MAXIMAL OR GREATEST HOMOMORPHIC IMAGES OF GIVEN TYPE

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1. Introduction. Let Q be a quasi-ordered set with respect to $\leqslant$; that is, the order $\leqslant$ is reflexive and transitive. An element a of Q is called maximal (minimal) if

$$x \geqslant a \ (x \leqslant a) \quad \text{implies} \quad x \leqslant a \ (x \geqslant a);$$

a is called greatest (smallest) if

$$a \geqslant x \ (a \leqslant x) \quad \text{for all } x \in Q.$$

Obviously a greatest (smallest) element is maximal (minimal). A greatest (smallest) element in a partially ordered set is unique, but it is not necessarily unique in a quasi-ordered set.

Let S be a groupoid (more generally an algebraic system with binary operations), and $\mathfrak{C}$ be the set of all congruences on S. We define two relations on $\mathfrak{C}$ as follows (let small Greek letters denote elements of $\mathfrak{C}$):

(1) $\rho \subseteq \sigma$ if and only if $x \,\rho\, y$ implies $x \,\sigma\, y$.

(2) $\rho \rightarrow \sigma$ if and only if S/ρ is homomorphic onto S/σ.

$\mathfrak{C}$ is a complete lattice with respect to $\subseteq$ in which the equality relation ι is smallest and the universal relation $\omega = S \times S$ is greatest. $\mathfrak{C}$ is a quasi-ordered set with respect to $\rightarrow$. A subset $\mathfrak{T}$ of $\mathfrak{C}$ is called a (congruence) type (on S) if

$$\omega \in \mathfrak{T}, \qquad \text{where } \omega = S \times S.$$

A type $\mathfrak{T}$ is called isomorphically closed (iso-closed) if and only if

$$\rho \in \mathfrak{T}, \sigma \in \mathfrak{C}, S/\rho \cong S/\sigma \quad \text{imply} \quad \sigma \in \mathfrak{T}.$$

A subset $\mathfrak{F}$ of $\mathfrak{T}$ is iso-closed relative to $\mathfrak{T}$ if and only if

$$\rho \in \mathfrak{F}, \sigma \in \mathfrak{T}, S/\rho \cong S/\sigma \quad \text{imply} \quad \sigma \in \mathfrak{F}.$$

A type $\mathfrak{T}$ is called a basic type if and only if

$$\text{for any } \{\rho_\alpha\} \subseteq \mathfrak{T}, \qquad \phi \neq \bigcap\nolimits_\alpha \rho_\alpha \in \mathfrak{T}.$$

Proposition 1.7 in **(1)** says that if $\mathfrak{T}$ is a basic type, then there is a smallest element in $\mathfrak{T}$ with respect to $\rightarrow$. However, the converse is not true. If $\mathfrak{T}$ is defined by the property "having zero," then we have examples; see **(4; 5)**.

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Let Δ_ρ denote the partition of S induced by a congruence ρ. We call Δ_ρ a decomposition. Customarily we define

$$\Delta_\rho \geqslant \Delta_\sigma \quad \text{if and only if } \rho \subseteq \sigma.$$

We say that Δ_ρ is greater than Δ_σ, i.e. Δ_ρ is a refinement of Δ_σ; if S/ρ is homomorphic onto S/σ, we say that S/ρ is greater than S/σ.

Let $\mathfrak{T}$ be a (congruence) type on S. Any $\rho \in \mathfrak{T}$ is called a $\mathfrak{T}$-congruence; S/ρ is called a $\mathfrak{T}$-homomorphic image of S if $\rho \in \mathfrak{T}$; Δ_ρ is called a $\mathfrak{T}$-decomposition of S if $\rho \in \mathfrak{T}$. An element ρ_0 of $\mathfrak{T}$ is said to be a smallest (minimal) $\mathfrak{T}$-congruence on S if ρ_0 is smallest (minimal) with respect to $\subseteq$; S/ρ_0 is called a greatest (maximal) $\mathfrak{T}$-homomorphic image of S if ρ_0 is smallest (minimal) with respect to $\rightarrow$. Clearly a congruence ρ_0 is a smallest (minimal) $\mathfrak{T}$-congruence on S if and only if Δ_{ρ_0} is a greatest (maximal) $\mathfrak{T}$-decomposition of S.

The subjects "smallest congruence of given type" and "greatest decomposition of given type" have been studied by many people (**2**; **3**; **6–9**; **1**, p. 18), but no systematic study of maximal homomorphic image of given type has been done. This paper is partly a generalization of the result in (**5**). We remark that the term "maximal" has been used in many papers up to this time in the sense of "greatest." However, in this paper we distinguish "greatest" from "maximal."

2. Maximal homomorphic images. For simplicity, we shall mean by "$S/\rho \rightarrow S/\sigma$" the statement that S/ρ is homomorphic onto S/σ.

LEMMA 1. *If $\rho \subseteq \sigma$, then $S/\rho \rightarrow S/\sigma$.*

Proof. See (**1**, Corollary 1.6a, p. 17).

LEMMA 2. *If $S/\rho \rightarrow S/\sigma$, then there is a congruence ρ' on S such that*

$$\rho \subseteq \rho' \quad \textit{and} \quad S/\rho' \cong S/\sigma.$$

Proof. Let f be the homomorphism $S \rightarrow S/\rho$ and g be the homomorphism $S/\rho \rightarrow S/\sigma$. Then $S \rightarrow S/\sigma$ under fg: $x(fg) = (xf)g$, $x \in S$. If ρ' denotes the congruence on S induced by fg, then $\rho \subseteq \rho'$ and $S/\rho' \cong S/\sigma$.

THEOREM 1. *Let $\mathfrak{T}$ be an iso-closed (congruence) type on S. S has a maximal $\mathfrak{T}$-homomorphic image if and only if there is a non-empty subclass $\mathfrak{F}$ of $\mathfrak{T}$ such that the following conditions are satisfied:*

(1.1) *$\mathfrak{F}$ is iso-closed relative to $\mathfrak{T}$.*

(1.2) *If $\xi \in \mathfrak{F}$, $\eta \in \mathfrak{T}$, and $\eta \subseteq \xi$, then $\eta \in \mathfrak{F}$.*

(1.3) *If $\xi, \eta \in \mathfrak{F}$ and if $\eta \subseteq \xi$, then $S/\xi \rightarrow S/\eta$.*

If such an $\mathfrak{F}$ exists, then, for any $\xi \in \mathfrak{F}$, S/ξ is a maximal $\mathfrak{T}$-homomorphic image of S.

Proof. Suppose that S has a maximal $\mathfrak{T}$-homomorphic image, say S/ξ_0, $\xi_0 \in \mathfrak{T}$. We define $\mathfrak{F}$ by

$$\mathfrak{F} = \{\xi \in \mathfrak{T} \mid S/\xi \rightarrow S/\xi_0\}.$$

To prove (1.1), let $\eta \in \mathfrak{T}$ such that $S/\eta \cong S/\xi$ for some $\xi \in \mathfrak{F}$. Then $S/\eta \to S/\xi \to S/\xi_0$; hence $S/\eta \to S/\xi_0$, i.e. $\eta \in \mathfrak{F}$. To prove (1.2) let $\xi \in \mathfrak{F}$, $\eta \in \mathfrak{T}$, and $\eta \subseteq \xi$. Then $S/\eta \to S/\xi \to S/\xi_0$ by Lemma 1; therefore, $S/\eta \to S/\xi_0$, that is $\eta \in \mathfrak{F}$. To prove (1.3), let $\xi, \eta \in \mathfrak{F}$ and $\eta \subseteq \xi$. We see that

$$S/\eta \to S/\xi \to S/\xi_0$$

and so $S/\eta \to S/\xi_0$. By maximality of S/ξ_0, we have $S/\xi_0 \to S/\eta$. Together with $S/\xi \to S/\xi_0$, we have $S/\xi \to S/\eta$.

Conversely, suppose that there is a subclass $\mathfrak{F}$ of $\mathfrak{T}$ satisfying (1.1), (1.2), and (1.3). Let ξ be any element of $\mathfrak{F}$. We shall prove that S/ξ is a maximal $\mathfrak{T}$-homomorphic image of S. Suppose that $\eta \in \mathfrak{T}$ and $S/\eta \to S/\xi$. By Lemma 2 there is a congruence η' on S such that $\eta \subseteq \eta'$ and $S/\eta' \cong S/\xi$. Since $\mathfrak{T}$ is iso-closed, $\eta' \in \mathfrak{T}$. By (1.1), $\eta' \in \mathfrak{F}$; furthermore, by (1.2), $\eta \in \mathfrak{F}$. At last we have $S/\eta' \to S/\eta$ by (1.3). Together with $S/\xi \to S/\eta'$, we have $S/\xi \to S/\eta$. Thus S/ξ is a maximal $\mathfrak{T}$-homomorphic image of S.

(1.1) and (1.2) are equivalent to:

(1.4) If $\xi \in \mathfrak{F}$, $\eta \in \mathfrak{T}$, and if $S/\eta \to S/\xi$, then $\eta \in \mathfrak{F}$.

Remark. In Theorem 1 we assumed that $\mathfrak{T}$ is iso-closed. This restriction does not lose generality in the following sense. Let $\mathfrak{T}$ be a congruence type on S and $\bar{\mathfrak{T}}$ be the iso-closure of $\mathfrak{T}$, that is,

$$\bar{\mathfrak{T}} = \mathfrak{T} \cup \{\rho' \in \mathfrak{C} \mid S/\rho' \cong S/\rho,\ \rho \in \mathfrak{T}\}.$$

Then there is a maximal $\bar{\mathfrak{T}}$-homomorphic image of S if and only if there is a maximal $\mathfrak{T}$-homorphic image of S. For each $\xi \in \bar{\mathfrak{F}}$, S/ξ is a maximal $\mathfrak{T}$-homomorphic image of S, where $\bar{\mathfrak{F}}$ is given for $\bar{\mathfrak{T}}$ by Theorem 1.

3. Normal families and greatest homomorphic images. A subclass $\mathfrak{F}$ of an iso-closed type $\mathfrak{T}$ is called a normal family in $\mathfrak{T}$ if and only if $\mathfrak{T}$ satisfies (1.1), (1.2), and (1.3). Let $\mathfrak{N}$ be the system of all non-empty normal families in $\mathfrak{T}$:

$$\mathfrak{N} = \{\mathfrak{F}_\alpha \mid \alpha \in \Xi\}.$$

THEOREM 2. *$\mathfrak{N}$ is closed with respect to set union and non-empty intersection: If $\{\mathfrak{F}_\alpha\} \subset \mathfrak{N}$ is a (collection of) normal families in $\mathfrak{T}$, then $\bigcup_\alpha \mathfrak{F}_\alpha$ and $\bigcap_\alpha \mathfrak{F}_\alpha$ (if $\neq \emptyset$) are normal families in $\mathfrak{T}$.*

Proof. (1.1), (1.2) for $\bigcup_\alpha \mathfrak{F}_\alpha$ and (1.1), (1.2), (1.3) for $\bigcap_\alpha \mathfrak{F}_\alpha$ are obtained immediately. We shall prove (1.3) for $\bigcup_\alpha \mathfrak{F}_\alpha$. Suppose $\xi, \eta \in \bigcup_\alpha \mathfrak{F}_\alpha$ and that $\eta \subseteq \xi$. We may assume that $\xi \in \mathfrak{F}_\alpha$ and $\eta \in \mathfrak{F}_\beta$. Since $\mathfrak{F}_\alpha$ satisfies (1.2), $\eta \in \mathfrak{F}_\alpha$. Again by (1.3) for $\mathfrak{F}_\alpha$, we get $S/\xi \to S/\eta$.

In particular $\bigcup_{\alpha \in \Xi} \mathfrak{F}_\alpha$, the set union of all normal families in $\mathfrak{T}$, is also a normal family which is the greatest normal family in $\mathfrak{T}$. Let us put

$$\mathfrak{F}_0 = \bigcup_{\alpha \in \Xi} \mathfrak{F}_\alpha.$$

This is the set of all $\mathfrak{T}$-congruences ξ for which S/ξ is a maximal $\mathfrak{T}$-homomorphic image of S.

Let $\mathfrak{F}_1$ be a non-empty subset of $\mathfrak{T}$. $\mathfrak{F}_1$ is called homomorphically irreducible (hom-irreducible) if and only if

$$\text{for every } \rho, \eta \in \mathfrak{F}_1, \qquad S/\rho \rightarrow S/\eta \quad \text{and} \quad S/\eta \rightarrow S/\rho.$$

A hom-irreducible class is a class modulo the equivalence defined by $\rho \rightarrow \eta$ and $\eta \rightarrow \rho$.

LEMMA 3. *If $\mathfrak{F}$ is a normal family, then its hom-irreducible classes are also normal families.*

Proof. Let $\mathfrak{G}$ be a hom-irreducible class of $\mathfrak{F}$. To show (1.1), let $\xi \in \mathfrak{G}$, $\eta \in \mathfrak{T}$, $S/\eta \cong S/\xi$. Since $\mathfrak{F}$ is iso-closed, $\eta \in \mathfrak{F}$. Clearly $S/\eta \rightarrow S/\xi$ and $S/\xi \rightarrow S/\eta$; hence $\eta \in \mathfrak{G}$. To show (1.2), let $\xi \in \mathfrak{G}$ and $\eta \subseteq \xi$. Obviously $\eta \in \mathfrak{F}$ and $S/\eta \rightarrow S/\xi$. Since ξ is in $\mathfrak{G}(\subseteq \mathfrak{F})$, S/ξ is a maximal $\mathfrak{T}$-homomorphic image and hence $S/\xi \rightarrow S/\eta$, so $\eta \in \mathfrak{G}$. To show (1.3), suppose $\xi, \eta \in \mathfrak{G}$ and $\eta \subseteq \xi$. Clearly $S/\xi \rightarrow S/\eta$ since $\xi, \eta \in \mathfrak{F}$.

THEOREM 3. *Let $\mathfrak{T}$ be a fixed iso-closed type. The following statements are equivalent:*

(2.1) *$\bigcap_{\alpha \in \Xi} \mathfrak{F}_\alpha \neq \emptyset$ and it is hom-irreducible.*
(2.2) *$\bigcap_{\alpha \in \Xi} \mathfrak{F}_\alpha \neq \emptyset$.*
(2.3) *Any normal family is hom-irreducible.*
(2.4) *$\bigcup_{\alpha \in \Xi} \mathfrak{F}_\alpha$ is hom-irreducible.*
(2.5) *$\mathfrak{T}$ has a unique normal family and it is hom-irreducible.*
(2.6) *$\mathfrak{T}$ has a unique normal family.*

Proof. (2.1) $\Rightarrow$ (2.2) is trivial. First we shall prove (2.2) $\Rightarrow$ (2.3). Suppose that some normal family $\mathfrak{F}$ is not hom-irreducible under the assumption (2.2). This means that $\mathfrak{F}$ has more than one non-empty irreducible class, say $\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_1 \cap \mathfrak{G}_2 = \emptyset$. By Lemma 3, both $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are normal families. Then

$$\bigcap_{\alpha \in \Xi} \mathfrak{F}_\alpha \subseteq \mathfrak{G}_1 \cap \mathfrak{G}_2 = \emptyset \quad \text{and hence} \quad \bigcap_{\alpha \in \Xi} \mathfrak{F}_\alpha = \emptyset.$$

This is a contradiction. (2.3) $\Rightarrow$ (2.4) is obvious, since $\bigcup_{\alpha \in \Xi} \mathfrak{F}_\alpha$ is also a normal family by Theorem 2. Next we shall prove (2.4) $\Rightarrow$ (2.5). The assumption (2.4) implies that, by the definition,

$$\text{for any two } \rho, \sigma \in \bigcup_\alpha \mathfrak{F}_\alpha, \qquad S/\rho \rightarrow S/\sigma \quad \text{and} \quad S/\sigma \rightarrow S/\rho.$$

Accordingly, every normal family is hom-irreducible.

Let $\mathfrak{F}_0 = \bigcup_{\alpha \in \Xi} \mathfrak{F}_\alpha$ and let $\mathfrak{F}_\alpha$ be any normal family. Take any $\xi \in \mathfrak{F}_0$ and $\eta \in \mathfrak{F}_\alpha$. Then $S/\xi \rightarrow S/\eta$, that is, there is an $\eta_0 \in \mathfrak{C}$ such that $\xi \subseteq \eta_0$, $S/\eta_0 \cong S/\eta$. Since $\mathfrak{T}$ is iso-closed, $\eta_0 \in \mathfrak{T}$; since $\mathfrak{F}_\alpha$ is iso-closed relative to $\mathfrak{T}$, $\eta_0 \in \mathfrak{F}_\alpha$. We see that $\xi \in \mathfrak{F}_\alpha$ since $\mathfrak{F}_\alpha$ is a normal family. Thus we have $\mathfrak{F}_0 \subseteq \mathfrak{F}_\alpha$; therefore $\mathfrak{F}_\alpha = \mathfrak{F}_0$. (2.5) $\Rightarrow$ (2.1) is obvious. Finally we shall prove that (2.5) and (2.6) are equivalent. (2.5) $\Rightarrow$ (2.6) is obvious. (2.6) $\Rightarrow$ (2.5) is obtained in the following way: (2.6) $\Rightarrow$ (2.2) $\Rightarrow$ (2.3) $\Rightarrow$ (2.4) $\Rightarrow$ (2.5). This leads to a conclusion that the unique normal family is hom-irreducible.

THEOREM 4. *Let $\mathfrak{T}$ be an iso-closed type (of congruences) on a groupoid S. S has a greatest $\mathfrak{T}$-homomorphic image if and only if $\mathfrak{T}$ has a unique normal family $\mathfrak{F}_0$ and $\mathfrak{F}_0$ satisfies*

(3) *for any $\xi \in \mathfrak{T}$ there is an $\eta \in \mathfrak{F}_0$ such that $S/\eta \rightarrow S/\xi$.*

Proof. Suppose that S has a greatest $\mathfrak{T}$-homomorphic image S/ρ_0, $\rho_0 \in \mathfrak{T}$. S/ρ_0 is a maximal $\mathfrak{T}$-homomorphic image. By Theorem 1, $\mathfrak{N} \neq \emptyset$, that is, there is at least one normal family. Clearly $\rho_0 \in \mathfrak{F}_0 = \bigcup_{\alpha \in \Xi} \mathfrak{F}_\alpha$. Let $\mathfrak{N} = \{\mathfrak{F}_\alpha | \alpha \in \Xi\}$, $\mathfrak{F}_\alpha$ be any normal family, and let $\xi \in \mathfrak{F}_\alpha$. Since S/ρ_0 is greatest, $S/\rho_0 \rightarrow S/\xi$. Since $\mathfrak{F}_\alpha$ is a normal family, $\rho_0 \in \mathfrak{F}_\alpha$ for all $\alpha \in \Xi$, as we proved again and again by using Theorem 1. Therefore, $\bigcap_{\alpha \in \Xi} \mathfrak{F}_\alpha \neq \emptyset$. By Theorem 3, $\mathfrak{T}$ has a unique normal family. The condition (3) is clearly satisfied as ρ_0 is regarded as η.

Conversely, suppose that there is a unique normal family $\mathfrak{F}_0$ and $\mathfrak{F}_0$ satisfies (3). By Theorem 3, $\mathfrak{F}_0$ is hom-irreducible. We shall prove that for any $\rho \in \mathfrak{F}_0$, S/ρ is greatest. Let $\xi \in \mathfrak{T}$. By (3) there is $\eta \in \mathfrak{F}_0$ such that $S/\eta \rightarrow S/\xi$. On the other hand $S/\rho \rightarrow S/\eta$ by the irreducibility of $\mathfrak{F}_0$. Hence, we have $S/\rho \rightarrow S/\xi$ for all $\xi \in \mathfrak{T}$.

THEOREM 5. *Let $\mathfrak{T}$ be an iso-closed congruence type on a groupoid S. S has a greatest $\mathfrak{T}$-homomorphic image if and only if there is a non-empty subclass $\mathfrak{F}$ of $\mathfrak{T}$ such that:*

(4.1) *For any $\eta \in \mathfrak{T}$ and for any $\xi \in \mathfrak{F}$ there are $\eta_1 \in \mathfrak{T}$ and $\xi_1 \in \mathfrak{F}$ such that*

$$\xi_1 \subseteq \xi, \qquad \xi_1 \subseteq \eta_1, \qquad S/\eta_1 \cong S/\eta.$$

(4.2) *If $\xi, \eta \in \mathfrak{F}$ and if $\eta \subseteq \xi$, then $S/\xi \rightarrow S/\eta$.*

Proof. Suppose that S has a greatest $\mathfrak{T}$-homomorphic image S/ξ_0. $\mathfrak{F}$ is defined to be the class of all $\xi \in \mathfrak{T}$ such that $\xi \subseteq \xi_0$. For the proof of (4.1), let η be any element of $\mathfrak{T}$. Since S/ξ_0 is greatest, $S/\xi_0 \rightarrow S/\eta$ and hence there is a congruence σ on S such that $\xi_0 \subseteq \sigma$, $S/\sigma \cong S/\eta$ by Lemma 2. Since $\mathfrak{T}$ is iso-closed, $\sigma \in \mathfrak{T}$; hence $\xi \subseteq \sigma$ and $S/\sigma \cong S/\eta$. For the proof of (4.2), take $\xi, \eta \in \mathfrak{F}$ such that $\eta \subseteq \xi$. Since $\eta \subseteq \xi \subseteq \xi_0$, $S/\xi \rightarrow S/\xi_0$. Also since S/ξ_0 is greatest, $S/\xi_0 \rightarrow S/\eta$; hence $S/\xi \rightarrow S/\eta$.

Conversely, suppose (4.1) and (4.2) are satisfied by a subclass $\mathfrak{F}$. Let ξ be any element of $\mathfrak{F}$. We shall prove that S/ξ is a greatest $\mathfrak{T}$-homomorphic image of S. Let ζ be any element of $\mathfrak{T}$. By (4.1) there are $\xi_1 \in \mathfrak{F}$ and $\zeta_1 \in \mathfrak{T}$ such that $\xi_1 \subseteq \xi$, $\xi_1 \subseteq \zeta_1$, and $S/\zeta_1 \cong S/\zeta$. By (4.2) $S/\xi \rightarrow S/\xi_1$; clearly $S/\xi_1 \rightarrow S/\zeta_1$ by Lemma 1. Therefore $S/\xi \rightarrow S/\zeta_1$ and hence $S/\xi \rightarrow S/\zeta$.

A type $\mathfrak{T}$ is said to be lower directed if for any $\xi, \eta \in \mathfrak{T}$ there is a $\zeta \in \mathfrak{T}$ such that

(5) $\zeta \subseteq \xi \cap \eta$.

Statement (5) is a necessary condition for S to have a greatest $\mathfrak{T}$-homomorphic image.

THEOREM 6. *Let $\mathfrak{T}$ be an iso-closed lower-directed type of congruence on a groupoid S. S has a greatest $\mathfrak{T}$-homomorphic image if and only if $\mathfrak{T}$ has a unique normal family.*

Proof. Suppose that $\mathfrak{T}$ has a unique normal family $\mathfrak{F}_0$. Let $\xi \in \mathfrak{T}$ and $\zeta \in \mathfrak{F}_0 \subseteq \mathfrak{T}$. Since $\mathfrak{T}$ is lower-directed, there is an $\eta \in \mathfrak{T}$ such that $\eta \subseteq \zeta \cap \xi$. By (1.2) in Theorem 1, $\eta \in \mathfrak{F}_0$ because $\mathfrak{F}_0$ is a normal family. Immediately we have $S/\eta \to S/\xi$ since $\eta \subseteq \xi$. We have proved that (3) is fulfilled. Therefore S has a greatest $\mathfrak{T}$-homomorphic image by Theorem 4. The converse is obvious by Theorem 4.

4. Three relations on congruences. Let $\mathfrak{C}$ be the set of all congruences on a groupoid S. The three relations $\subseteq, \to, \cong$ on $\mathfrak{C}$ are denoted as follows:

$$\xi\, A\, \eta \quad \text{if and only if } \xi \subseteq \eta,$$
$$\xi\, B\, \eta \quad \text{if and only if } S/\xi \cong S/\eta,$$
$$\xi\, C\, \eta \quad \text{if and only if } S/\xi \to S/\eta.$$

Clearly $A \subseteq C, B \subseteq C$. C is completely determined by A and B by the following theorem:

THEOREM 7. *$C = A \cdot B = B \cdot A \cdot B$ and C is the quasi-ordering generated by A and B.* ($A \cdot B$ is a product of relations in the sense of (**1**, p. 13).)

To prove this theorem, we need the following lemma:

LEMMA 4. *Let ρ be a quasi-ordering and σ be a reflexive relation. Then the following statements are equivalent:*

(6.1) *$\rho \cdot \sigma$ is a quasi-ordering.*

(6.2) *$\rho \cdot \sigma = \sigma \cdot \rho \cdot \sigma$.*

(6.3) *$\rho \cdot \sigma$ is a quasi-ordering generated by ρ and σ.*

Proof. This is straightforward.

Proof of Theorem 7. Suppose $\xi\, C\, \eta$, that is $S/\xi \to S/\eta$. By Lemma 2 there is $\rho' \in \mathfrak{C}$ such that $\xi \subseteq \rho'$ (and hence $S/\xi \to S/\rho'$) and $S/\rho' \cong S/\eta$. Therefore we have $C \subseteq A \cdot B$. It remains to show that $A \cdot B \subseteq C$. Since $A \subseteq C$ and $B \subseteq C$, we have $A \cdot B \subseteq C \cdot C \subseteq C$ because C is transitive. Thus we have $C = A \cdot B$. We know that C is a quasi-ordering. The remaining part can be proved as the application of Lemma 4.

As a corollary to Lemma 4, we have

COROLLARY.

(7.1) *Let ρ and σ be quasi-orderings. $\rho \cdot \sigma$ is a quasi-ordering if and only if $\rho \cdot \sigma = \sigma \cdot \rho \cdot \sigma$.*

(7.2) *Let ρ, σ be equivalences. $\rho \cdot \sigma$ is an equivalence if and only if $\rho \cdot \sigma = \sigma \cdot \rho$.*

(7.3) *Let ρ, σ be congruences. $\rho \cdot \sigma$ is a congruence if and only if $\rho \cdot \sigma = \sigma \cdot \rho$.*

5. Examples. Considering the relationship between decompositions and homomorphic images, there are seven possible cases as Table I shows. For each case we shall give an example.

TABLE I*

Case No.	Greatest decomposition	Maximal decomposition	Greatest homomorphic image	Maximal homomorphic image
1	×	×	×	×
2	0	×	×	×
3	0	×	0	×
4	0	×	0	0
5	0	0	×	×
6	0	0	0	×
7	0	0	0	0

*× means "exists"; 0 means "does not exist."

Case 1. Let $\mathfrak{T}$ be a type given by implication or identity, for example, $\mathfrak{T}$ defined by $\{x^2 = x, xy = yx\}$ or by $\{xy = xz \Rightarrow y = z\}$.

Case 2. Let S be the direct product of two copies of a cyclic group G of order prime: $S = G \times G$. Let $\mathfrak{T}$ be a type defined by "cyclic group." Then G is a greatest $\mathfrak{T}$-homomorphic image of S under each projection $S \rightarrow G$.

Case 3. Consider a right group $S = G \times I$, G a group, I a right zero semigroup, and $|G| > 1$, $|I| > 1$. Let $\mathfrak{T}$ be defined by "idempotent semigroup or group." We can understand this example by being aware of the fact that any congruence ρ on S is determined by a congruence σ on G and a congruence τ on I in the following sense:

$$(x, a)\ \rho\ (y, b) \text{ if and only if } x\ \sigma\ y \text{ and } a\ \tau\ b.$$

The projections $S \rightarrow G$ and $S \rightarrow I$ give maximal $\mathfrak{T}$-decompositions and maximal $\mathfrak{T}$-homomorphic images.

Case 4. Let S be a group defined by the restricted direct product

$$S = G_p \times G_{p^2} \times \ldots \times G_{p^n} \times \ldots$$

where G_{p^n} is a cyclic group of order p^n, p prime. Let $\mathfrak{T}$ be "cyclic group." A projection $S \rightarrow G_{p^n}$ $(n = 1, 2, \ldots)$ gives a maximal $\mathfrak{T}$-decomposition, but no G_{p^n} is a maximal $\mathfrak{T}$-homomorphic image.

Case 5. Let $S = \{\ldots, a_{-i}, \ldots, a_{-2}, a_{-1}, a_0, \ldots, a_i, \ldots\}$ be a semigroup in which $a_i a_j = a_k$, $k = \min(i, j)$. Let $\mathfrak{T}$ be "having zero." This example was discussed in **(5)**.

Case 6. Let G be the semigroup of all positive integers under addition and I be the same semigroup as S in the example for Case 5. Let S be the direct product of G and I. Let $\mathfrak{T}$ be "idempotent semigroup with zero or group." For simplicity, let $\mathfrak{T}_1$ denote $\mathfrak{T}$ in Case 5 and $\mathfrak{T}_2$ denote $\mathfrak{T}$ in Case 6. Then a greatest $\mathfrak{T}_1$-homomorphic image of I is a maximal $\mathfrak{T}_2$-homomorphic image of S; a group $G/(n)$ is a $\mathfrak{T}_2$-homomorphic image of S. Clearly there is neither maximal $\mathfrak{T}_2$-decomposition nor greatest $\mathfrak{T}_2$-homomorphic image of S.

Case 7. Let S be $\{1, 2, 3, 4, \ldots\}$ in which

$$i \cdot j = \max\{i, j\}.$$

Let $\mathfrak{T}$ be "having zero."

Addendum. We remark that the two conditions (4.1) and (4.2) in Theorem 5 are equivalent to the following single condition:

(4.3) *For any* $\eta \in \mathfrak{T}$ *and for any* $\xi \in \mathfrak{F}$ *there is an* $\eta_1 \in \mathfrak{T}$ *such that* $\xi \subseteq \eta_1$ *and* $S/\eta_1 \cong S/\eta$.

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