

LEMMA 3.3. *Let B and p be as in the second paragraph of Definition 3.1.*

(1) $D(u)$ is equal to the infimum of the lengths of paths $\alpha \mapsto u_\alpha$ in $U(pBp)$ with $u_0 = u$ and $\det(u_1) = 1$.

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(2) *If* $\operatorname{rank}\big(p(x)\big)$ *is a constant* k, *then*

$$D(u) = \inf\{k^{-1}\|\eta\| : \eta\colon X \longrightarrow \mathbb{R} \textit{ is continuous with } \exp(i\eta) = \det(u)\}.$$

(3) *If* $u \in U_0(pBp)$ *then* $D(u) < \infty$.

(4) *Let* $C = \bigoplus_{t=1}^{r} C(Y_t) \otimes M_{m(t)}$ *and let* $\varphi\colon B \longrightarrow C$ *be a homomorphism. If* $\det_B(u) = 1$ *then* $\det_C\big(\varphi(u)\big) = 1$.

(5) *Let* $\varphi\colon B \longrightarrow C$ *be as in (4). Then for any* $u \in U(pBp)$, *we have* $D\big(\varphi(u)\big) \leq D(u)$.

PROOF. (1) We may clearly reduce to the case in which B and p are as in the first paragraph of Definition 3.1. For one inequality, let $\det\big(u \exp(ia)\big) = 1$. Then $\alpha \longmapsto u_\alpha = u\exp(i\alpha a)$ is a path of length $\|a\|$ from $u_0 = u$ to u_1 which satisfies $\det(u_1) = 1$. For the other inequality, let $\alpha \longmapsto u_\alpha$ be a path as in (1). Let $\eta\colon [0,1] \times X \longrightarrow \mathbb{R}$ be a continuous function such that $\det\big(u_\alpha(x)\big) = \exp\big(i\eta(\alpha,x)\big)$ and $\eta(1,x) = 0$ for all x. For each fixed x, let ℓ_x be the length of the path $\alpha \longmapsto u_\alpha(x)$ in $p(x)M_n p(x) \cong M_k$. Then, using the previous lemma,

$$|\eta(0,x)| \leq \ell\Big(\exp\big(i\eta(-,x)\big)\Big) \leq k\ell_x \leq k\ell(u).$$

Therefore $a(x) = -k^{-1}\eta(0,x)p(x)$ is a selfadjoint element of pBp with $\|a\| \leq \ell(u)$ and $\det\big(u\exp(ia)\big) = 1$. This proves (1).

(2) If $\det\big(u\exp(ia)\big) = 1$ then $\eta(x) = -\operatorname{tr}\big(a(x)\big)$ satisfies $\exp(i\eta) = \det\big(\exp(ia)\big)^{-1} = \det(u)$, and $|\eta(x)| \leq k\|a(x)\|$. This shows that $D(u)$ is greater than or equal to the infimum in (2). The reverse inequality is obtained by observing that if $\exp(i\eta) = \det(u)$, then $a = -k^{-1}\eta \cdot 1$ satisfies $\|a\| = k^{-1}\|\eta\|$ and $\det\big(u\exp(ia)\big) = 1$.

(3) If $u \in U_0(pBp)$ then by [28], Proposition 2.11, there is a path of finite length in $U(pBp)$ connecting u to p, the identity of pBp. Since $\det(p) = 1$, the result follows from (1).

(4) The condition $\det_C\big(\varphi(u)\big) = 1$ can be checked at each point of $\coprod Y_t$ separately. Thus, it suffices to verify it for the composition of φ with each point evaluation. This reduces us to the case $C = M_m$.

The kernel of φ has the form $\bigoplus_{t=1}^{s} C_0(U_t) \otimes M_{n(t)}$ for open subsets $U_t \subset X_t$. Hence it is clear that the image $\bar{u}$ of u in $B/\operatorname{Ker}(\varphi)$ satisfies $\det(\bar{u}) = 1$. (Of course, this determinant is relative to the subalgebra $\bar{p}[B/\operatorname{Ker}(\varphi)]\bar{p}$, where $\bar{p}$ is the image of p.) We may therefore replace B by $B/\operatorname{Ker}(\varphi)$. But now B is finite dimensional, hence a direct sum of matrix algebras. Further replacing B by pBp and C by $\varphi(p)C\varphi(p)$, we reduce to the case in which φ is a unital homomorphism from $\bigoplus_{t=1}^{s} M_{n(t)}$ to M_m.

In this situation, we have $u = (u_1, \ldots, u_s)$ with $\det(u_t) = 1$ for each t, using the usual determinant. Furthermore, $\varphi(u)$ is unitarily equivalent to a direct sum of the u_t, with various multiplicities, and so also has determinant 1. This proves (4).

(5) This follows from (1) and (4), since a homomorphism cannot increase the lengths of paths. ∎

Recall from [28] that for a C^*-algebra A and $u \in U_0(A)$, the C^*-exponential length $\operatorname{cel}(u)$ to equal to the infimum of the lengths of paths in $U(A)$ from u to 1.

LEMMA 3.4. *For each integer $d \geq 0$, there is an integer $K(d)$ with the following property. Let X be a compact metric space of dimension at most d, let $B = C(X) \otimes M_n$, let $p \in B$ be a projection with constant rank $k \geq K(d)$, and let $u \in U_0(pBp)$. Then* $\mathrm{cel}(u) \leq D(u) + 6\pi$. *(Both* $\mathrm{cel}(u)$ *and $D(u)$ are taken in pBp.)*

PROOF. Theorem 4.5 of [22] states that there is for each $\varepsilon > 0$ a number $M_0(d, \varepsilon)$ such that whenever $k \geq M_0(d, \varepsilon)$, E is a k-dimensional hermitian vector bundle over a compact metric space X of dimension at most d, and $u \in U_0\Big(\Gamma\big(L(E)\big)\Big)$, the there are selfadjoint $h_1, h_2, h_3 \in \Gamma\big(L(E)\big)$ such that $\|u - \exp(ih_1)\exp(ih_2)\exp(ih_3)\| < \varepsilon$. Here $L(E)$ is the bundle with fiber $L(E)_x = L(E_x)$, the endomorphisms of the fiber E_x, and $\Gamma\big(L(E)\big)$ is the C^*-algebra of sections of E. (If $E_x = p(x)\mathbb{C}^n$, then $\Gamma\big(L(E)\big) = pBp$.) We choose $K(d)$ to be the number $M_0(d, 1)$ used in the proof of this theorem.

If E is such a bundle, let $\mathrm{SU}(E)$ denote the bundle whose fiber $\mathrm{SU}(E)_x$ is $\big\{u \in U\big(L(E_x)\big) : \det(u) = 1\big\}$, and let $\Gamma_0\big(\mathrm{SU}(E)\big)$ denote the set of homotopically trivial sections of $\mathrm{SU}(E)$. Note that $\Gamma_0\big(\mathrm{SU}(E)\big) \subset U_0\Big(\Gamma\big(L(E)\big)\Big) \cap \Gamma\big(\mathrm{SU}(E)\big)$, but it is not clear that we have equality here. An examination of the proof of Theorem 4.5 of [22] shows that if $\dim(E) \geq K(d)$ and $u \in \Gamma_0\big(\mathrm{SU}(E)\big)$, then the selfadjoint elements h_1, h_2, h_3 actually constructed there (satisfying $\|u - \exp(ih_1)\exp(ih_2)\exp(ih_3)\| < 1$) in addition satisfy

$$(*) \qquad \|h_1\| \leq 2\pi \quad \text{and} \quad \|h_2,\|, \|h_3\| \leq \pi.$$

(One must go back to the proof of Theorem 3.3 of [22] for all three of these, and further back to the proof of Corollary 5 of [21] to see that in the approximation $\|a \oplus a^* - \exp(ih)\| < \varepsilon$, we may take $\|h\| \leq \pi$.) We can then write

$$u = \exp(ih_1)\exp(ih_2)\exp(ih_3)\exp(ih_4)$$

with

$$(**) \qquad \|ih_4\| \leq 2\arcsin(1/2) < \pi.$$

Now let $p \in B = C(X) \otimes M_n$ have rank $k \geq K(d)$, assume $\dim(X) \leq d$, and let $v \in U_0(pBp)$. Let $\varepsilon > 0$. By definition there is selfadjoint $a \in pBp$ with

$$(***) \qquad \|a\| \leq D(u) + \varepsilon$$

and $\det\big(v\exp(ia)\big) = 1$. We obviously still have $v\exp(ia) \in U_0(pBp)$, so there is a continuous path $t \longmapsto w_t$ in $U_0(pBp)$ with $w_0 = 1$ and $w_1 = v\exp(ia)$. Choose a continuous function η: $[0, 1] \times X \longrightarrow \mathbb{R}$ such that $\det\big(w_t(x)\big) = \exp\big(i\eta(t, x)\big)$ and $\eta(0, x) = 0$ for all x. Since $\det\big(w_1(x)\big) = 1$ for all x, we have $\eta(1, x) \in 2\pi\mathbb{Z}$ for all $x \in X$. Partitioning X, we assume without loss of generality that $\eta(1, x)$ is constant, equal, say to $2\pi m$.

Define $u = v\exp(ia)\exp(-2\pi i m/k)$. Then the path $t \longmapsto u_t$, defined by

$$u_t(x) = v(x)\exp\big(ia(x)\big)\exp\big(-\eta(x)/k\big),$$

is a path in $\Gamma\big(\mathrm{SU}(E)\big)$, where $E = p(X \times \mathbb{C}^n)$, from $u_0 = 1$ to $u_1 = u$. Therefore we may write u as a product of four exponentials as above. So

$$v = \exp(ih_1)\exp(ih_2)\exp(ih_3)\exp(ih_4)\exp(-2\pi im/k)\exp(-ia),$$

where $\|h_j\|$ and $\|a\|$ are estimated as in $(*)$, $(**)$, and $(***)$. We can certainly write $\exp(-2\pi im/k) = \exp(i\lambda)$ with $|\lambda| \leq \pi$, and so we get (using [28])

$$\mathrm{cel}(v) \leq \|h_1\| + \|h_2\| + \|h_3\| + \|h_4\| + |\lambda| + \|-a\| \leq 6\pi + D(u) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the lemma is proved. ∎

4. **Projections and unitaries.** In this section we assemble some results on the rank, comparison, and subdivision of projections in simple C^*-algebras with no dimension growth. We also give what amounts to a cancellation theorem for unitaries. These results are mostly folklore, but we have not found suitable statements and proofs in the literature.

The closest thing to our first lemma that we have found in the literature is Lemma 4.1 of [13]. (Lemma F of [3] is also related, but does not address the key point here.)

LEMMA 4.1. *Let A be a simple unital direct limit as in Definition 1.1, except that we make no assumptions on the dimensions of the spaces. Let $p \in A_{i_0}$ be a nonzero projection. Then, using Notation 1.3, we have*

$$(*) \qquad \lim_{i\to\infty} \inf_{x\in X_i} \mathrm{rank}\big(p(x)\big) = \infty.$$

PROOF. Suppose the conclusion fails. We follow Notation 1.3. Since the maps of the direct system are injective, the limit in $(*)$ is equal to the supremum. Let k be this supremum. Then in fact $\mathrm{rank}\big(p(x)\big) \leq k$ for all $i \geq i_0$ and all $x \in X_i$. To see this, suppose we had $\mathrm{rank}\big(p(x)\big) > k$ for some i and some $x \in X_i$. Then there is a closed and open subset $Y \subset X_i$ such that $\mathrm{rank}(p) > k$ on Y. Define $p_0 \in A_i$ by

$$p_0(x) = \begin{cases} p(x) & x \in Y \\ 0 & x \notin Y \end{cases}.$$

By Proposition 2.1 of [8] there is $j \geq i$ such that $p_0(x) \neq 0$ for all $x \in X_j$. Then $\mathrm{rank}\big(p(x)\big) \geq \mathrm{rank}\big(p_0(x)\big) > k$ for all $x \in X_j$, because for each $x \in X_j$ the rank of $p_0(x)$ must be of the form $\sum_{\ell=1}^m r_\ell \,\mathrm{rank}\big(p_0(x_\ell)\big)$ for integers $r_\ell \geq 0$ and points $x_\ell \in X_i$. This contradicts the definition of k.

We conclude that $\mathrm{rank}(p) \leq k$ in every A_i for $i \geq i_0$. The hereditary subalgebra pAp is simple, and stably isomorphic to A by Theorem 2.8 of [6]. However, the polynomial identity argument from the proof of Lemma 4.1 of [13] shows that pAp is finite dimensional. Since A is infinite dimensional and not isomorphic to K, this is a contradiction. ∎

We note that an alternate, more C^*-algebraic, way to show that pAp is finite dimensional is to construct a nonzero homomorphism from pAp to M_k. This is not too difficult, using Proposition 2.1 of [8] as in the first part of our proof to find j_0 such that for all $j \geq j_0$ and $x \in X_j$, we have $\operatorname{rank}\bigl(p(x)\bigr) = k$.

We also note that Lemma 4.1 shows that simple C^*-algebras with no dimension growth in our sense do in fact have slow dimension growth in the sense of [4]. (This is not true without simplicity, since $C(X)$ satisfies our definition of no dimension growth whenever X is a finite dimensional compact metric space with infinitely many points.)

LEMMA 4.2. *Let $\varphi\colon C(X) \otimes M_n \longrightarrow \bigoplus_{t=1}^{r} C(Y_t) \otimes M_{m(t)}$ be a homomorphism, and let $p, q \in C(X) \otimes M_n$ be nonzero projections with constant ranks. Then for each t and each $y \in Y_t$, we have*

$$\varphi(p)(y) = \varphi(q)(y) = 0 \quad or \quad \frac{\operatorname{rank}\bigl(\varphi(p)(y)\bigr)}{\operatorname{rank}\bigl(\varphi(q)(y)\bigr)} = \frac{\operatorname{rank}(p)}{\operatorname{rank}(q)}.$$

PROOF. In the same manner as in the proof of part (4) of Lemma 3.3, we reduce to the case X finite, φ unital, $t = 1$, and $Y_1 = \{y\}$. Let $X = \{x_1, \ldots, x_k\}$. Then φ is essentially a map $\bigoplus_{i=1}^{k} M_n \longrightarrow M_m$. It is determined up to unitary equivalence by the partial multiplicities r_i $(1 \leq i \leq k)$ with which each copy of M_n is embedded in M_m. We then have

$$\operatorname{rank}\bigl(\varphi(p)\bigr) = \sum_{1}^{k} r_i \operatorname{rank}\bigl(p(x_i)\bigr) = \operatorname{rank}(p) \sum_{i=1}^{k} r_i,$$

and similarly for q in place of p. The result follows. ∎

The next two results are also true for simple direct limit C^*-algebras with slow dimension growth. One substitutes Lemma F of [4] for our Lemma 4.1 in the proofs. They are also both presumably known; certainly, their proofs are completely standard. However, the most closely related result we have found is the rather more complicated Lemma 1.7 of [29].

LEMMA 4.3. *Let A be simple with no dimension growth. Let $e, p \in A_0$ be projections with*

$$N \operatorname{rank}\bigl(e(x)\bigr) < \operatorname{rank}\bigl(p(x)\bigr)$$

for all $x \in X_0$. Then there exist i and N orthogonal projections $f_1, \ldots, f_N \leq p$ in A_i, all Murray-von Neumann equivalent in A_i to e.

PROOF. By partitioning the spaces X_{0t} into closed and open sets on which $\operatorname{rank}(e)$ and $\operatorname{rank}(p)$ are constant, we may assume these ranks are constant on each X_{0t}. Then $A_0 = \bigoplus_{t=1}^{s(0)} C(X_{0t}) \otimes M_{n(0,t)}$, and with respect to this direct sum decomposition we can write $p = (p_1, \ldots, p_{s(0)})$ and $e = (e_1, \ldots, e_{s(0)})$. It suffices to find, for each fixed t, an integer i and N orthogonal projections $f_{t1}, \ldots, f_{tN} \leq p_t$ in A_i, each of which is Murray-von Neumann equivalent to e_t.

Let d be the dimension bound as in Notation 1.3. Let $\alpha = \frac{\operatorname{rank}(p_t(x))}{N \operatorname{rank}(e_t(x))}$, for $x \in X_{0t}$; note that α is constant and $\alpha > 1$. By Lemma 4.1 there is i such that

$$(*) \qquad \alpha - \frac{d}{2 \operatorname{rank}\bigl(e_t(x)\bigr)} > 1$$

for all r and $x \in X_{ir}$. As in the previous paragraph, we may assume that $\mathrm{rank}(e_t)$ and $\mathrm{rank}(p_t)$ are constant on each X_{ir}. In A_i we write $p_t = (p_{t1}, \ldots, p_{t,s(i)})$ and $e_t = (e_{t1}, \ldots, e_{t,s(i)})$. By the previous lemma, $\mathrm{rank}(p_{tr}) = N\alpha \,\mathrm{rank}(e_{tr})$, so that $(*)$ implies

$$\mathrm{rank}(p_{tr}) \geq N \,\mathrm{rank}(e_{tr}) + d/2.$$

Now apply Theorem 2.5(c) of [13] a total of N times to obtain N projections f_{trj} ($j = 1, \ldots, N$) with f_{trj} Murray-von Neumann equivalent to e_{tr} and

$$f_{trj} \leq p_{tr} - \sum_{\ell=1}^{j-1} f_{tr\ell}.$$

Take $f_{tj} = \sum_{r=1}^{s(i)} f_{trj}$. ∎

LEMMA 4.4. *Let A be simple with no dimension growth, and let $A = \varinjlim A_i$ as in Notation 1.3.*

(1) If $p_1, \ldots, p_k$ are nonzero projections in A_{i_0} for some i_0, then there exist $i \geq i_0$ and nonzero projections $q, q_1, \ldots, q_k \in A_i$ such that q is Murray-von Neumann equivalent to each q_j and $q_j \leq p_j$.

(2) If p is a nonzero projection in A_{i_0} for some i_0, and $k \geq 1$, then there exist $i \geq i_0$ and orthogonal nonzero projections $q_1, \ldots, q_k \in A_i$ which are all mutually Murray-von Neumann equivalent and satisfy $q_j \leq p$.

PROOF. Without loss of generality take $i_0 = 0$ in both parts. Also let d be the dimension bound as in Notation 1.3, and let d_0 be an integer with $d_0 \geq (d-1)/2$.

(1) Using Lemma 4.1, choose i such that $\inf_{x \in X_i} \mathrm{rank}\big(p(x)\big) \geq d_0 + 1$ for all k. Let q be a constant rank one projection on each X_{it}. Then $q|_{X_{it}}$ is Murray-von Neumann equivalent in $C(X_{it}) \otimes M_{n(i,t)}$ to a subprojection q_{kt} of p_k by Theorem 2.6(a) of [13] (which is the version for compact spaces of Theorem 8.1.2 of [15]). Take $q_k = (q_{k,1}, \ldots, q_{k,s(i)})$.

(2) Using Lemma 4.1, choose i such that $\inf_{x \in X_i} \mathrm{rank}\big(p(x)\big) \geq d_0 + k$. On each X_{it}, Theorem 2.6(a) of [13] shows that there is a subprojection of $p|_{X_{it}}$ which is Murray-von Neumann equivalent to a trivial projection e of rank k. Obviously e is the sum of k nonzero orthogonal Murray-von Neumann equivalent subprojections. Combining the results over the X_{it} for $1 \leq t \leq s(i)$ gives the desired conclusion. ∎

The pieces of the proof of the following lemma and its corollary are scattered over several places in [11]. Here we put them all together in the same place. The lemma is still true without separability. We don't prove this because we don't need it and the proof is longer.

LEMMA 4.5. *Let A be a simple separable C^*-algebra with $\mathrm{tsr}(A) = 1$, let $p \in A$ be a projection, and let $u = U(pAp)$ and $v \in U_0\big((1-p)A(1-p)\big)$. If $u + v \in U_0(A)$, then $u \in U_0(pAp)$.*

PROOF. Since A is simple, pAp is a full hereditary subalgebra. Therefore $K_1(pAp) \longrightarrow K_1(A)$ is an isomorphism, by Proposition 1.2 of [19]. Consequently $[u] = 0$ in $K_1(pAp)$.

Since A is separable and pAp is a full hereditary subalgebra, Theorem 2.8 of [6] implies that $K \otimes A \cong K \otimes pAp$. Theorem 3.6 of [26] therefore implies $\mathrm{tsr}(pAp) = 1$. Now $[u] = 0$ in $K_1(pAp)$ and Theorem 2.10 of [27] imply $u \in U_0(pAp)$. (We use the fact that the inclusion of the unitary group in the invertible group is a homotopy equivalence.) ∎

COROLLARY 4.6. *Lemma 4.5 holds for any simple C^*-algebra with no dimension growth.*

PROOF. Such an algebra A satisfies $\mathrm{tsr}(A) = 1$ by [8]. ∎

5. **Splitting off exponentials.** In this section, as in Section 3, we let $U(A)$ denote the unitary group of a C^*-algebra, and we let $U_0(A)$ denote its identity component. Let A be simple with no dimension growth, and let $u \in U(A)$. In this section, we show how to split off, as approximate direct summands, a piece with given finite spectrum (but not necessarily a large summand) and a very large summand which, while not necessarily having finite spectrum, is at least an exponential. The two lemmas (Lemmas 5.2 and 5.3) are counterparts of steps in the proof of Theorem 2.6 of [11]. However, we measure size by rank in matrix algebras rather than by values of traces. This change frees us from the assumption that there are only finitely many extreme traces. It also plays a major role in avoiding the assumption of real rank 0, since we are able to use the selection result from [8] (Lemma 1.4 of this paper) instead.

The first lemma is a necessary preliminary result on the distribution of eigenvalues.

LEMMA 5.1. *Let A be simple with no dimension growth. Using Notation 1.3, let $p \in A_j$ be a projection, let $u \in pA_jp$ be unitary, and let $\varepsilon > 0$. Assume that every arc in the unit circle S^1 of length at least $\varepsilon/2$ has nonempty intersection with* $\mathrm{sp}(u)$. *Then there exist $i_0 \in \mathbb{N}$ and $\alpha > 0$ such that for any $i \geq i_0$, $1 \leq t \leq s(i)$, and $x \in X_{it}$, every closed arc in S^1 of length ε contains at least $\alpha \cdot n(i,t)$ eigenvalues of $u(x)$, including multiplicity.*

PROOF. Without loss of generality we may assume $j = 0$. Let $f\colon S^1 \longrightarrow [0,1]$ be a continuous function whose support is contained in the arc $\exp(i[0,\varepsilon])$ and such that $f(\zeta) = 1$ for $\zeta \in \exp(i[\varepsilon/4, 3\varepsilon/4])$. Let $T(A)$ be the set of (normalized) traces on A, with the weak* topology. Define

$$\alpha = \frac{1}{2}\inf\bigl\{\tau\bigl(f(\zeta u)\bigr) : \tau \in T(A), \zeta \in S^1\bigr\}.$$

(Functional calculus is evaluated in pAp.)

We show that $\alpha > 0$. Note first that $f(\zeta u)$ is positive and never 0, by the condition on $\mathrm{sp}(u)$ and the choice of f. Since A is simple, all traces are faithful, whence $\tau\bigl(f(\zeta u)\bigr) > 0$. Next, $(a,\tau) \longmapsto \tau(a)$ is jointly continuous on $A \times T(A)$: if $\tau_\lambda \longrightarrow \tau$ weak* and $a_\lambda \longrightarrow a$, then

$$|\tau_\lambda(a_\lambda) - \tau(a)| \leq \|a_\lambda - a\| + |\tau_\lambda(a) - \tau(a)| \longrightarrow 0.$$

Finally, $T(A) \times S^1$ is compact by Alaoglu's Theorem. Therefore $\alpha > 0$, as desired.

Now suppose the lemma is false with this choice of α. Discarding some of the terms in the direct limit, we may suppose that for every i there are $t(i)$ with $1 \leq t(i) \leq s(i)$,

$\zeta_i \in S^1$, and $x_i \in X_{i,t(i)}$, such that $u(x_i)$ has less than $\alpha \cdot n\big(i, t(i)\big)$ eigenvalues in the arc from ζ_i^{-1} to $\zeta_i^{-1} \cdot \exp(i\varepsilon)$. It follows that the usual trace on $M_{n(i,t(i))}$ satisfies $\mathrm{tr}\Big(f\big(\zeta_i u(x_i)\big)\Big) \leq \alpha \cdot n\big(i, t(i)\big)$.

For each i define a normalized trace τ_i on A_i by $\tau_i(a) = n\big(i, t(i)\big)^{-1} \mathrm{tr}\big(a(x_i)\big)$. Use the Hahn-Banach Theorem to find a functional σ_i on A such that $\|\sigma_i\| = 1$ and $\sigma_i|_{A_i} = \tau_i$. Passing again to a subsequence, we may assume $\zeta_i \longrightarrow \zeta$ and $\sigma_i \longrightarrow \tau$ weak* for some $\zeta \in S^1$ and some τ in the dual A^*. (Note that the unit ball of A^* is weak* metrizable, since A, being a direct limit of separable C^*-algebras, is separable.) Then τ is a normalized trace on $\bigcup_{i=1}^{\infty} A_i$, and therefore by continuity on A.

Using the same joint continuity argument as above, and the convergence $f(\zeta_i u) \longrightarrow f(\zeta u)$, we get

$$\tau\big(f(\zeta u)\big) = \lim_{i \longrightarrow \infty} \tau_i\big(f(\zeta_i u)\big) \leq \alpha.$$

This contradicts the choice of α. ∎

LEMMA 5.2. *Let A be simple with no dimension growth. Using Notation 1.3, let $u \in A_{i_0}$ be unitary with $\mathrm{sp}(u) = S^1$, let $\lambda_1, \ldots, \lambda_k \in S^1$, and let $\varepsilon > 0$. Then there exist $i \geq i_0$, nonzero orthogonal projections $p_1, \ldots, p_k \in A_i$, and a unitary $v \in (1 - \sum_{j=1}^k p_j) A_i (1 - \sum_{j=1}^k p_j)$, such that*

$$\Big\| u - \Big(v + \sum_{j=1}^{k} \lambda_j p_j\Big) \Big\| \leq \varepsilon.$$

PROOF. Without loss of generality we may assume that $\varepsilon/4 < 1/2$, and that no closed arc of length $\varepsilon/2$ contains more than one of the λ_j. For each j choose a closed arc I_j of length $\varepsilon/6$ containing λ_j and an open arc U_j of length $\varepsilon/4$ containing I_j. Note that the U_j are disjoint.

Since $\mathrm{sp}(u) = S^1$, every arc of length at least $\varepsilon/12$ must have nonempty intersection with $\mathrm{sp}(u)$. Using the previous lemma, choose $i_1 \geq i_0$ and $\alpha > 0$ such that for $i \geq i_1$ and $x \in X_{it}$, each I_j contains at least $\alpha n(i,t)$ eigenvalues of $u(x)$. Now use Lemma 4.1 (taking $p = 1$) to choose a fixed $i \geq i_1$ such that $\alpha n(i,t) \geq 1 + (d-1)/2$ for all t.

Temporarily fix t and j. For $x \in X_{it}$, let J_x be an open arc with $I_j \subset J_x \subset U_j$ whose endpoints are not eigenvalues of $u(x)$. Let V_x be an open neighborhood of x such that the endpoints of J_x are not eigenvalues of $u(y)$ for $y \in V_x$. Let χ_{J_x} be the characteristic function of J_x. Then using continuous functional calculus we can define a continuous projection valued function $q_x \colon V_x \longrightarrow M_{n(i,t)}$ by $q_x(y) = \chi_{J_x}\big(u_0(y)\big)$. Reducing the size of V_x if necessary, we may assume $\mathrm{rank}\big(q_x(y)\big) = \mathrm{rank}\big(q_x(x)\big)$ for $y \in V_x$. By the previous paragraph, this rank is at least $1 + (d-1)/2$. Apply Lemma 1.4 to obtain a continuous projection valued function $q \colon X_{it} \longrightarrow M_{n(i,t)}$ such that $q(x)\mathbb{C}^{n(i,t)} \subset \mathrm{span}\{q_y(x)\mathbb{C}^{n(i,t)} : x \in V_y\}$ and $\mathrm{rank}\big(q(x)\big) \geq 1$ for all x. Call this function e_{tj}.

For each t and $x \in X_{it}$, the unitary $u(x) \in M_{n(i,t)}$, the arcs U_j, the numbers λ_j, and the projections $e_{tj}(x)$ satisfy the hypotheses of Lemma 2.3, using $\varepsilon/4$ in place of ε. Therefore, with $e_t(x) = 1 - \sum_j e_{tj}(x)$ and

$$w_t(x) = e_t(x)u(x)e_t(x)\big(e_t(x)u(x)^* e_t(x)u(x)e_t(x)\big)^{-1/2} + \sum_j \lambda_j e_{tj}(x),$$

we get that $w_t(x)$ is a unitary satisfying $\|w_t(x) - u(x)\| \leq \varepsilon$. Set $w = (w_1, \ldots, w_{s(i)})$ and $p_j = (p_{1j}, \ldots, p_{s(i)j})$. Then the p_j are nonzero projections in A_i, and w is a unitary of the form $v + \sum_j \lambda_j p_j$ such that $\|u - w\| \leq \varepsilon$. ∎

LEMMA 5.3. *Let A be simple with no dimension growth. Let $p, q \in A_0$ be projections, let $N \in \mathbb{N}$, let $u \in U_0(qA_0q)$, and let $\varepsilon > 0$. Then there exist i, a projection $e \leq q$ in A_i, unitaries $w \in eA_ie$ and $v \in (q-e)A_i(q-e)$, such that:*

(1) $\|(v + w) - u\| < \varepsilon$.

(2) v is an exponential.

(3) $D_{A_i}(w) \leq 2\big(D_{A_0}(u) + 2\pi + 2\big)$.

(4) There are N orthogonal projections $f_1, \ldots, f_N \leq p$ in A_i, each of which is Murray-von Neumann equivalent to e.

The proof of this lemma is long and complicated, but the basic idea behind it is reasonably simple, so we explain it before starting the proof. (We also note that it does not differ greatly from the previous lemma.) For the purposes of this explanation, we make the simplifying assumptions that $q = 1$, and that each A_i has the form $C(X_i) \otimes M_{n(i)}$, with X_i connected and $n(i) \longrightarrow \infty$ as $i \longrightarrow \infty$. The connectedness assumption implies that all projections have constant rank. By Lemma 4.3, we only need to make sure that $N\,\mathrm{rank}(e) < \mathrm{rank}(p)$. By Lemma 4.2, it suffices to show how to construct e, *etc.*, in each A_i in such a way that $\mathrm{rank}(e)/n(i) \longrightarrow 0$ as $i \longrightarrow \infty$.

Let $\rho = \varepsilon/25$. Use Lemma 5.1 to find a number α such that for large i, any $\alpha n(i)$ cyclically consecutive eigenvalues of each $u(x)$ are in an arc of length less than ρ. (The lemma doesn't apply if $\mathrm{sp}(u) \neq S^1$, but then we can take $v = u$ and $e = 0$.) Assume this is true for all i. Fix i. By Lemma 1.5, a small perturbation u_0 of u has no eigenvalues (at any $x \in X_i$) of multiplicity m or larger, and we still have any $\alpha n(i)$ cyclically consecutive eigenvalues of each $u(x)$ in an arc of length 3ρ.

Assume for simplicity that $1/\alpha$ is an integer L. Choose continuous functions $\alpha_\ell\colon X_i \longrightarrow [0, 2\pi]$, for $0 \leq \ell \leq L$, such that $\alpha_0(x) = 0$ and $\alpha_\ell(x) = 2\pi$ for all x, and there are approximately $\alpha n(i)$ eigenvalues of $u_0(x)$ in the open arc $C_{x\ell}$ from $\exp\big(i\alpha_{\lambda-1}(x)\big)$ to $\exp\big(i\alpha_\ell(x)\big)$. (Note: The essential difference from the real rank 0 case is that the functions α_ℓ are *not* constant.) It is possible to do this so that we are never more than $3m$ eigenvalues short, and so that $\alpha_\ell(x) - \alpha_{\ell-1}(x) \leq 6\rho$.

Use Lemma 1.4 to choose a continuous projection e_ℓ of rank at least $\alpha n(i) - 3m - d/2$ whose range is in the span of the eigenvectors with eigenvalues in $C_{x\ell}$. Then we take

$$e = 1 - \sum e_\ell \quad \text{and} \quad v(x) = \sum \exp\left(i\frac{\alpha_{\ell-1}(x) + \alpha_\ell(x)}{2}\right)e_\ell(x),$$

and we obtain $w(x)$ from Lemma 2.3. Our error is now $4 \cdot 6\rho < 25\rho$, and $\operatorname{rank}(e) \leq L(3m + d/2)$, which is a constant that does not depend on $n(i)$. So $\operatorname{rank}(e)/n(i)$ goes to 0 for large i.

If we did not have to control the determinant, this would be the entire proof. Unfortunately, this procedure gives no control over $D(w)$. The first improvement is not to take $\alpha_0(x) = 0$, but to choose $\alpha_0(x)$ in such a way as to force $\det(v)$ to be close to $\det(u)$. Since $\|u-(v+w)\|$ is small, we get $D(u)$ close to $D(v+w)$. This argument gives $D(w+1-e) < \varepsilon$. Unfortunately, it only gives $D(w) \leq \varepsilon n(i)/\operatorname{rank}(e)$. Since $n(i)/\operatorname{rank}(e)$ is large and making ε small forces $n(i)$ to get larger, this estimate is simply not good enough. The solution is roughly speaking as follows. We only need $\operatorname{rank}(e)/n(i)$ to be less than a previously given nonzero constant, namely $1/N$ times the ratio of the ranks of p and q (here, $q = 1$). If $n(i)$ is much larger than necessary, we can transfer some of the summands in v over to w, thus increasing $\operatorname{rank}(e)$. So we are able to assume $n(i)/\operatorname{rank}(e)$ is bounded by a large, but finite, constant. Now carry out the argument with a value of ε so small that $\varepsilon n(i)/\operatorname{rank}(e)$ is at most, say, 1.

In the estimate on $D(w)$, the term $2D(u)$ comes from the determinant of the summands in v that are transferred to w as above. The remaining parts are related to the fact that there is no reason for $n(i)/L$ to be an integer. When it is not, comparing $\det(u)$ and $\det(v)$ becomes rather messy.

PROOF OF LEMMA 5.3. To simplify the notation, we will write $D_i(a)$ for $D_{A_i}(a)$. Note that Lemma 3.3(5) implies, for any a,

(1) $$D_{i_1}(a) \leq D_{i_2}(a) \quad \text{for} \quad i_1 \geq i_2.$$

Write
$$A_0 = \bigoplus_{t=1}^{s(0)} C(X_{0t}) \otimes M_{n(0,t)}$$
in such a way that the components of both p and q in each summand have constant rank. Let g_t be the identity of $A_{0t} = C(X_{0t}) \otimes M_{n(0,t)}$, and set $p_t = g_t p g_t$, $q_t = g_t q g_t$, and $u_t = g_t u g_t$. Then it suffices to find e_t, w_t, and v_t for p_t, q_t, and u_t as in the statement of the lemma, all in some $A_{i(t)}$, for each t separately. We may therefore fix t, and for convenience simply take $p = p_t$, $q = q_t$, and $u = u_t$. This allows us to assume that $\operatorname{rank}(p)$ and $\operatorname{rank}(q)$ are constant.

Choose $\delta > 0$ such that

(2) $$\delta < \min\left(\frac{1}{2}, \frac{\operatorname{rank}(p)}{N \operatorname{rank}(q)}\right).$$

Choose $\rho > 0$ such that

(3) $$\rho < \min(1/13, \varepsilon/25)$$

and

(4) $$2\arcsin(25\rho/2) + 12\rho < \delta/4.$$

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If $\mathrm{sp}(u) \neq S^1$ then u has a logarithm of norm at most 2π, so we can take $e = 0$. Otherwise, Lemma 5.1 provides i_0 and an integer L such that for every $i \geq i_0$, $1 \leq t \leq s(i)$, and $x \in X_{it}$, any closed arc in S^1 of length ρ contains $L^{-1}n(i,t)$ eigenvalues of $u(x)$, counting multiplicity.

Let d and m be as in Notation 1.3. Choose an integer d_0 such that

$$d_0 \geq (d-1)/2. \tag{5}$$

Set

$$s = d_0 + 3m. \tag{6}$$

Choose a number T_0 such that $k \geq T_0$ implies

(7)

$$\left|1-\left(1-\frac{s}{kL^{-1}-1}\right)\left(1-\frac{s}{kL^{-1}}\right)^{-1}\right|, \left|1-\left(1-\frac{s}{kL^{-1}+1}\right)\left(1-\frac{s}{kL^{-1}}\right)^{-1}\right| < \frac{\rho}{D_0(u)+2\pi+1}.$$

Let T be an integer satisfying

$$T \geq \max\bigl(T_0, L(4m+1), 2\pi m/\rho, 2(Ls+1)/\delta\bigr). \tag{8}$$

Use Lemma 4.1 to choose $i_1 \geq i_0$ such that $\mathrm{rank}\bigl(q(x)\bigr) \geq T$ for all t and $x \in X_{i_1 t}$.

By partitioning the spaces, we may assume that $\mathrm{rank}(p)$ and $\mathrm{rank}(q)$ are constant on each $X_{i_1 t}$. Let g_t be the identity of $C(X_{i_1 t}) \otimes M_n(i_1, t)$. Then each g_t is a central projection, so $a \longmapsto g_t a g_t$ is a homomorphism. Lemma 3.3(5) and (1) yield

$$D_{i_1}(g_t u g_t) \leq D_{i_1}(u) \leq D_0(u).$$

It therefore suffices to find e_t, w_t, and v_t, corresponding as in the statement of the lemma to $g_t p g_t$, $g_t q g_t$, and $g_t u g_t$, and satisfying the estimate $D_i(v_t) \leq 2\bigl(D_{i_1}(g_t u g_t) + 2\pi + 2\bigr)$ in place of conclusion (3) of the lemma. Furthermore, the inequality (7) continues to hold using the smaller number $D_{i_1}(g_t u g_t)$ in place of $D_0(u)$, and, by Lemma 4.2, the inequality (2) continues to hold using $\mathrm{rank}(g_t p g_t)/\bigl(N\,\mathrm{rank}(g_t q g_t)\bigr)$ in place of $\mathrm{rank}(p)/\bigl(N\,\mathrm{rank}(q)\bigr)$. (We can obviously ignore the case $\mathrm{rank}(g_t q g_t) = 0$.)

We may therefore, without loss of generality, fix t and replace p, q, and u by $g_t p g_t$, $g_t q g_t$, and $g_t u g_t$. Thus $p, q \in B = C(X) \otimes M_n$, where $X = X_{i_1 t}$ and $n = n(i_1, t)$. We may furthermore assume $i_1 = 0$. Thus, in addition to the hypotheses of the lemma, we may assume that $\mathrm{rank}(p)$ is a constant, that $k = \mathrm{rank}(q)$ is a constant at least as large as the number T in (8), and that every closed arc in S^1 of length ρ contains at least n/L eigenvalues of each $u(x)$. In particular, the inequalities (3)–(7) all hold, as well as:

$$\delta < \min\left(\frac{1}{2}, \frac{\mathrm{rank}(p)}{N\,\mathrm{rank}(q)}\right). \tag{9}$$

$$k \geq 2\pi m/\rho. \tag{10}$$

$$k \geq 2(Ls+1)/\delta. \tag{11}$$

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From (9) and (11) we get:

(12) $$k \geq 2Ls.$$

Let r be the least integer satisfying $r \geq k/L$; then we get

(13) $$rL \geq k$$

and from $k \geq T \geq L(4m+1)$ in (8),

(14) $$r \geq 4m+1.$$

Finally, every closed arc of S^1 of length ρ contains at least r eigenvalues of each $u(x)$. (This follows because the number of eigenvalues is an integer and at least $nL^{-1} \geq kL^{-1}$.)

Use Lemma 3.3(2) to choose a continuous function $\eta: X \longrightarrow \mathbb{R}$ such that

(15) $$\exp(i\eta) = \det(u)$$

and

(16) $$\|\eta\| \leq k\big(D_0(u)+1\big).$$

By Lemma 1.5 there is a unitary $u_0 \in qBq$ such that $\|u - u_0\| < 2\sin(\rho/2)$ and no $u_0(x)$ has any eigenvalues of multiplicity greater than $m-1$. Let $x \in X$ and let I be any closed arc of length at least 3ρ. Then the central closed subarc J of length ρ must contain at least r eigenvalues of $u(x)$. By Theorem 13.6 of [2] (see Section 11 of [2] for the notation), $u_0(x)$ must have at least r eigenvalues within a distance $2\sin(\rho/2)$ of J, and thus in I. Therefore:

(17) For any $x \in X$, any r eigenvalues (counting multiplicity) of $u_0(x)$ which are consecutive in the cyclic order are contained in a closed arc in S^1 of length at most 3ρ.

For $\alpha \in \mathbb{R}$ let $\arg_\alpha: S^1 \longrightarrow [\alpha, \alpha+2\pi)$ be the branch of $-i\cdot\log$ with values in $[\alpha, \alpha+2\pi)$. (It is continuous everywhere except at $\exp(i\alpha)$.) For $x \in X$ set

$$f_x(\alpha) = \operatorname{tr}\Big(\arg_\alpha\big(u_0(x)\big)\Big).$$

Then f_x is a nondecreasing function whose range is a countable discrete subset of $\mathbb{R}$, with jumps at exactly those numbers α such that $\exp(i\alpha)$ is an eigenvalue of $u_0(x)$. Since eigenvalues of $u_0(x)$ have multiplicity at most $m-1$, these jumps are at most $2\pi(m-1)$. Since $f_x(\alpha) \longrightarrow \pm\infty$ as $\alpha \longrightarrow \pm\infty$, there is a number α_x, not a discontinuity of f_x, such that

$$\left|f_x(\alpha_x) - \frac{k}{k-Ls}\eta(x)\right| \leq 2\pi(m-1).$$

By the continuity of the eigenvalues of $u_0(x)$ (see Theorem 13.6 of [2] again), there is an open neighborhood U_x of x such that $y \in U_x$ implies $\exp(i\alpha_x)$ is not an eigenvalue of $u_0(y)$ and

(18) $$\left|f_y(\alpha_x) - \frac{k}{k-Ls}\eta(y)\right| \leq 2\pi m.$$

(Note: Theorem 13.6 of [2] really applies only to unitaries in a single matrix algebra, and here $u_0(x) \in q(x)M_n q(x)$. Strictly speaking, we must therefore choose a neighborhood of x over which q is trivial, that is, unitarily equivalent to a constant projection, before applying this theorem.)

Let $\{g_\nu\}$ be a locally finite partition of unity subordinate to the open cover $\{U_x : x \in X\}$ of X, with g_ν supported in $U_{x(\nu)}$. Define $\alpha_0(y) = \sum_\nu g_\nu(y)\alpha_{x(\nu)}$. Then $\alpha_0 \colon X \longrightarrow \mathbb{R}$ is continuous. The set of numbers α_x satisfying (18) is convex, so, on substituting the definition of f_x, we have:

$$\left|\mathrm{tr}\Big(\arg_{\alpha_0(x)}[u_0(x)]\Big) - \frac{k}{k-Ls}\eta(x)\right| < 2\pi m \quad \text{for all } x \in X. \tag{19}$$

Since r is the least integer with $r \geq k/L$, there exist integers $r_1, \ldots, r_L$, each equal to either r or $r-1$, such that $\sum_{\ell=1}^{L} r_\ell = k$. We are now going to construct continuous functions $\alpha_\ell \colon X \longrightarrow \mathbb{R}$, for $1 \leq \ell \leq L$, such that the following properties hold for all $x \in X$ and all ℓ:

$$\alpha_L(x) = \alpha_0(x) + 2\pi. \tag{20}$$

$$\alpha_0(x) < \alpha_1(x) < \cdots < \alpha_{L-1}(x) < \alpha_L(x). \tag{21}$$

$$\alpha_\ell(x) - \alpha_{\ell-1}(x) \leq 6\rho. \tag{22}$$

(23) The open arc from $\exp\big(i\alpha_0(x)\big)$ to $\exp\big(i\alpha_\ell(x)\big)$ (taken in the positive direction) contains between $(\sum_{i=1}^{\ell} r_i) - m$ and $(\sum_{i=1}^{\ell} r_i) + m$ eigenvalues of $u_0(x)$.

(24) The open arc from $\exp\big(i\alpha_{\ell-1}(x)\big)$ to $\exp\big(i\alpha_\ell(x)\big)$ contains between $r_\ell - 3m$ and $r_\ell + 2m$ eigenvalues of $u_0(x)$.

Before constructing the functions α_ℓ, let us observe that properties (22) and (24) follow from the other three. Indeed, it folllows from (23) that the half-open arc $\exp\Big(i\big[\alpha_{\ell-1}(x), \alpha_\ell(x)\big)\Big)$ contains between $r_\ell - 2m$ and $r_\ell + 2m$ eigenvalues of $u_0(x)$. Since $\exp\big(i\alpha_{\ell-1}(x)\big)$ is an eigenvalue with multiplicity at most $m-1$, this gives (24). Similarly, the closed arc from $\exp\big(i\alpha_{\ell-1}(x)\big)$ to $\exp\big(i\alpha_\ell(x)\big)$ contains at most $r_\ell + 3m$ eigenvalues of $u_0(x)$. Since $r_\ell \leq r$, inequality (14) implies that $r_\ell + 3m \leq 2r - 2$ (as $m \geq 1$), so (17) implies that this arc has length at most 6ρ. This proves (22).

We now turn to the construction of the functions α_ℓ. Let $x \in X$. Choose $\lambda > 0$ such that $u_0(x)$ has no eigenvalues in the open arc from $\exp\Big(i\big(\alpha_0(x) - 2\lambda\big)\Big)$ to $\exp\big(i\alpha_0(x)\big)$. Let U_x be an open neighborhood of x such that $\exp\Big(i\big(\alpha_0(y) - \lambda\big)\Big)$ is not an eigenvalue of $u_0(y)$ for $y \in U_x$. Let $a(y) = \arg_{\alpha_0(y)-\lambda}\big(u_0(y)\big)$; then a is a continuous function from U_x to M_n, with $a(y) \in q(y)M_n q(y) \cong M_k$. Restricting the size of U_x, we may assume that q is trivial over U_x, and we may thus regard a as a function from U_x to M_k. Now let $\beta_1(y) \leq \beta_2(y) \leq \cdots \leq \beta_k(y)$ be the eigenvalues of $a(y)$, in increasing order. The functions $\beta_1, \ldots, \beta_k$ are continuous by Theorem 8.1 of [2].

For $1 \leq \ell \leq L-1$, let $j(\ell)$ be the least integer satisfying $j(\ell) \geq \sum_{i=1}^{\ell} r_i$ and $\beta_{j(\ell)+1}(x) > \beta_{j(\ell)}(x)$. That is, we go out to the eigenvalue in position $\sum_1^\ell r_i$ on the list, and then continue to the next j such that $\beta_{j+1}(x)$ is strictly larger than $\beta_j(x)$. Because the multiplicities of the eigenvalues are at most $m-1$, we have

$$\sum_1^\ell r_i \leq j(\ell) \leq \Big(\sum_1^\ell r_i\Big) + m - 2.$$

Now set $\alpha_\ell^{(x)} = \frac{1}{2}\big(\beta_{j(\ell)}(x) + \beta_{j(\ell)+1}(x)\big)$.

Since the functions β_ℓ are continuous, there is an open set V_x with $x \in V_x \subset U_x$ such that for $y \in V_x$, no $\alpha_\ell^{(x)}$ is an eigenvalue of $u_0(y)$, and $a(y)$ has the same number of eigenvalues as $a(x)$ in each interval $(\alpha_{\ell-1}^{(x)}, \alpha_\ell^{(x)})$, for $2 \leq \ell \leq L-1$. We will also choose V_x so small that a corresponding condition is satisfied for $\ell = 1$, as described in the next two paragraphs. The condition depends on whether $\alpha_0(x)$ is an eigenvalue of $a(x)$.

If $\alpha_0(x)$ is an eigenvalue of $a(x)$, let μ be the smallest eigenvalue of $a(x)$ which is strictly greater than $\alpha_0(x)$, and choose $\gamma \in \big(\alpha_0(x), \mu\big)$. Since α_0 is continuous, we can require V_x to be small enough that $\alpha_0(y) < \gamma$ for $y \in V_x$, that $a(x)$ and $a(y)$ have the same number of eigenvalues in $(\gamma, \alpha_1^{(x)})$, that γ is not an eigenvalue of $a(y)$, and that $a(y)$ has the same number of eigenvalues in $\big(\alpha_0(y) - \lambda, \gamma\big)$ as $a(x)$ does in $\big(\alpha_0(x) - \lambda, \gamma\big)$. Note that we do in fact have $\gamma < \alpha_1^{(x)}$, because there are at least $r_1 - (m-1) > 0$ eigenvalues of $a(x)$ in $(\gamma, \alpha_1^{(x)})$. If the multiplicity of $\alpha_0(x)$ as an eigenvalue is $j(0) \leq m-1$, then for $y \in V_x$ somewhere between 0 and $j(0)$ of the eigenvalues $\beta_1(y), \ldots, \beta_{j(0)}(y)$ are in $\big(\alpha_0(y) - \lambda, \alpha_0(y)\big]$. The number of them in this interval is exactly the number of eigenvalues $\beta_1(y), \ldots, \beta_{j(\ell)}(y)$ which are not in $\big(\alpha_0(y), \alpha_\ell^{(x)}\big)$. Therefore, for $y \in V_x$,

(25) the number of eigenvalues of $a(y)$ in the interval $\big(\alpha_0(y), \alpha_\ell^{(x)}\big)$ is between $(\sum_1^\ell r_i) - (m-1)$ and $(\sum_1^\ell r_i) + m - 2$.

Now suppose $\alpha_0(x)$ is not an eigenvalue of $a(x)$. Impose the same additional restrictions on V_x as in the previous paragraph, except with $\alpha_0(y)$ in place of γ. Then $a(y)$ has no eigenvalues in $\big(\alpha_0(y) - \lambda, \alpha_0(y)\big]$, and the same number of eigenvalues in $\big(\alpha_0(y), \alpha_1^{(x)}\big)$ as $a(x)$ does in $\big(\alpha_0(x), \alpha_1^{(x)}\big)$. Therefore $a(y)$ has exactly $j(\ell)$ eigenvalues in $\big(\alpha_0(x), \alpha_\ell^{(x)}\big)$ for each ℓ. Thus, (25) holds in this case also.

In either case there are at most $m-1$ eigenvalues of $a(y)$ in $\big(\alpha_0(y) - \lambda, \alpha_0(y)\big]$. Therefore $a(y)$ has at most

$$\Big(\sum_{\ell=1}^{L-1} r_\ell\Big) + 2m - 3 = k - r_L + 2m - 3 < k$$

eigenvalues in $\big(\alpha_0(y) - \lambda, \alpha_{L-1}^{(x)}\big)$. (The last step in the inequality follows from $r_\ell \geq r-1$ and (14).) So $\alpha_0(y) - \lambda + 2\pi > \alpha_{L-1}^{(x)}$. From the definition of $a(y)$, we can use (25) to conclude that $u_0(y)$ has between $(\sum_1^\ell r_i) - (m-1)$ and $(\sum_1^\ell r_i) + m - 2$ eigenvalues in the open arc from $\exp\big(i\alpha_0(y)\big)$ to $\exp(i\alpha_\ell^{(x)})$, for $y \in V_x$.

Consider the set of numbers $\alpha \in \big(\alpha_0(y), \alpha_0(y) + 2\pi\big)$ which can be substituted for $\alpha_\ell^{(x)}$ in the last statement. This set is clearly an interval. Therefore the statement holds with $\alpha_\ell^{(x)}$ replaced by the number $\alpha_\ell(y)$ constructed as follows. Let $\{g_\nu\}$ be a locally finite partition of unity subordinate to the open cover $\{V_x : x \in X\}$, with g_ν supported in $V_{x(\nu)}$, and set

$$\alpha_\ell(y) = \sum_\nu g_\nu(y)\alpha_\ell^{(x(\nu))}.$$

But the statement at the end of the previous paragraph, with $\alpha_\ell(y)$ in place of $\alpha_\ell^{(x)}$, implies (23) for $1 \leq \ell \leq L-1$. Setting $\alpha_L(x) = \alpha_0(x) + 2\pi$ gives (23) for $\ell = L$ as well, since the multiplicity of $\exp\big(i\alpha_0(x)\big)$ is at most $m-1$.

It is clear from the construction that

$$\alpha_0(y) \leq \alpha_1^{(x)} \leq \alpha_2^{(x)} \leq \cdots \leq \alpha_{L-1}^{(x)} \leq \alpha_0(y) + 2\pi$$

for $y \in V_x$. Therefore

$$\alpha_0(y) \leq \alpha_1(y) \leq \cdots \leq \alpha_{L-1}(y) \leq \alpha_L(y) = \alpha_0(y) + 2\pi$$

for all y. This, combined with (23), is enough to yield (24), using the argument given after the statements of (20)–(24). But $r_\ell \geq r - 1 \geq 4m$ by (14), so each arc $\exp\Big(i \cdot \big(\alpha_{\ell-1}(y), \alpha_\ell(y)\big)\Big)$ must then contain at least $m \geq 1$ eigenvalues of $u_0(x)$. So we must have strict inequalities above, proving (21). Finally, (20) is true by definition. The functions α_ℓ are clearly continuous, so this completes the proof of the existence of $\alpha_1, \ldots, \alpha_L$.

Temporarily fix ℓ with $1 \leq \ell \leq L$. For $x \in X$, let J_x be the open arc from $\exp\big(i\alpha_{\ell-1}(x)\big)$ to $\exp\big(i\alpha_\ell(x)\big)$. Let I_x be an open arc with $\bar{I}_x \subset J_x$ and such that I_x contains all the eigenvalues of $u_0(x)$ which are in J_x. By continuity (using Theorem 13.6 of [2] over trivializing open sets again), there is an open neighborhood U_x of x such that for $y \in U_x$, the arcs I_x and $\bar{I}_x$ contain the same number of eigenvalues of $u_0(y)$ as of $u_0(x)$. In particular, for such y the endpoints of I_x are not eigenvalues of $u_0(y)$. We may furthermore choose U_x so small that $\bar{I}_x \subset J_y$ for $y \in U_x$.

If $v \in U\big(q(x)M_nq(x)\big)$ and $J \subset S^1$, we will temporarily let $\mathrm{proj}(v, J)$ denote the projection onto the linear span of the eigenvectors of v whose eigenvalues are in J. Define $f_{\ell x}\colon U_x \longrightarrow M_n$ by $f_{\ell x}(y) = \mathrm{proj}\big(u(y), I_x\big)$. This function can be obtained from continuous functional calculus and is therefore continuous. It furthermore has constant rank, by (24) at least $r_\ell - 3m$. Apply Lemma 1.4 to the functions $f_{\ell x}$ defined on the open cover $\{U_x : x \in X\}$, to obtain a projection $f_\ell\colon X \longrightarrow M_n$ such that $\mathrm{rank}\big(f_\ell(x)\big) \geq r_\ell - 3m - d_0$ for all x, and

$$f_\ell(x) \leq \mathrm{proj}\Big(u_0(x), \exp\Big[i \cdot \big(\alpha_{\ell-1}(x), \alpha_\ell(x)\big)\Big]\Big).$$

We now want to find $e_\ell \leq f_\ell$ whose rank is exactly $r_\ell - 3m - d_0$. For convenience, assume $\mathrm{rank}(f_\ell)$ is constant on X. (This loses no generality, since we can partition X into subsets on which f_ℓ has constant rank, and construct e_ℓ separately on each of the subsets.) We have

$$\operatorname{rank}(f_\ell) \geq r_\ell - 3m - d_0 = r_\ell - s \geq r - 1 - s \geq s - 1 \geq d_0.$$

Here, the second step uses (6), the fourth step uses $r \geq 2s$ (which follows from (12) and (13)), and the last step uses (6) and $m \geq 1$. We can now apply Theorem 2.5(a) of [13], which is the generalization to compact spaces of Theorem 8.1.2 of [15], to obtain a trivial subprojection of f_ℓ with rank equal to $\operatorname{rank}(f_\ell) - (r_\ell - 3m - d_0)$. The orthogonal complement of this subprojection in $f_\ell B f_\ell = f_\ell\big(C(X) \otimes M_n\big)f_\ell$ is a projection $e_\ell \leq f_\ell$ of rank exactly $r_\ell - 3m - d_0 = r_\ell - s$.

We now define

$$\gamma_\ell(x) = \big(\alpha_{\ell-1}(x) + \alpha_\ell(x)\big)/2, \tag{26}$$

$$v_\ell(x) = \exp\big(i\gamma_\ell(x)\big)e_\ell(x), \tag{27}$$

$$e_0(x) = q(x) - \sum_{\ell=1}^{L} e_\ell(x), \tag{28}$$

$$w_0(x) = e_0(x)u_0(x)e_0(x) \cdot \Big(e_0(x)u_0(x)^* e_0(x)u_0(x)e_0(x)\Big)^{-1/2},$$

and

$$u_1 = w_0 + \sum_{\ell=1}^{L} v_\ell. \tag{29}$$

Since $e_\ell(x) \leq \operatorname{proj}\Big(u_0(x), \exp\big[i \cdot \big(\alpha_{\ell-1}(x), \alpha_\ell(x)\big)\big]\Big)$, the relations (3), (22), and (26) enable us to apply Lemma 2.3 to obtain $\|u_0 - u_1\| \leq 4 \cdot 6\rho = 24\rho$. Therefore:

$$\|u - u_1\| \leq 24\rho + \|u - u_0\| \leq 24\rho + 2\sin(\rho/2) < 25\rho. \tag{30}$$

If we did not have to estimate $D(w)$, we could stop here, taking $e = e_0$, $w = w_0$, and $v = \sum_{\ell=1}^{L} v_\ell$. The rest of the proof is therefore devoted to modifications needed to control $D(w)$. Most of it consists of estimating $D(w_0)$, which we do next.

The inequality (30) implies $\|u_1^* u - 1\| < 25\rho$, and $25\rho < 2$ by (3). So $u_1^* u = \exp(ib)$ with $\|b\| \leq 2\arcsin(25\rho/2)$. Since

$$\det\big(u_1(x)\big) = \det\big(w_0(x)\big) \prod_{\ell=1}^{L} \exp\big(i(r_\ell - s)\gamma_\ell(x)\big),$$

we can write

$$\det\big(w_0(x)\big) = \exp\big(i\sigma(x)\big), \tag{31}$$

where, using (15),

$$\sigma(x) + \sum_{\ell=1}^{L}(r_\ell - s)\gamma_\ell(x) + \operatorname{tr}\big(b(x)\big) = \eta(x).$$

We have

$$\left|\operatorname{tr}\big(b(x)\big)\right| \leq k\|b\| \leq 2\arcsin(25\rho/2)k.$$

Using this, and then (19) and $(k-Ls)/k \leq 1$, we get

$$\begin{aligned}|\sigma(x)| &\leq 2\arcsin(25\rho/2)k + \left|\eta(x) - \sum_{\ell=1}^{L}(r_\ell - s)\gamma_\ell(x)\right| \\ &\leq 2\arcsin(25\rho/2)k + 2\pi m + \frac{k-Ls}{k}\left|\operatorname{tr}\big(\arg_{\alpha_0(x)}[u_0(x)]\big) - \sum_{\ell=1}^{L}\frac{k(r_\ell - s)}{k-Ls}\gamma_\ell(x)\right|.\end{aligned}$$

We now estimate the last term in this inequality. For convenience of notation, we omit the letter x. Substituting the definition (26) of γ_ℓ, we see that this term is dominated by

$$\left|\operatorname{tr}\big(\arg_{\alpha_0(x)}(u_0)\big) - \sum_{\lambda=1}^{L} r_\ell\Big(\frac{\alpha_{\ell-1}+\alpha_\ell}{2}\Big)\right| + \Big(\sum_{\ell=1}^{L} r_\ell\left|\frac{\alpha_{\ell-1}+\alpha_\ell}{2}\right|\Big)\sup_\ell\left|1 - \frac{k(r_\ell - s)}{(k-Ls)r_\ell}\right|. \tag{32}$$

(We have dropped a factor $(k-Ls)/k$ from the first term in (32) because $(k-Ls)/k \leq 1$.)

To estimate the first term of (32), let the eigenvalues of $\arg_{\alpha_0}(u_0)$ be $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k$, and let $\mu_1' \leq \mu_2' \leq \cdots \leq \mu_k'$ be the sequence

$$\frac{\alpha_0+\alpha_1}{2},\ldots,\frac{\alpha_0+\alpha_1}{2},\frac{\alpha_1+\alpha_2}{2},\ldots,\frac{\alpha_1+\alpha_2}{2},\ldots,\frac{\alpha_{L-1}+\alpha_L}{2},\ldots,\frac{\alpha_{L-1}+\alpha_L}{2},$$

in which the term $(\alpha_{\ell-1}+\alpha_\ell)/2$ is repeated r_ℓ times. Let $j(\ell)$ be the least integer satisfying $\mu_{j(\ell)+1} \geq \alpha_\ell$. (Take $j(L) = k$.) Since $\exp(i\alpha_0)$ can occur as an eigenvalue of u_0 with multiplicity at most $m-1$, relation (23) implies that

$$\Big(\sum_{\ell=1}^{L} r_\ell\Big) - m \leq j(\ell) \leq \Big(\sum_{\ell=1}^{L} r_\ell\Big) + 2m - 1.$$

Since each r_i is either r or $r-1$, inequality (14) now implies

$$\sum_{i=1}^{\ell-1} r_\ell \leq j(\ell) \leq \sum_{i=1}^{\ell+1} r_\ell.$$

Using this result for both $j(\ell)$ and $j(\ell-1)$, we see that if $j(\ell-1)+1 \leq j \leq j(\ell)$, then μ_j' must be one of

$$\frac{\alpha_{\ell-2}+\alpha_{\ell-1}}{2},\ \frac{\alpha_{\ell-1}+\alpha_\ell}{2},\quad \text{or}\quad \frac{\alpha_\ell+\alpha_{\ell+1}}{2}.$$

(The first of these is impossible if $\ell = 1$, the last if $\ell = L$.) For j in this range, we have $\alpha_{\ell-1} \leq \mu_j \leq \alpha_\ell$, so (22) implies $|\mu_j - \mu_j'| < 9\rho$. Therefore the first term of (32) is at most $9\rho k$.

The first factor in the second term of (32) is dominated by $k\sup_\ell|\alpha_\ell|$. If $\alpha_0 \geq 0$ then

$$\begin{aligned}\sup_\ell|\alpha_\ell| = \alpha_0 + 2\pi &\leq \frac{1}{k}\operatorname{tr}\big(\arg_{\alpha_0}(u_0)\big) + 2\pi \leq \frac{1}{k-Ls}\eta + \frac{2\pi m}{k} + 2\pi \\ &\leq 2\big(D_0(u)+1\big) + 2\pi + 2\pi = 2\big(D_0(u) + 2\pi + 1\big).\end{aligned}$$

The third step uses (19), and the fourth step uses (12) and (16) on the first term and (3) and (10) (which imply $k \geq m$) on the second term. If $\alpha_L \leq 0$, then $\sup_\ell |\alpha_\ell| = -\alpha_L + 2\pi$, and essentially the same argument applies and yields the same result. If $\alpha_0 < 0 < \alpha_L$, then $|\alpha_\ell| \leq 2\pi$ for all ℓ. Thus, in any case we have

$$\sup_\ell |\alpha_\ell| \leq 2\big(D_0(u) + 2\pi + 1\big), \tag{33}$$

and the first factor in the second term of (32) is at most $2\big(D_0(u) + 2\pi + 1\big)k$.

For the second factor, we have

$$\sup_\ell \left| 1 - \frac{k(r_\ell - s)}{(k - Ls)r_\ell} \right| = \sup_\ell \left| 1 - \left(1 - \frac{s}{r_\ell}\right)\left(1 - \frac{s}{kL^{-1}}\right)^{-1} \right|.$$

Each r_ℓ is either r or $r - 1$. The expression inside the absolute value signs on the right is monotone in r_ℓ, so it suffices to estimate it at the endpoints of an interval containing r and $r - 1$. Since r is the least integer with $r \geq kL^{-1}$, we use the interval $[kL^{-1}, kL^{-1} + 1]$. Now (7) gives the upper bound $\rho / \big[2\big(D_0(u) + 2\pi + 1\big)\big]$.

Putting together our estimates, we obtain

$$\begin{aligned} |\alpha(x)| &\leq 2\arcsin(25\rho/2)k + 2\pi m + 9\rho k \\ &\quad + 2\big(D_0(u) + 2\pi + 1\big)k\Big[\rho \Big/ \Big(2\big(D_0(u) + 2\pi + 1\big)\Big)\Big] \\ &= k[2\arcsin(25\rho/2) + 11\rho] + 2\pi m \leq k[2\arcsin(25\rho/2) + 12\rho]. \end{aligned}$$

The last step follows from (10). Now (4) implies

$$|\sigma(x)| \leq k\delta/4. \tag{34}$$

This completes our estimate of $D(w_0)$.

Let R be the least integer satisfying $R \geq k\delta/4$. We will absorb the error in the determinant, as estimated in (34), by increasing the rank of e_0 by R. We have

$$R + Ls \leq \frac{1}{4}k\delta + 1 + Ls \leq \frac{3}{4}k\delta < \frac{\operatorname{rank}(p)}{n}, \tag{35}$$

using (11) and (9). Also,

$$R + Ld_0 \leq R + Ls \leq \frac{3}{4}k\delta \leq \frac{3}{8}k < k - Ls, \tag{36}$$

using (6), (9), (12), and the first two steps from (35).

Using (36), we can choose integers r'_ℓ with $0 \leq r'_\ell \leq r_\ell - s - d_0$ such that $\sum_{\ell=1}^{L} r'_\ell = R$. By Theorem 2.5(a) of [13], there is a trivial subprojection e'_ℓ of e_ℓ with rank exactly r'_ℓ. (Recall that $\operatorname{rank}(e_\ell) = r_\ell - s$.) Now define:

$$e = e_0 + \sum_{\ell=1}^{L} e'_\ell,$$

$$w(x) = w_0(x) + \sum_{\ell=1}^{L} \exp\big(i\gamma_\ell(x)\big)e'_\ell(x),$$

and

$$v(x) = \sum_{\ell=1}^{L} \exp\bigl(i\gamma_\ell(x)\bigr)\bigl(e_\ell(x) - e'_\ell(x)\bigr).$$

Observe that $\operatorname{rank}(e) = R+Ls$, so that $N \operatorname{rank}(e) < \operatorname{rank}(p)$ by (35). Therefore Lemma 4.3 yields i and N orthogonal projections $f_1, \ldots, f_N \leq p$ in A_i, each Murray-von Neumann equivalent to e. We claim that i, e, w, and v satisfy the conclusion of the lemma.

For conclusion (1), note that

$$v + w = w_0 + \sum_{\ell=1}^{L} v_\ell = u_1,$$

simply by comparing the definition with (27), (28), and (29). Therefore (30) and (3) imply conclusion (1). It is clear from the definition that $v = \exp(ih)$, with $h(x) = \sum_{\ell=1}^{L} \gamma_\ell(x)\bigl(e_\ell(x) - e'_\ell(x)\bigr)$. This verifies conclusion (2). Conclusion (4) has already been done. It remains only to verify conclusion (3).

Using (31), we have $\det(w) = \exp(i\lambda)$, where

$$\lambda(x) = \sigma(x) + \sum_{\ell=1}^{L} r'_\ell \gamma_\ell(x).$$

Using (26) and (33), we have

$$|\gamma_\ell(x)| \leq \sup_\ell |\alpha_\ell(x)| \leq 2\bigl(D_0(u) + 2\pi + 1\bigr).$$

Combining this with (34), we get

$$\begin{aligned} |\lambda(x)| &\leq k\delta/4 + 2\bigl(D_0(u) + 2\pi + 1\bigr) \sum_{\ell=1}^{L} r'_\ell \\ &< R + 2R\bigl(D_0(u) + 2\pi + 1\bigr) < 2(R + Ls)\bigl(D_0(u) + 2\pi + 2\bigr). \end{aligned}$$

Therefore

$$D_i(w) \leq D_0(w) = \frac{1}{\operatorname{rank}(e)} \sup_x |\lambda(x)| \leq 2\bigl(D_0(u) + 2\pi + 2\bigr).$$

as desired. This verifies conclusion (3), and completes the proof. ∎

6. **The main theorem.** In this section, we assemble the results from previous sections and prove our main result. We then discuss several examples, some possible generalizations of the theorem, and related questions. As before, $U(A)$ is the unitary group of A and $U_0(A)$ is its identity component.

THEOREM 6.1. *Let A be simple with no dimension growth. Then* $\operatorname{cer}(A) \leq 1 + \varepsilon$.

PROOF. The proof is adapted from the last part of the proof of Theorem 2.6 of [11], which in turn uses methods of [21].

We have to prove that $\exp(iA_{sa})$ is dense in $U_0(A)$. Write $A = \varinjlim A_i$ as in Notation 1.3. Then, as is well known,

$$U_0(A) = \overline{\bigcup_{i=1}^{\infty} U_0(A_i)}.$$

So it suffices to show each $U_0(A_i)$ is contained in $\overline{\exp(iA_{sa})}$. Without loss of generality we may take $i = 0$.

Let $u \in U_0(A_0)$, let $\varepsilon > 0$, and let $\rho = \min\bigl(1, \varepsilon/(2e^{2\pi} + 5)\bigr)$. Choose k such that $2\pi/k < \rho$.

If $\mathrm{sp}(u) \neq S^1$, then u is already an exponential, so assume $\mathrm{sp}(u) = S^1$. For $j = 0, \ldots, k-1$ set $\lambda_j = 2\pi ij/k$. By Lemma 5.2 there are i, nonzero orthogonal projections $p_0, \ldots, p_{k-1} \in A_i$, and a unitary $u_0 \in (1 - \sum_j p_j)A_i(1 - \sum_j p_j)$ such that

$$\Bigl\| u - \Bigl(u_0 + \sum_j p_j\Bigr)\Bigr\| < \rho.$$

Using Lemma 4.4(1) (increasing i if necessary), we can assume the p_j are mutually Murray-von Neumann equivalent. (Find mutually equivalent nonzero subprojections and absorb what is left over into u_0.) Further increasing i if necessary, we can use Lemma 4.4(2) to find a nonzero projection p and two orthogonal projections $p_{01}, p_{02} \leq p_0$ which are both Murray-von Neumann equivalent to p. Then there are also orthogonal projections $p_{j1}, p_{j2} \leq p_j$ which are both Murray-von Neumann equivalent to p. Again absorbing leftovers into u_0, we may assume $p_j = p_{j1} + p_{j2}$. Now renumber the terms of the direct limit so as to take $i = 0$.

Define $q = 1 - \sum_j p_j$. Observe that $u_0 \in U_0(qAq)$ by Corollary 4.6. Therefore $D_{A_0}(u_0) < \infty$. Choose an integer

$$N > \Bigl[2\bigl(D_{A_0}(u_0) + 2\pi + 2\bigr) + 6\pi + 1\Bigr]/\rho.$$

Apply Lemma 5.3, with p and q as given, ρ in place of ε, u_0 in place of u, and $2N$ in place of N, to obtain i, $e \leq q$ in A_i, $w \in U(eA_ie)$, $v \in (q-e)A_i(q-e)$, and projections $f_1, \ldots, f_{2N}$, as in the conclusion of the lemma. Since $\rho < 2$, we get $\|(v + w) - u_0\| < 2$. Since v is an exponential, we can use Corollary 4.6 to conclude $w \in U_0(eAe)$. Increasing i if necessary, we have $w \in U_0(eA_ie)$. (Note that increasing i does not affect the conclusions of Lemma 5.3, since $j \geq 1$ implies $D_{A_j}(w) \leq D_{A_i}(w)$ by Lemma 3.3(5).)

Let d be the dimension bound as in Notation 1.3, and choose $K(d)$ as in Lemma 3.4. By Lemma 4.1, we can further increase i so as to have

$$\inf_{x \in X_i} \mathrm{rank}\bigl(e(x)\bigr) \geq K(d).$$

Lemma 3.4 therefore provides a continuous path $t \longmapsto w(t)$ in $U_0(eA_ie)$, with $w(0) = w$ and $w(1) = 1$ (really e), with length at most

$$\mathrm{cel}(w) + 1 \leq D_{A_i}(w) + 6\pi + 1 \leq 2\bigl(D_{A_0}(u_0) + 2\pi + 2\bigr) + 6\pi + 1 < N\rho.$$

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Therefore there are $0 = t_0 < t_1 < \cdots < t_N = 1$ such that the unitaries $w_\ell = w(t_\ell)$ satisfy $\|w_\ell - w_{\ell-1}\| < \rho$.

We have orthogonal projections $f_1, \ldots, f_{2N} \leq p$, all Murray-von Neumann equivalent to e. Since p is Murray-von Neumann equivalent to each p_{j1} and p_{j2}, we can find orthogonal projections $f_{1,j,r}, \ldots, f_{2N,j,r} \leq p_{jr}$ $(0 \leq j \leq k-1, 1 \leq r \leq 2)$, each Murray-von Neumann equivalent to e. Set

$$g_j = p_{j1} + p_{j2} - \sum_{l=1}^{2N} (f_{\ell j1} + f_{\ell j2}).$$

Then

$$v + w + \sum_j \lambda_j p_j = \Big(v + \sum_j \lambda_j q_j\Big) + \Big(w + \sum_{\ell,j,r} \lambda_j f_{\ell jr}\Big).$$

The first term on the right is an exponential because v is. Furthermore,

$$\Big\|u - \Big(v + w + \sum_j \lambda_j p_j\Big)\Big\| \leq \Big\|u - \Big(u_0 + \sum_j \lambda_j p_j\Big)\Big\| + \|u_0 - (v + w)\| < \rho + \rho = 2\rho.$$

Therefore it suffices to prove that there is h with

$$\Big\|\exp(ih) - \Big(w + \sum_{\ell,j,r} \lambda_j f_{\ell jr}\Big)\Big\| < \varepsilon - 2\rho.$$

Since the $f_{\ell jr}$ are all Murray-von Neumann equivalent to e, and $w \in U_0(eAe)$, this last estimate can be regarded as happening in $M_{4Nk+1}(B)$, with $B = eAe$.

At this point, we no longer need anything special about the structure of B. Our problem is reduced as follows. Given B, $w = w_0, w_1, \ldots, w_N = 1$ in $U(B)$ with $\|w_\ell - w_{\ell-1}\| < \rho$, given $\lambda_j = 2\pi ij/k$ for $j = 0, \ldots, k-1$, and given

$$y = \mathrm{diag}(\lambda_0, \ldots, \lambda_0, \lambda_1, \ldots, \lambda_1, \ldots, \lambda_{k-1}, \ldots, \lambda_{k-1}) \in M_{4Nk}(B)$$

(where diag( ) is the diagonal matrix with the given entries, and each λ_j occurs $4N$ times), we want a selfadjoint $h \in M_{4Nk+1}(B)$ such that

$$(*) \qquad \|\exp(ih) - w \oplus y\| < \varepsilon - 2\rho,$$

where $w \oplus y$ has the obvious meaning $\begin{pmatrix} w & 0 \\ 0 & y \end{pmatrix} \in M_{4Nk+1}(B)$.

Let

$$z = \mathrm{diag}(w_0^*, w_1, w_1^*, \ldots, w_{N-1}, w_{N-1}^*, w_N) \in M_{2N}(B),$$

and let

$$z_0 = \mathrm{diag}(w_0^*, w_0, w_1^*, w_1, \ldots, w_{N-1}^*, w_{N-1}) \in M_{2N}(B).$$

Then z_0 can be written in the form $y \oplus y^*$ (up to unitary equivalence). Therefore there is a selfadjoint $a \in M_{2N}(B)$ such that $\|\exp(ia) - z_0\| < \rho$, by Corollary 5 of [21]. An examination of the proof of that corollary shows we may take $\|a\| \leq \pi$. Furthermore, $\|z - z_0\| < \rho$, so $\|\exp(ia) - z\| < 2\rho$.

Now let

$$b = \mathrm{diag}(-2\pi, \ldots, -2\pi, 0, \ldots, 0, -2\pi + 2\pi/k, \ldots, -2\pi + 2\pi/k,$$
$$2\pi/k, \ldots, 2\pi/k, \ldots, -2\pi/k, \ldots, -2\pi/k, 2\pi - 2\pi/k, \ldots, 2\pi - 2\pi/k) \in M_{4Nk}(B).$$

In this expression, each entry is repeated $2N$ times. Note that $\exp(ib) = y$. Since $\rho \leq 1$ and $2\pi/k < \rho$, we have $k \geq 2$. Therefore we can apply Lemma 2.4, using $M_{2N}(B)$ for A, to find a projection g and a unitary s in $M_{4Nk}(B)$ such that

$$sgs^* = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$$

(1 repeated $2N$ times, 0 repeated $2N(2k-1)$ times), $\|gb - bg\| \leq 2\pi/k < \rho$, and

$$sgbgs^* = \mathrm{diag}(a, 0, \ldots, 0).$$

In $M_{4Nk}(B)$, we have

$$\begin{aligned}\left\|b - \big(gbg + (1-g)b(1-g)\big)\right\| &\leq \|gb(1-g)\| + \|(1-g)bg\| \\ &\leq \|g\|\,\|gb - bg\| + \|bg - gb\|\,\|g\| < 2\rho.\end{aligned}$$

Therefore, taking exponentials in the appropriate corners,

$$\left\|y - \Big(\exp(igbg) + \exp\big(i(1-g)b(1-g)\big)\Big)\right\| < 2\rho e^{2\pi}$$

by Lemma 2.1, since $\|b\|$, $\|gbg + (1-g)b(1-g)\| \leq 2\pi$. Consequently,

$$\begin{aligned}\left\|y - \Big(s^*zs + \exp\big(i(1-g)b(1-g)\big)\Big)\right\| &< 2\rho e^{2\pi} + \|s^*zs - \exp(igbg)\| \\ &= 2\rho e^{2\pi} + \|z - \exp(isgbgs^*)\| \\ &= 2\rho e^{2\pi} + \|z - \exp(ia)\| < (2e^{2\pi} + 2)\rho.\end{aligned}$$

(Again, the various exponentials are evaluated in the corners of $M_{4Nk}(B)$ in which the elements live.)

In $M_{2N+1}(B)$, the elements $w \oplus z$ and $1 \oplus z_0$ are unitarily equivalent via a permutation matrix, since $w_0 = w$ and $w_N = 1$. We have already found a such that $\|\exp(ia) - z_0\| < \rho$. Therefore there is c such that $\|\exp(ic) - w \oplus z\| < \rho$. Therefore

$$\begin{aligned}&\left\|w \oplus y - \exp\big(i[(1 \oplus s)^* c(1 \oplus s) + (1-g)b(1-g)]\big)\right\| \\ &\qquad = \left\|w \oplus y - \big[(1 \oplus s)^* \exp(ic)(1 \oplus s) \oplus \exp\big(i(1-g)b(1-g)\big)\big]\right\| \\ &\qquad < \rho + \left\|w \oplus y - (1 \oplus s^*)(w \oplus z)(1 \oplus s) \oplus \exp\big(i(1-g)b(1-g)\big)\right\| \\ &\qquad < (2e^{2\pi} + 3)\rho.\end{aligned}$$

Since $(2e^{2\pi} + 5)\rho \leq \varepsilon$ by the choice of ρ, we have verified $(*)$ with

$$h = (1 \oplus s)^* c(1 \oplus s) + (1-g)b(1-g).$$

This completes the proof. ■

This theorem gives a large class of examples of simple C^*-algebras which do not have real rank 0, but nevertheless have C^* exponential rank at most $1 + \varepsilon$. We note that a few such examples were known before, basically cases covered by our theorem in which in addition the dimension bound d is at most 2. (See for instance Example 3.7 of [20].) Those examples are, however, rather special.

EXAMPLE 6.2. Let X be an finite dimensional compact metric space with $\dim(X) > 0$. Then the construction of [12], with the numbers $\alpha(n)$ and $\nu(n)$ chosen so that $\lim_{t\to\infty} \omega_{t,1} > 0$ (see Section 4 of [12]) yields a simple unital C^*-algebra with real rank 1 and exponential rank at most $1 + \varepsilon$. (See Theorem 6 of [12].)

EXAMPLE 6.3 (Compare [5]). Let X be a finite dimensional connected compact metric space, and let $h: X \longrightarrow X$ be a minimal homeomorphism with more than one invariant probability measure. Define $\varphi_i: C(X, M_{2^i}) \longrightarrow C(X, M_{2^{i+1}})$ by

$$\varphi_i(f) = \begin{pmatrix} f & 0 \\ 0 & f \circ h \end{pmatrix}.$$

Then $A = \varinjlim_i C(X, M_{2^i})$ is a simple C^*-algebra with two different traces (obtained from the two invariant measures) which agree on all projections. (See [5].) Therefore A does not have real rank 0. But our theorem shows that $\mathrm{cer}(A) \leq 1 + \varepsilon$. (Example 3.7 of [20] is of this type, but it has C^* exponential rank at most $1 + \varepsilon$ by more elementary arguments, since $\dim(X) = 2$.)

Theorem 6.1 raises several questions.

QUESTION 6.4. In Theorem 6.1, can "no dimension growth" be replaced by "slow dimension growth" as in [4]?

Adapting the methods of our proof would seem to require knowing that if $e_i \in C(X_i) \otimes M_{n(i)}$ is a projection of constant rank, and $\dim(X_i) / \operatorname{rank}(e_i) \longrightarrow 0$, then

$$\sup_i \mathrm{cer}\Big(e_i [C(X_i) \otimes M_{n(i)}] e_i\Big) < \infty.$$

We think this is probably true, but it has not been proved. As special cases, one might ask whether the hypothesis $\dim(X) < \infty$ can be removed in Examples 6.2 and 6.3.

Since projections play a major role in our proof, even though the real rank is not 0, one might ask:

QUESTION 6.5. Let A be a simple C^*-algebra with stable rank 1 such that every nonzero hereditary subalgebra contains a nontrivial projection. Does it follow that $\mathrm{cer}(A) \leq 1 + \varepsilon$?

Of course, the following question also remains open:

QUESTION 6.6. Does there exist any simple C^*-algebra A at all such that $\mathrm{cer}(A) > 1 + \varepsilon$?

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In another direction, one can also ask:

QUESTION 6.7. Is simplicity necessary in Theorem 6.1?

We think it is possible that the tensor product $C(B_3)\otimes \varinjlim M_{2^i}$, of the continuous functions on the closed ball in $\mathbb{R}^3$ with the 2^∞ UHF algebra, has C^* exponential rank at least 2. This would show that simplicity is necessary. In this connection, we note that $C(B_3)\otimes M_n$ has real rank 1 for large enough n (by [1]), but can be shown to have exponential rank at least 2.

7. **Related invariants.** In this section, we discuss the Banach exponential rank, the C^* exponential length, and the C^* projective length of simple C^*-algebras with no dimension growth. For the convenience of the reader we briefly describe these three quantities, providing references for the detailed definitions. The Banach exponential rank $\operatorname{ber}(A)$ is defined in the same way as the C^* exponential rank except using the invertible group in place of the unitary group. (See Section 4 of [20].) The C^* exponential length $\operatorname{cel}(A)$ is the rectifiable diameter of $U_0(A)$, or alternatively $\sup\{\operatorname{cel}(u) : u \in U_0(A)\}$, where

$$\operatorname{cel}(u) = \inf\Big\{\sum_{j=1}^{n} \|h_j\| : h_j \in A_{\mathrm{sa}}, u = \prod_{j=1}^{n} \exp(ih_j)\Big\}.$$

See [28] for details. The C^* projective length is the supremum of the rectifiable diameters of the path components of the space of projections in A. (Note that the rectifiable distance between two projections p and q is $\inf\{\operatorname{cel}(u) : upu^* = q\}$. See [23] for details.)

In this section, we prove that $\operatorname{ber}(A) \leq 2$, and that usually $\operatorname{ber}(A) = 2$; however, there are simple C^*-algebras with real rank 1 satisfying $\operatorname{ber}(A) \leq 1+\varepsilon$. The C^* exponential length is π if A has real rank 0 and ∞ is A has real rank 1; we will provide the complete proof of this fact in [25]. The C^* projective length is $\pi/2$ if A has real rank 0. It is at most 2π in general, and in the real rank 1 case most likely either π or $\pi/2$.

PROPOSITION 7.1. *Let A be simple with no dimension growth. Then* $\operatorname{ber}(A) \leq 2$.

PROOF. This is immediate from Lemma 11 of [21] and Theorem 6.1. ■

It is at least possible to have $\operatorname{ber}(A) \leq 1+\varepsilon$ in this situation, even after the AF algebras are excluded. We will provide the first known example of a simple C^*-algebra A with $\operatorname{ber}(A) \leq 1+\varepsilon$ but which is not AF. This example has real rank 1. Its construction requires the following lemma.

LEMMA 7.2. *For any n, we have* $\operatorname{ber}\big(C([-1,1])\otimes M_n\big) \leq 1+\varepsilon$.

We note that $\operatorname{ber}\big(C(S^1)\otimes M_2\big) \geq 2$. See Example 4.9 of [20].

PROOF OF LEMMA 7.2. The argument will follow the pattern in Section 2 of [20] for C^* exponential rank. The main step in the proof is to show that the set of matrices in M_n with a repeated eigenvalue is the union of a finite collection of submanifolds, each of codimension at least 2. (This is the analog of Lemma 2.4 of [20].)

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Given this, we can prove the lemma as follows. Let $a \in C([-1,1]) \otimes M_n$ be invertible, and let $\varepsilon > 0$. We can approximate a to within $\varepsilon/2$ by a smooth invertible element b. The proof of the Transversality Homotopy Theorem ([14], p. 70) shows that we can approximate b to within $\varepsilon/2$ by a function $c: [-1,1] \longrightarrow M_n$ which is transverse to each of the finitely many submanifolds in the previous paragraph. If $\|b - c\|$ is small enough, then c will also be invertible. Counting dimensions shows that transversality implies empty intersection, so $c(t)$ has distinct eigenvalues for every $t \in [-1,1]$. Using (locally) holomorphic functional calculus with the characteristic functions of small neighborhoods of the eigenvalues, we can write

$$(*) \qquad c(t) = \sum_{k=1}^{n} \lambda_k(t) e_k(t),$$

where $\lambda_k: [-1,1] \longrightarrow \mathbb{C} - \{0\}$ and $e_k: [-1,1] \longrightarrow M_n$ are continuous functions such that $e_k(t)$ is always a rank 1 idempotent. Since $[-1,1]$ is simply connected, the right hand side of $(*)$ is obviously an exponential. By the choice of b and c, we have $\|a - c\| < \varepsilon$.

It remains only to establish the claim about the set X of matrices with a repeated eigenvalue. I am indebted to Dan Grayson, Jens Jantzen, and Sergey Yuzvinsky for the following argument. It replaces a much longer one based explicitly on Jordan forms.

For $a \in M_n$ let $D(a)$ be the discriminant of the characteristic polynomial $p_a(\lambda) = \det(a - \lambda)$ of a. If we write $p_a(\lambda) = \alpha_n \lambda^n + \cdots + \alpha_0$, and let its roots, the eigenvalues of a, be $\lambda_1, \ldots, \lambda_n$ (repeated according to multiplicity), then $D(a) = \alpha_n^{2n-2} \prod_{1 \le j < k \le n} (\lambda_j - \lambda_k)^2$. (See the end of Section 5.7 of [30].) Therefore $D(a) = 0$ if and only if $a \in X$. Furthermore, $D(a)$ is a polynomial in the coefficients α_i ([30], Section 5.9), and therefore in the entries a_{jk} of a. It follows that X is a complex algebraic variety in $M_n = \mathbb{C}^{n^2}$, of complex codimension 1.

Every algebraic variety has a stratification into finitely many locally closed smooth subvarieties. (See, in [18], the discussion following Lemma 1.15.) Furthermore, a smooth subvariety of $\mathbb{C}^m$ is actually a complex manifold. (See, in [18], the remarks immediately after Corollary 1.26.) In the case of X, the resulting smooth manifolds must all have complex codimension at least 1, and hence real codimension at least 2, as desired. ∎

EXAMPLE 7.3. Let $X = [-1,1]$, let $(x_0, x_1, \ldots)$ be a sequence in X such that $\{x_i, x_{i+1}, \ldots\}$ is dense in X for each i, let $d_i = 2^i + 1$, and let $n_i = d_1 \cdot d_2 \cdot \cdots \cdot d_i$. Let $A_i = C(X) \otimes M_{n_i}$, and define $\varphi_i: A_i \longrightarrow A_{i+1}$ by $\varphi_i(a) = \operatorname{diag}\bigl(a, \ldots, a, a(x_i)\bigr)$, the block diagonal matrix in which a is repeated $d_{i+1} - 1$ times and $a(x_i)$ is the constant function from X to M_{n_i} with value $a(x_i)$. Set $A = \varinjlim A_i$. One easily checks that

$$\sum_{i=1}^{\infty} \log\left(1 - \frac{1}{d_i}\right) \ge -\sum_{i=1}^{\infty} \frac{1}{d_i - 1} = -1,$$

so that

$$\prod_{i=1}^{\infty} \left(1 - \frac{1}{d_i}\right) \ge e^{-1}.$$

Therefore A has real rank 1 by Theorem 6 of [12]. In particular, A is not AF. Also, A is simple by Lemma 1 of [12]. By the previous lemma, $\mathrm{ber}(A_i) \leq 1 + \varepsilon$ for all i. It follows that $\mathrm{ber}(A_i) \leq 1 + \varepsilon$, by an obvious approximation argument. (The analogous result for cer is Proposition 1.7 of [20].)

As we will now show, Example 7.3 is rather special. Again, we need a lemma. This one is a generalization of Example 4.9 of [20]. We denote the invertible group of A by $\mathrm{inv}(A)$ and its identity component by $\mathrm{inv}_0(A)$.

LEMMA 7.4. *Let A be a unital C^*-algebra, let $p \in A$ be a projection, and let $u \in U(pAp)$ and $v \in U\big((1-p)A(1-p)\big)$. Let α, β, and γ satisfy $0 < \alpha < \gamma < \beta$, and let $a \in \mathrm{inv}(A)$ be given by $a = \alpha u + \beta v$. If there is $x \in A$ such that $\|\exp(x) - a\| < \min(\gamma - \alpha, \beta - \gamma)$, then $u \in U_0(pAp)$ and $v \in U_0\big((1-p)A(1-p)\big)$.*

PROOF. Assume x exists. Let $b_t = ta + (1-t)\exp(x)$. Let $\lambda \in \mathbb{C}$ satisfy $|\lambda| = \gamma$. Since a is normal, and since the spectral radius of a normal element is equal to its norm, one verifies that $\|(\lambda - a)^{-1}\|^{-1} \leq \min(\gamma - \alpha, \beta - \gamma)$. For $t \in [0,1]$, we therefore have

$$\|(\lambda - b_t) - (\lambda - a)\| = t\|\exp(x) - a\| < \|(\lambda - a)^{-1}\|^{-1},$$

whence $\lambda \notin \mathrm{sp}(b_t)$.

Define $\chi(\lambda) = 0$ if $|\lambda| > \gamma$ and $\chi(\lambda) = 1$ if $|\lambda| < \gamma$. Then χ is holomorphic on a neighborhood of each $\mathrm{sp}(b_t)$, and $t \longmapsto e_t = \chi(b_t)$ defines a continuous path of idempotents with $e_1 = p$. Let $t \longmapsto z_t$ be a continuous path of invertible elements with $z_1 = 1$ and $z_t e_t z_t^{-1} = p$. Then $t \longmapsto c_t = z_t(e_t b_t e_t)z_t^{-1}$ is a continuous path in $\mathrm{inv}(pAp)$ with $c_1 = \alpha u$. Furthermore, $\exp(e_0 x e_0) = e_0 b_0 e_0$ in the Banach algebra $e_0 A e_0$, so that c_0 is an exponential in pAp. It follows that $\alpha u \in \mathrm{inv}_0(pAp)$, whence $u \in U_0(pAp)$.

A similar argument shows that $v \in U_0\big((1-p)A(1-p)\big)$. ∎

THEOREM 7.5. *Let A be a simple C^*-algebra with no dimension growth. If $K_1(A) \neq 0$ then* $\mathrm{ber}(A) = 2$.

PROOF. A certainly contains nontrivial projections, so choose one, and call it p. As in the proofs of Lemma 4.5 and Corollary 4.6, $K_1(pAp) \longrightarrow K_1(A)$ and $K_1\big((1-p)A(1-p)\big) \longrightarrow K_1(A)$ are isomorphisms and $\mathrm{tsr}(A) = \mathrm{tsr}(pAp) = \mathrm{tsr}\big((1-p)A(1-p)\big) = 1$. Theorem 2.10 of [27] implies that if $\mathrm{tsr}(B) = 1$ then $U(B)/U_0(B) \longrightarrow K_1(B)$ is an isomorphism. Therefore we can choose $\eta \neq 0$ in $K_1(A)$, and we can choose $u \in U(pAp)$ and $v \in U\big((1-p)A(1-p)\big)$ such that the images in $K_1(A)$ of $[u]$ and $[v]$ are η and $-\eta$ respectively. Furthermore, $u + v \in U_0(A)$. Therefore $a = 2u + v \in \mathrm{inv}_0(A)$, but, taking $\gamma = 3/2$, the lemma shows $\|\exp(x) - a\| \geq 1/2$ for all $x \in A$. Thus $\mathrm{ber}(A) \geq 2$.

We have $\mathrm{ber}(A) \leq 2$ by Proposition 7.1. ∎

In particular, the irrational rotation algebras A_θ are covered by this theorem [9], and so satisfy $\mathrm{ber}(A_\theta) = 2$. (This can also be proved directly from Lemma 7.4.) Similary, the Bunce-Deddens algebras have Banach exponential rank 2.

The proof of Theorem 7.5 also shows that if A is simple with slow dimension growth (as in [4]), and $K_1(A) \neq 0$, then $\mathrm{ber}(A) \geq 2$. We do not know whether $\mathrm{ber}(A)$ is $1 + \varepsilon$ or 2 in case $K_1(A) = 0$ and A has no dimension growth, but with a dimension bound greater than 1.

Lemma 7.4 also yields the following result.

THEOREM 7.6. *Let A be a unital C^*-algebra, and suppose $U(A)$ is not connected. Then* $\mathrm{ber}\big(M_n(A)\big) \geq 2$ *for any* $n \geq 2$.

PROOF. Let $u \in U(A) - U_0(A)$, and let $a = \mathrm{diag}(2u, u^*, 1, \ldots, 1)$. Then $a \in \mathrm{inv}_0\big(M_n(A)\big)$, and Lemma 7.4 implies $\|a - \exp(x)\| \geq 1/2$ for any $x \in A$. ∎

This theorem does not hold for $n = 1$, since $U\big(C(X)\big)$ need not be connected but $\mathrm{ber}\big(C(X)\big)$ is always 1.

We next turn to the C^* exponential length. For this quantity, we can give a complete answer.

THEOREM 7.7. *Let A be a simple C^*-algebra with no dimension growth. If A has real rank* 0 *then* $\mathrm{cel}(A) = \pi$, *and if A has real rank* 1, *then* $\mathrm{cel}(A) = \infty$.

The real rank 0 case is easy. (Let $u \in U_0(A)$. Using $\mathrm{cer}(A) \leq 1 + \varepsilon$ and real rank zero, choose $a \in A_{\mathrm{sa}}$ with finite spectrum such that $\|\exp(ia) - u\| < \varepsilon$. Using finiteness of the spectrum, it is easy to find $b \in A_{\mathrm{sa}}$ with $\exp(ib) = \exp(ia)$ and $\mathrm{sp}(b) \subset (-\pi, \pi]$. Since $\varepsilon > 0$ is arbitrary, this shows that the rectifiable distance from u to 1 is at most π.)

We omit the proof of the real rank 1 case; it will follow from a more general theorem which we will prove elsewhere. (See [25].) We point out, however, that is is not too hard to show directly that the algebra A in Example 7.3 satisfies $\mathrm{cel}(A) = \infty$. In fact, using Lemma 3.2 and some approximation arguments in direct limits, one can show that the image in A of the function $u_0(t) = \exp(iNt)$ in A_0 satisfies $\mathrm{cel}(u) \geq N/e$.

Finally, we discuss the C^* projective length. The obvious result, comparable to the estimate $3 + \varepsilon$ for C^* exponential rank, is:

PROPOSITION 7.8. *Let A be as in Notation 1.3, with no dimension growth. Assume*

$$\lim_{i \to \infty} \inf_{1 \leq t \leq s(i)} n(i, t) = \infty.$$

Then $\mathrm{cpl}(A) \leq 2\pi$.

PROOF. Let d be the dimension bound. It is shown in [24] that for all sufficiently large n, every compact space X of dimension at most d satisfies $\mathrm{cpl}\big(C(X) \otimes M_n\big) \leq 2\pi$. Now take direct limits. ∎

Just as for exponential rank, simplicity presumably enables one to reduce this estimate. In fact, the methods used to prove Theorem 6.1 should also yield $\mathrm{cpl}(A) \leq \pi$ when A is simple with no dimension growth. The point is that we must estimate, not $\mathrm{cel}(u)$ for an arbitrary u, but only

$$(*) \qquad \inf\{\mathrm{cel}(u) : upu^* = q\},$$

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for pairs (p,q) of equivalent projections. Making small perturbations, we may assume $p,q \in A_i$ for some i, and that there is $u \in U_0(A_i)$ such that $upu^* = q$. This relation is not affected if we factor out the determinant. Therefore

$$\mathrm{cpl}(a) \leq \sup\Big\{\mathrm{cel}(u) : u \in \bigcup_{i=1}^{\infty} A_i, \det(u) = 1\Big\}.$$

We believe that suitable modifications of Lemma 5.2 and 5.3 will show that if $\det(u) = 1$, then $\mathrm{cel}(u) \leq \pi$ (even though, for general u, $\mathrm{cel}(u)$ can be arbitrarily large).

Actually, there is reason to think one can do better. If A has real rank 0, then it follows from [31] that $\mathrm{cpl}(A) \leq \pi/2$. Furthermore, the algebra A of Example 7.3 (which has real rank 1) satisfies $\mathrm{cpl}(A) \leq \pi/2$. In fact, we have the following result:

PROPOSITION 7.9. *Let A be a unital direct limit as in Notation 1.3, not necessarily simple, and assume that the dimension bound d is* 1. *Then* $\mathrm{cpl}(A) \leq \pi/2$.

PROOF. By Proposition 2.11 of [23], it suffices to prove that if $\dim(X) \leq 1$ then $\mathrm{cpl}\big(C(X) \otimes M_n\big) \leq \pi/2$. Since X is an inverse limit of finite complexes of dimension at most 1 ([10], Theorem 1.13.5), it suffices to prove $\mathrm{cpl}\big(C(X) \otimes M_n\big) \leq \pi/2$ when X is such a finite complex. We may further assume X is connected.

Let $p_0 \in M_n$ be a projection. We claim that the set of projections $q_0 \in M_n$ such that $\mathrm{rank}(p_0) = \mathrm{rank}(q_0)$ and $\|p_0 - q_0\| = 1$ is the union of finitely many submanifolds, each of codimension at least 2 in the set of all projections with the same rank as p_0. Given this claim, let $p, q \in C(X) \otimes M_n$ be unitarily equivalent projections. Using the method of proof of Lemma 2.5 of [20] (as in the proof of Lemma 1.5), we can find arbitrarily small perturbations p_0 and q_0 of p and q which satisfy $\|p_0 - q_0\| < 1$. It follows from [23] (see Lemma 2.3 and the proof of Theorem 2.4) that $\mathrm{cpl}(p_0, q_0) \leq \pi/2$. Since $\|p - p_0\|$ and $\|q - q_0\|$ are arbitrarily small, it follows that $\mathrm{cpl}(p,q) \leq \pi/2$. Thus, the claim implies the proposition.

It remains only to prove the claim. So let $p_0 \in M_n$ be a projection of rank k. Let P be the set of all projections of rank k, and let Q be the set of projections of rank $n-k$. Then P and Q are submanifolds of M_n and $p \longmapsto 1-p$ is a diffeomorphism from Q onto P. Let G be the set of $p \in P$ satisfying

$$(*) \qquad p_0\mathbb{C}^n \cap (1-p)\mathbb{C}^n = 0$$

and

$$(**) \qquad (1-p_0)\mathbb{C}^n \cap p\mathbb{C}^n = 0.$$

If $p \in P$, then $(*)$ implies that for every nonzero $\xi \in p_0\mathbb{C}^n$ there is a nonzero $\eta \in p\mathbb{C}^n$ such that the angle between ξ and η is less than $\pi/2$. Also $(**)$ implies the same statement with p and p_0 interchanged. By compactness of the unit sphere, there is $\theta < \pi/2$ such that for $\xi \in p_0\mathbb{C}^n - \{0\}$ there is $\eta \in p\mathbb{C}^n - \{0\}$ such that the angle between ξ and η is at most θ, and similarly with p and p_0 interchanged. Therefore $\|p - p_0\| < 1$ by Lemma 4.6 of [25].

We now complete the proof of the claim by showing that $P - G$ is the union of finitely many submanifolds of codimension at least 2. By Lemma 1.6 of [24], the set of $p \in P$ which do not satisfy $(**)$ is the union of finitely many submanifolds, each of dimension at most $2k(n-k)-2$. Similarly, the set $\{q \in Q : p_0\mathbb{C}^n \cap q\mathbb{C}^n \neq 0\}$ is the union of finitely many submanifolds of dimension at most $2k(n-k)-2$. Applying the diffeomorphism $q \longmapsto 1-q$, we conclude that the set of projections $p \in P$ not satisfying $(*)$ is also such a finite union of submanifolds. Since $\dim(P) = 2k(n-k)$, this completes the proof. ■

The real rank 0 case and this proposition suggest the following question.

QUESTION 7.10. Let A be simple with no dimension growth, and suppose A has real rank 1. Does it follow that $\mathrm{cpl}(A) \leq \pi/2$?

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