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# A NOTE ON LAGRANGE INTERPOLATION FOR $|x|^\lambda$ AT EQUIDISTANT NODES

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In this note, we discuss the exceptional set $E \subseteq [-1,1]$ of points $x_0$ satisfying the inequality

$$\liminf_{n\to\infty} n^{-1}\log\bigl||x|^\lambda - L_n(f_\lambda, x_0)\bigr| < \frac{1}{2}\bigl[(1+x_0)\log(1+x_0) + (1-x_0)\log(1-x_0)\bigr],$$

where $\lambda > 0, \lambda \neq 2, 4, \dots$ and $L_n(f_\lambda, .)$ is the Lagrange interpolation polynomial of degree at most $n$ to $f_\lambda(x) := |x|^\lambda$ on the interval $[-1,1]$ associated with the equidistant nodes. It is known that $E$ has Lebesgue measure zero. Here we show that $E$ contains infinite families of rational and irrational numbers.

## 1. INTRODUCTION

Let $L_n(f, .)$ be the Lagrange interpolation polynomial of degree at most $n$ to a continuous function $f$ on $[-1,1]$ associated with the equidistant nodes $x_{j,n} := -1 + 2j/n, j = 0, 1, \dots, n, n \in \mathbb{N}$ and let $f_\lambda(x) := |x|^\lambda$.

In 1916 Bernstein ([**1, 2, 7**]) proved the surprising result that the sequence $L_n(f_1, x_0)$ is divergent as $n \to \infty$ for every $x_0 \in [-1,1]$, apart from the values $x_0 = -1, 0, 1$. Since the endpoints $\pm 1$ are interpolation points for every index $n$ the sequence of the interpolation polynomials cannot diverge there. On the other hand, for the point zero it is proved in Natanson ([**7**, pp.30–35]) that $\lim\limits_{n\to\infty} L_n(f_1, 0) = 0$. For other results in this direction, see also [**9**]. The classical result of Bernstein was revisited in the 1990s and 2000s. In particular, the rate of this divergence process was discussed in ([**3, 5, 6, 8, 10**]). More precisely, the following $n$th root asymptotic relation for $0 < |x_0| < 1$

$$\limsup_{n\to\infty} \frac{1}{n}\log\bigl|\,|x|^\lambda - L_n(f_\lambda, x_0)\bigr| = \frac{1}{2}\bigl[(1+x_0)\log(1+x_0) + (1-x_0)\log(1-x_0)\bigr] \tag{1}$$

was established for $\lambda = 1$ by Byrne, Mills and Smith [**3**] and for $\lambda = 3$ by the second author [**8**]. Recently, the first author [**5**] proved the conjecture posed in [**8**] that (1) holds for all $\lambda > 0$, $\lambda \neq 2, 4, \dots$ .

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Li and Mohapatra [6] showed that for almost every $x_0 \in [-1,1]$,

$$\lim_{n=p_k+1\to\infty} \frac{1}{n}\log\big||x| - L_n(f_1, x_0)\big| = \frac{1}{2}\big[(1+x_0)\log(1+x_0) + (1-x_0)\log(1-x_0)\big],$$

where $(p_k)_{k=1}^{\infty}$ is the increasing sequence of all positive prime numbers. The following generalisation and strengthening of this result was proved in [5]:

$$\lim_{n\to\infty} \frac{1}{n}\log\big||x|^{\lambda} - L_n(f_\lambda, x_0)\big| = \frac{1}{2}\big[(1+x_0)\log(1+x_0) + (1-x_0)\log(1-x_0)\big], \tag{2}$$

for almost every $x_0 \in [-1,1], \lambda > 0, \lambda \neq 2, 4, \ldots$ In this note, we discuss the exceptional set $E \subseteq [-1,1]$ of Lebesgue measure zero for which (2) does not hold for each $x_0 \in E$, that is

$$\liminf_{n\to\infty} \frac{1}{n}\log\big||x|^{\lambda} - L_n(f_\lambda, x_0)\big| < \frac{1}{2}\big[(1+x_0)\log(1+x_0) + (1-x_0)\log(1-x_0)\big], \tag{3}$$

for $\lambda > 0, \lambda \neq 2, 4, \ldots$ and $x_0 \in E$. It is easy to see that $E = E_o \cup E_e$, where $E_o$ and $E_e$ are the subsets of $[-1,1]$ for which (3) holds with $\liminf_{n\to\infty}$ replaced by $\liminf_{n=2N-1\to\infty}$ and by $\liminf_{n=2N\to\infty}$, respectively. In the following theorem we show that $E_o$ and $E_e$ contain certain infinite families of rational and irrational numbers.

**THEOREM 1.**

(a) *Any rational number $x_0 \in (-1,1)$ belongs to $E_e$.*

(b) *For any odd $k$ and odd $m > 0$, satisfying $|k| < m$, we have $x_0 := k/m \in E_o$.*

(c) *There exists an infinite family $\{\beta_R\}_{R\geqslant 1}$ of irrational numbers such that $\beta_R \in E_o \cap E_e$ for all real $R \geqslant 1$.*

To prove Theorem 1 we shall need the following result on Diophantine approximation:

**LEMMA 2.**

(a) *For any real $R \geqslant 1$ there exists an irrational number $\beta_R \in (0, 1/3)$ and two sequences of odd numbers $\big(p_n(R)\big)_{n=1}^{\infty}$ and $\big(q_n(R)\big)_{n=1}^{\infty}$ such that*

$$0 < \left|\beta_R - \frac{p_n(R)}{q_n(R)}\right| < \frac{1}{3^{Rq_n(R)}}. \tag{4}$$

(b) *The family $\{\beta_R\}_{R\geqslant 1}$ is infinite.*

PROOF: (a) To this end select a real number $R \geqslant 1$ and let us define an increasing sequence $\big(a_m(R)\big)_{m=1}^{\infty}$ by the following recurrence formula:

$$a_1(R) := 3,$$
$$a_{m+1}(R) := 3^{[Ra_m]+1}, \quad \forall m \geqslant 1,$$

where $[x]$ stands for the integral part of $x$. Then (with the shortened notation $a_j = a_j(R)$) we set

$$\beta_R := \sum_{m=1}^{\infty}(-1)^{m+1}a_m^{-1}.$$

Since $(a_m)_{m=1}^{\infty}$ is an increasing sequence, $\beta_R$ is a well defined element and one simply checks that $0 < \beta_R < 1/3$. Next since $a_M/a_m$ is odd for $M \geqslant m \geqslant 1$, we can define the odd numbers

$$q_n(R) := a_{2n-1},$$
$$p_n(R) := \sum_{m=1}^{2n-1}(-1)^{m+1}a_{2n-1}/a_m,$$

for $n \in \mathrm{N}$. Then $p_n(R)/q_n(R) = \sum_{m=1}^{2n-1}(-1)^{m+1}a_m^{-1}$, and we have

$$0 < \frac{1}{a_{2n}} - \frac{1}{a_{2n+1}} < \left|\beta_R - \frac{p_n(R)}{q_n(R)}\right| = \left|\sum_{m=2n}^{\infty}(-1)^{m+1}\frac{1}{a_m}\right| \leqslant \frac{1}{a_{2n}}$$
$$= \frac{1}{3^{[Ra_{2n-1}]+1}} < \frac{1}{3^{Rq_n(R)}}.$$

This establishes the mentioned inequalities in (4). It remains to show that $\beta_R$ is irrational. Indeed, assuming that $\beta_R$ is rational, say $\beta_R = a/b$, and taking account of the left-hand inequality in (4), we obtain by a standard argument in Diophantine approximation the following estimate

$$\left|\beta_R - \frac{p_n(R)}{q_n(R)}\right| \geqslant \frac{1}{bq_n(R)}.$$

This obviously contradicts the right-hand inequality in (4) for $n$ sufficiently large. Therefore statement (a) of the lemma follows.

(b) This statement follows from the inequalities

(5) $$\beta_1 < \beta_2 < \cdots < \beta_n < \cdots$$

To prove (5), a standard argument on $\beta_R$ leads us to the both-sided estimates

(6) $$3^{-3R}(1 - 3^{-78}) \leqslant 3^{-3R} - 3^{-R3^{3R+1}} < 1 - 3\beta_R < 3^{-3R},$$

which are valid for all integers $R \geqslant 1$. Since the family of intervals $\left\{[3^{-3R}(1 - 3^{-78}), 3^{-3R}]\right\}_{R=1}^{\infty}$ is mutually disjoint, then (6) implies (5) and thus the lemma is proved. ∎

Proof of Theorem 1: (a) If $x_0 = k/m$ is a rational number with $m > 0$ and $|k| < m$, then for any $p = 1, 2, \ldots$, the node $x_{p(m+k),2pm}$ coincides with $x_0$. Hence

(7) $$0 = |x_0|^\lambda - L_{N_p}(f_\lambda, x_0) < \left((1 + x_0)^{1+x_0}(1 - x_0)^{1-x_0}\right)^{pm},$$

for $p = 1, 2, \ldots$, where $N_p = 2pm$. Thus $x_0 \in E_e$.

(b) Similarly, if $x_0 = k/m$ is rational for odd $k$ and $m$ with $m > 0$ and $|k| < m$, we find that the node

$$x_{(2p+1)(k+m)/2,(2p+1)m} = x_0.$$

Therefore (7) is again valid for $p = 1, 2, \ldots$, and $N_p = (2p+1)m$. That is, we have shown that $x_0 \in E_o$.

(c) Let $\{\beta_R\}_{R\geqslant 1}$ be the infinite family of irrational numbers from the lemma. To prove that $\beta_R \in E_o \cap E_e$ for all $R \geqslant 1$, we first show that the following inequalities hold:

$$\liminf_{n\to\infty}\left|\cos\left(\pi\beta_R\frac{2n-1}{2}\right)\right|^{1/(2n-1)} < 3^{-R}, \tag{8}$$

$$\liminf_{n\to\infty}\left|\sin(\pi\beta_R n)\right|^{1/(2n)} < 3^{-(R/2)}. \tag{9}$$

Indeed, let $\left(p_n(R)\right)_{n=1}^{\infty}$ and $\left(q_n(R)\right)_{n=1}^{\infty}$ be the sequences of odd numbers from the lemma. Then combining $|\sin x| \leqslant |x|$ together with (4), we get

$$\begin{aligned}
\liminf_{n\to\infty}\left|\cos\left(\pi\beta_R\frac{2n-1}{2}\right)\right|^{1/(2n-1)} &\leqslant \liminf_{n\to\infty}\left|\cos\left(\frac{\pi\beta_R q_n(R)}{2}\right)\right|^{1/(q_n(R))} \\
&= \liminf_{n\to\infty}\left|\sin\left[\frac{\pi q_n(R)}{2}\left(\beta_R - \frac{p_n(R)}{q_n(R)}\right)\right]\right|^{1/(q_n(R))} \\
&\leqslant \liminf_{n\to\infty}\left[\left(\frac{\pi q_n(R)}{2}\right)^{1/(q_n(R))}\left|\beta_R - \frac{p_n(R)}{q_n(R)}\right|^{1/(q_n(R))}\right] \\
&\leqslant 3^{-R}.
\end{aligned}$$

This implies (8). Next using (4) again, we obtain

$$\begin{aligned}
\liminf_{n\to\infty}\left|\sin(\pi\beta_R n)\right|^{1/(2n)} &\leqslant \liminf_{n\to\infty}\left|\sin(\pi\beta_R q_n)\right|^{1/(2q_n(R))} \\
&= \liminf_{n\to\infty}\left|\sin\left[\pi q_n(R)\left(\beta_R - \frac{p_n(R)}{q_n(R)}\right)\right]\right|^{1/(2q_n(R))} \\
&\leqslant 3^{-(R/2)}.
\end{aligned}$$

Thus (9) follows.

To proceed further, we use the following asymptotics for the interpolation errors established in ([**5**, Theorem 4]):

(10)

$$|x_0|^\lambda - L_{2n-1}(f_\lambda, x_0) = B_1(2n-1)^{-\lambda-2}x_0^{-2}\cos\left(\frac{\pi(2n-1)x_0}{2}\right)\varphi_{2n-1}(x_0)\left(1+\alpha_{n,1}(x_0)\right),$$

$$|x_0|^\lambda - L_{2n}(f_\lambda, x_0)$$

(11)

$$|x_0|^\lambda - L_{2n}(f_\lambda, x_0) = B_2 n^{-\lambda-1}x_0^{-1}\sin(\pi n x_0)\varphi_{2n}(x_0)\left(1+\alpha_{n,2}(x_0)\right),$$

where $0 < |x_0| < 1, \lambda > 0, B_1 = B_1(\lambda)$ and $B_2 = B_2(\lambda)$ are some constants,

$$\varphi_N(x) := \sqrt{1-x^2}\left[(1+x)^{1+x}(1-x)^{1-x}\right]^{N/2},$$

and the error terms $\alpha_{n,i}(x)$ satisfy the estimates

$$|\alpha_{n,i}(x)| \leqslant C_i n^{-(1/3)}, \quad i = 1, 2. \tag{12}$$

Here $C_i$ is independent of $n$, $i = 1, 2$. Then using (8), (10) and (12), we have for $R \geqslant 1$

$$\liminf_{n\to\infty} \left|\beta_R{}^\lambda - L_{2n-1}(f_\lambda, \beta_R)\right|^{1/(2n-1)} \leqslant 3^{-R}\left[(1+\beta_R)^{1+\beta_R}(1-\beta_R)^{1-\beta_R}\right]^{1/2}.$$

Thus $\beta_R \in E_o$. Furthermore using (9), (11) and (12), we get for $R \geqslant 1$

$$\liminf_{n\to\infty} \left|\beta_R{}^\lambda - L_{2n}(f_\lambda, \beta_R)\right|^{1/(2n)} \leqslant 3^{-(R/2)}\left[(1+\beta_R)^{1+\beta_R}(1-\beta_R)^{1-\beta_R}\right]^{1/2}.$$

This shows that $\beta_R \in E_e$.This completes the proof of the theorem. □

REMARK 3. The theorem is new even for $\lambda = 1$.

REMARK 4. If we drop the condition in statement (a) of the lemma that $p_n(R)$ and $q_n(R)$ are odd numbers, then the existence of $\beta_R$ satisfying (4) is well known in Diophantine approximation (see [**4**]).

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